Exact Solutions of the Free Lateral Vibration Period of Beams Subjected to a Concentrated Load and a Full Uniform Load simultaneously

By

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Synopsis

This paper gives the exact solutions of the period of the free lateral vibration of various types of beams, which are loaded by a concentrated load and a full uniform load, simultaneously, and also shows the tables from which the exact values of the period can easily obtained if a ratio of a concentrated load to a uniform load and the position of a concentrated load are given.

I. Frequency Equation of the Period of the Free Lateral Vibration of Beams subjected to a Concentrated Load and a Uniform Load simultaneously

The differential equations for the free lateral vibration of a beam are given by the following equations. In these equations, a is the distance of the loaded point of a concentrated load P from the left support of a beam, and b is the distance of P from the right support.

$$\left.\begin{aligned} &\frac{\partial^4 y_1}{\partial x^4}+\lambda^2\frac{\partial^2 y_1}{\partial t^2}=0 \qquad (x=0\sim a)\,, \\ &\frac{\partial^4 y_2}{\partial x^4}+\lambda^2\frac{\partial^2 y_2}{\partial t^2}=0 \qquad (x=0\sim b\,, \end{aligned}\right\} \tag{1}$$

in which $\lambda^2=\dfrac{w}{gEI}$

w = intensity of uniform load,
g = gravity acceleration,
E = modulus of elasticity,
I = moment of inertia of the cross section of beam.

The general solutions of equations (1) become as follows:

$$y_1 = u_1 \cdot q\,, \qquad y_2 = u_2 \cdot q\,, \tag{2}$$

in which

$$\left.\begin{aligned} u_1 &= A_1 \cos\beta x + B_1 \sin\beta x + C_1 \cosh\beta x + D_1 \sinh\beta x\,, \\ u_2 &= A_2 \cos\beta x + B_2 \sin\beta x + C_2 \cosh\beta x + D_2 \sinh\beta x\,, \end{aligned}\right\} \tag{3}$$

$$q = A \cos nt + B \sin nt\,, \tag{4}$$

$$n = \frac{\beta^2}{\lambda} = \beta^2 \sqrt{\frac{gEI}{w}}\,, \tag{5}$$

$A_1 \sim D_2$, A, B = arbitrary constants.

Then the period of the free lateral vibration is given by

$$T = \frac{2\pi}{n} = \frac{2\pi}{(\beta l)^2} \sqrt{\frac{wl^4}{gEI}}\,, \tag{6}$$

in which l is the length of a beam.

For the determination of the period of the vibration, 8 conditions are required. These conditions consist of four end conditions and further four conditions at the loaded point of a concentrated load P. The conditions at the point loaded by P are as follows:

$$(u_1)_{x=a} = (u_2)_{x=b}\,, \tag{7 a}$$

$$\left(\frac{du_1}{dx}\right)_{x=a} = -\left(\frac{du_2}{dx}\right)_{x=b}, \tag{7 b}$$

$$\left(\frac{d^2u_1}{dx^2}\right)_{x=a} = \left(\frac{d^2u_2}{dx^2}\right)_{x=b}, \tag{7 c}$$

$$EI\left(\frac{\partial^3 y_1}{\partial x^3}\right)_{x=a} + EI\left(\frac{\partial^3 y_2}{\partial x^3}\right)_{x=b} = \frac{P}{g}\left(\frac{\partial^2 y_1}{\partial t^2}\right)_{x=a}. \tag{7 d}$$

These conditions can be applied for any beams. From equation (7 d) one obtains

$$\left(\frac{d^3u_1}{dx^3}\right)_{x=a} + \left(\frac{d^3u_2}{dx^3}\right)_{x=b} = -k\beta^4 l (u_1)_{x=a} \tag{7 d'}$$

in which k is the ratio of a concentrated load P to uniform load.

By these four equations (7a)$\sim$(7d) and four end conditions in each beam, the frequency equation for each beam can be obtained. In this paper, a simple beam, a cantilever beam and a fixed beam are studied.

A. Simple Beam

In this case, end conditions are given as follows:

$$\left.\begin{aligned}(u_1)_{x=0}=0\,,\quad \left(\frac{d^2u_1}{dx^2}\right)_{x=0}=0\,,\\ (u_2)_{x=0}=0\,,\quad \left(\frac{d^2u_1}{dx^2}\right)_{x=0}=0\,.\end{aligned}\right\} \tag{8}$$

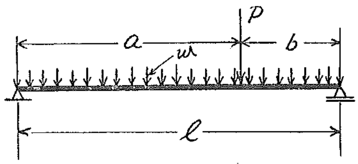

Fig. 1.

Substitution of equation (3) into (8) gives

$$A_1=0\,,\qquad C_1=0$$

for the left side of loaded point of the beam.

Therefore,

$$\left.\begin{aligned}u_1&=B_1\sin\beta x+D_1\sinh\beta x\,,\\ \frac{du_1}{dx}&=\beta(B_1\cos\beta x+D_1\cosh\beta x)\,,\\ \frac{d^2u_1}{dx^2}&=\beta^2(-B_1\sin\beta x+D_1\sinh\beta x)\,,\\ \frac{d^3u_1}{dx^3}&=\beta^3(-B_1\cos\beta x+D_1\cosh\beta x)\,.\end{aligned}\right\} \tag{9 a}$$

Similarly, for the right side of the loaded point,

$$A_2=0\,,\qquad C_2=0\,.$$

Therefore,

$$\left.\begin{aligned}u_2&=B_2\sin\beta x+D_2\sinh\beta x\,,\\ \frac{du_2}{dx}&=\beta(B_2\cos\beta x+D_2\cosh\beta x)\,,\\ \frac{d^2u_2}{dx^2}&=\beta^2(-B_2\sin\beta x+D_2\sinh\beta x)\,,\\ \frac{d^3u_2}{dx^3}&=\beta^3(-B_2\cos\beta x+D_2\cosh\beta x)\,.\end{aligned}\right\} \tag{9 b}$$

Substituting these equations in the conditions (7a), (7b), (7c) and (7d′), one has the following equations:

$$B_1\sin\beta a+D_1\sinh\beta a=B_2\sin\beta b+D_2\sinh\beta b\,, \tag{10 a}$$

$$B_1\cos\beta a+D_1\cosh\beta a=-B_2\cos\beta b-D_2\cosh\beta b\,, \tag{10 b}$$

$$-B_1 \sin \beta a + D_1 \sinh \beta a = -B_2 \sin \beta b + D_2 \sinh \beta b\,, \qquad (10\text{ c})$$

$$-B_1 \cos \beta a + D_1 \cosh \beta a - B_2 \cos \beta b + D_2 \cosh \beta b$$
$$= -k\beta l (B_1 \sin \beta a + D_1 \sinh \beta a)\,. \qquad (10\text{ d})$$

From equations (10 a) and (10 c) the following relations are obtained:

$$B_2 = B_1 \frac{\sin \beta a}{\sin \beta b}\,, \qquad (11) \qquad D_2 = D_1 \frac{\sinh \beta a}{\sinh \beta b}\,. \qquad (12)$$

Substituting these relations into (10 b) and (10 d) and remembering that

$$\cos \beta a \sin \beta b + \sin \beta a \cos \beta b = \sin \beta l\,,$$

$$\cosh \beta a \sinh \beta b + \sinh \beta a \cosh \beta b = \sinh \beta l\,,$$

one has

$$B_1 \sinh \beta b \sin \beta l + D_1 \sin \beta b \sinh \beta l = 0\,, \qquad (13)$$

$$B_1 (\sinh \beta b \sin \beta l - k\beta l \sin \beta b \sinh \beta b \sin \beta a)$$
$$+ D_1 (-\sin \beta b \sinh \beta l + k\beta l \sin \beta b \sinh \beta b \sinh \beta a) = 0\,. \qquad (14)$$

A solution for the constants B_1 and D_1, different from zero, can be obtained only in the case when the determinant of equations (13) and (14) is equal to zero. In this manner the following frequency equation for the calculation of βl is obtained:

$$\begin{vmatrix} \sinh \beta b \sin \beta l & \sin \beta b \sinh \beta l \\ \sinh \beta b \sin \beta l & -\sin \beta b \sinh \beta l \\ \quad -k\beta l \sin \beta b \sinh \beta b \sin \beta a & \quad + k\beta l \sin \beta b \sinh \beta b \sinh \beta a \end{vmatrix} = 0$$

or

$$2 \sin \beta l \sinh \beta l$$
$$+ k\beta l (\sin \beta l \sinh \beta a \sinh \beta b - \sinh \beta l \sin \beta b \sin \beta a) = 0\,. \qquad (15)$$

B. Cantliever Beam

End conditions of cantilever beam are as follows:

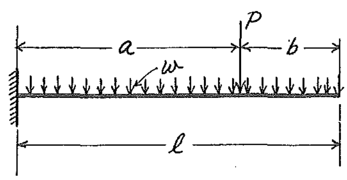

Fig. 2.

for the left side of loaded point

$$(u_1)_{x=0} = 0\,, \quad \left(\frac{du_1}{dx}\right)_{x=0} = 0\,, \qquad (16)$$

for the right side of loaded point

$$\left(\frac{d^2u_2}{dx^2}\right)_{x=0} = 0\,, \qquad \left(\frac{d^3u_2}{dx^3}\right)_{x=0} = 0\,. \tag{17}$$

Substituting the equations (3) into the conditions (16) and (17), one obtains

$$C_1 = -A_1\,,\quad D_1 = -B_1\,,\quad C_2 = A_2\,,\quad D_2 = B_2\,.$$

Accordingly.

$$u_1 = A_1(\cos\beta x - \cosh\beta x) + B_1(\sin\beta x - \sinh\beta x)\,, \tag{18}$$

$$u_2 = A_2(\cos\beta x + \cosh\beta x) + B_2(\sin\beta x + \sinh\beta x)\,. \tag{19}$$

Substituting (18) and (19) into equations (7a) (7d), which are the conditions of the loaded point, one has

$$A_1(\cos\beta a - \cosh\beta a) + B_1(\sin\beta a - \sinh\beta a)$$
$$= A_2(\cos\beta b + \cosh\beta b) + B_2(\sin\beta b + \sinh\beta b)\,, \tag{20 a}$$

$$-A_1(\sin\beta a + \sinh\beta a) + B_1(\cos\beta a - \cosh\beta a)$$
$$= A_2(\sin\beta b - \sinh\beta b) - B_2(\sin\beta b + \sinh\beta b)\,, \tag{20 b}$$

$$A_1(\cos\beta a + \cosh\beta a) + B_1(\cos\beta a + \cosh\beta a)$$
$$= A_2(\cos\beta b - \cosh\beta b) + B_2(\sin\beta b - \sinh\beta b)\,, \tag{20 c}$$

$$A_1(\sin\beta a - \sin\beta a) - B_1(\cos\beta a + \cosh\beta a)$$
$$+ A_2(\sin\beta b + \sinh\beta b) - B_2(\cos\beta b - \cosh\beta b)$$
$$= -k\beta l\left[A_1(\cos\beta a - \cosh\beta a) + B_1(\sin\beta a - \sinh\beta \mathrm{a})\right] \tag{20 d}$$

From equations (20a) and (20c) one obtains

$$A_2 = -\frac{A_1(\sin\beta b\cosh\beta a + \cos\beta a\sinh\beta b) + B_1(\sin\beta b\sinh\beta a + \sin\beta a\sinh\beta b)}{\sin\beta b\cosh\beta b - \cos\beta b\sinh\beta b}\,, \tag{21}$$

$$B_2 = \frac{A_2(\cos\beta b\cosh\beta a + \cos\beta a\cosh\beta b) + B_1(\cos\beta b\sinh\beta a + \sin\beta a\cosh\beta b)}{\sin\beta b\cosh\beta b - \cos\beta b\sinh\beta b}\,. \tag{22}$$

Substitution of (21) and (22) into equations (20b) and (20d) gives

$$A_1(\cos\beta a + \cosh\beta a + \cos\beta l\cosh\beta b$$
$$+ \sin\beta l\sinh\beta b - \sin\beta b\sinh\beta l + \cos\beta b\cosh\beta l)$$
$$+ B_1(\sin\beta a + \sinh\beta a + \sin\beta l\cosh\beta b$$
$$- \cos\beta l\sinh\beta b - \sin\beta b\cosh\beta l + \cos\beta b\sinh\beta l) = 0\,, \tag{23}$$

$$A_1\Big[(\cos\beta a-\cosh\beta a)(1+k\beta l\Delta)-(\cos\beta l\cosh\beta b+\sin\beta l\sinh\beta b)$$
$$-(\sin\beta b\sinh\beta l-\cos\beta b\cosh\beta l)\Big]+B_1\Big[(\sin\beta a-\sinh\beta a)(1+k\beta l\Delta)$$
$$-(\sin\beta l\cosh\beta b-\cos\beta l\sinh\beta b)-(\sin\beta b\cosh\beta l-\cos\beta b\sinh\beta l)\Big]=0, \tag{24}$$

in which

$$\Delta=\sin\beta b\cosh\beta b-\cos\beta b\sinh\beta b\,.$$

When the notations,

$$\cos\beta l\cosh\beta b+\sin\beta l\sinh\beta b=A\,,$$
$$\sin\beta b\sinh\beta l-\cos\beta b\cosh\beta l=B\,,$$
$$\sin\beta l\cosh\beta b-\cos\beta l\sinh\beta b=C\,,$$
$$\sin\beta b\cosh\beta l-\cos\beta b\sinh\beta l=D\,,$$

are employed, equations (23) and (24) can be written in the following forms:

$$A_1(\cos\beta a+\cosh\beta a+A-B)+B_1(\sin\beta a+\sinh\beta a+C+D)=0\,, \tag{23'}$$

$$A_1\Big[(\cos\beta a-\cosh\beta a)(1+k\beta l\Delta)-(A+B)\Big]$$
$$+B_1\Big[(\sin\beta a-\sinh\beta a)(1+k\beta l\Delta)-(C+D)\Big]=0\,. \tag{24'}$$

When A_1 and B_1 are eliminated from these equations the frequency equation for the calculation of βl can be obtained as follows:

$$\begin{vmatrix} (\cos\beta a+\cosh\beta a+A-B) & (\sin\beta a+\sinh\beta a+C-D) \\ \Big[(\cos\beta a-\cosh\beta a)(1+k\beta l\Delta) & \Big[(\sin\beta a-\sinh\beta a)(1+k\beta l\Delta \\ \quad -(A+B)\Big] & \quad -(C+D)\Big] \end{vmatrix}=0$$

or

$$2\Big[(\sin\beta a\cosh\beta a-\cos\beta a\sinh\beta a)+(A\sin\beta a+B\sin\beta a$$
$$-C\cos\beta a-D\cosh\beta a)+(BC-AD)\Big]$$
$$+\Big[2(\sin\beta a\cosh\beta a-\cos\beta a\sinh\beta a)+A\sin\beta a-B\sin\beta a-A\sinh\beta a$$
$$-C\cos\beta a+D\cos\beta a+C\cosh\beta a-D\cosh\beta a)\Big]k\beta l\Delta=0$$

or

$$2(\cos\beta l\cosh\beta l+1)-k\beta l(\sin\beta a\cosh\beta a-\cos\beta a\sinh\beta a)$$
$$-\sin\beta b\cosh\beta b+\cos\beta b\sinh\beta b-\sinh\beta l\cos\beta b\cos\beta a$$
$$+\sin\beta l\cosh\beta b\cosh\beta a=0 \tag{25}$$

C. Fixed Beam

In this case, the end conditions are: for the left side of loaded point

$$(u_1)_{x=0}=0\,,\ \left(\frac{du_1}{dx}\right)_{x=0}=0\,, \tag{26}$$

for the right side of loaded point

$$(u_2)_{x=0}=0\,,\ \left(\frac{du_2}{dx}\right)_{x=0}=0\,. \tag{27}$$

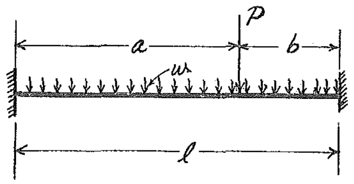

Fig. 3.

Application of equations (3) to equations (26) and (27) gives

$$C_1=-A\,,\quad D_1=-B_1\,,\quad C_2=-A_2\,,\quad D_2=-B_2\,.$$

Accordingly,

$$u_1=A_1(\cos\beta x-\cosh\beta x)+B_1(\sin\beta x-\sinh\beta x) \tag{28}$$

$$u_2=A_2(\cos\beta x-\cosh\beta x)+B_2(\sin\beta x-\sinh\beta x)\,. \tag{29}$$

Substituting these relations into equations (7 a)~(7 d′), which are the equations for the conditions at the loaded point, one has

$$A_1(\cos\beta a-\cosh\beta a)+B_1(\sin\beta a-\sinh\beta a)$$
$$=A_2(\cos\beta b-\cosh\beta b)+B_2(\sin\beta b-\sinh\beta b)\,, \tag{30 a}$$

$$-A_1(\sin\beta a+\sinh\beta a)+B_1(\cos\beta a-\cosh\beta a)$$
$$=A_2(\sin\beta b+\sinh\beta b)-B_2(\cos\beta b-\cosh\beta b)\,, \tag{30 b}$$

$$A_1(\cos\beta a+\cosh\beta a)+B_1(\sin\beta a+\sinh\beta a)$$
$$=A_2(\cos\beta b+\cosh\beta b)+B_2(\sin\beta b+\sinh\beta b)\,, \tag{30 c}$$

$$A_1(\sin\beta a-\sinh\beta a)-B_1(\cos\beta a+\cosh\beta a)$$
$$+A_2(\sin\beta b-\sinh\beta b)-B_2(\cos\beta b+\cosh\beta b)$$
$$=-k\beta l\left[A_1(\cos\beta a-\cosh\beta a)+B_1(\sin\beta a-\sinh\beta a)\right] \tag{30 d}$$

From equations (30 a) and (30 c) one has

$$A_2=\frac{A_1(\cos\beta a\sinh\beta b-\cosh\beta a\sin\beta b)+B_1(\sin\beta a\sinh\beta b-\sinh\beta a\sin\beta b)}{\cos\beta b\sinh\beta b-\cosh\beta b\sin\beta b}\,, \tag{31}$$

$$B_2 = \frac{A_1(\cosh\beta a\cos\beta b - \cos\beta a\cosh\beta b) + B_1(\sinh\beta a\cos\beta b - \sin\beta a\cosh\beta b)}{\cos\beta b\sinh\beta b - \cosh\beta b\sin\beta b}\,, \tag{32}$$

Substitution of (31) and (32) into equations (30 b) and (30 d) gives

$$\begin{aligned} &A_1\big[\cos\beta a + \cosh\beta a - \sin\beta l\sinh\beta b - \cos\beta l\cosh\beta b - \cosh\beta l\cos\beta b \\ &\quad + \sinh\beta l\sin\beta b\big] + B_1\big[\sin\beta a + \sinh\beta a + \cos\beta l\sinh\beta b \\ &\quad - \sin\beta l\cosh\beta b + \cosh\beta l\sin\beta b - \sinh\beta l\cos\beta b\big] = 0\,, \end{aligned} \tag{33}$$

$$\begin{aligned} &A_1\big[\sin\beta l\sinh\beta b + \cos\beta l\cosh\beta b - \cosh\beta l\cos\beta b + \sinh\beta l\sin\beta b \\ &\quad + (\cos\beta a - \cosh\beta a)(1 + k\beta\Delta)\big] + B_1\big[\sin\beta l\cosh\beta b - \cos\beta l\sinh\beta b \\ &\quad - \sinh\beta l\cos\beta b + \cosh\beta l\sin\beta b + (\sin\beta a - \sinh\beta a)(1 + k\beta l\Delta)\big] = 0 \end{aligned} \tag{34}$$

in which

$$\Delta = \cos\beta b\sinh\beta b - \cosh\beta b\sin\beta b\,.$$

When A_1 and B_1 are eliminated from these equations, the frequency equations for the cantilever beam acted by a concentrated load and a uniform load simultaneously can be obtained as follows:

$$\begin{vmatrix} \big[(\cos\beta a + \cosh\beta a) - A + B\big] & \big[(\sin\beta a + \sinh\beta a) + C + D\big] \\ \big[(\cos\beta a - \cosh\beta a)(1 + k\beta l\Delta) + A + B\big] & \big[(\sin\beta a - \sinh\beta a)(1 + k\beta l\Delta) - C + D\big] \end{vmatrix} = 0$$

in which

$$\begin{aligned} A &= \sin\beta l\sinh\beta b + \cos\beta l\cosh\beta b\,, \\ B &= \sinh\beta l\sin\beta b - \cosh\beta l\cos\beta b\,, \\ C &= -\sin\beta l\cosh\beta b + \cos\beta l\sinh\beta b\,, \\ D &= -\sinh\beta l\cos\beta b + \cosh\beta l\sin\beta b \end{aligned}$$

or

$$\begin{aligned} &\big[(\cos\beta a + \cosh\beta a - (A - B)\big]\big[(\sin\beta a - \sinh\beta a)(1 + k\beta l\Delta) - (C - D)\big] \\ &\quad - \big[\sin\beta a + \sinh\beta a) + (C + D)\big]\big[(\cos\beta a - \cosh\beta a)(1 + k\beta l\Delta) \\ &\quad + (A + B)\big] = 0 \end{aligned}$$

or

$$2(\cos\beta l\cosh l\beta-1)+k\beta l(\sin\beta a\cosh\beta a-\cos\beta a\sinh\beta a$$
$$+\sin\beta b\cosh\beta b-\cos\beta b\sinh\beta b+\sinh\beta l\cos\beta b\cos\beta a$$
$$-\sin\beta l\cosh\beta b\cosh\beta a)=0\,. \qquad (35)$$

When the exact values of βl are determined from the frequency equations (15), (25) and (35), the periods of the free vibration are obtained by the following formula:

$$T=\frac{2\pi}{n}=\frac{2\pi}{(\beta l)^2}\sqrt{\frac{wl^4}{gEI}}\,.$$

II. Approximate Formulae

It is very laborious to determine the values of βl from the frequency equations. If the approximate values are previously known from the approximate formulae, the exact values of βl can rapidly be calculated. If one denotes that T_1 is the period for only a uniform load or a dead load, and T_2 is the period for only a concentrated load, Dunkerlay's approximate formula is

$$T=\sqrt{T_1^2+T_2^2}$$

for the period of the free lateral vibration of the beams acted by a concentrated load and uniform load simultaneously.

T_1 and T_2 in the above formula are expressed as follows:

$$T_1=\frac{2\pi}{(\beta_1 l)^2}\sqrt{\frac{wl^4}{gEI}}$$

$$T_2=2\pi\sqrt{\frac{P}{gc}}\,.$$

In the expression T_2, l/c is a spring constant. The well known values of $\beta_1 l$ for the fundamental mode of vibration are as follows:

for simple beam $\quad \beta_1 l=\pi$

for fixed beam $\quad \beta_1 l=4.73004$

for cantilever beam $\quad \beta_1 l=1.87510$

and the values of spring constant are:

for simple beam $\quad \dfrac{1}{c}=\dfrac{a^2b^2}{3EIl}\,,$

for fixed beam $\dfrac{1}{c} = \dfrac{a^3b^3}{3EIl^3}$,

for cantilever beam $\dfrac{1}{c} = \dfrac{l^3}{3EI}\left(\dfrac{a}{l}\right)^3$.

Accordingly, T_2 is rewritten as follows:

$$T_2 = 2\pi\sqrt{\alpha}\sqrt{\frac{Pl^3}{gEI}},$$

in which

$$\alpha = \frac{1}{3}\left(\frac{a}{l}\right)^2\left(\frac{b}{l}\right)^2 \quad \text{for simple beam ,}$$

$$\alpha = \frac{1}{3}\left(\frac{a}{l}\right)^3\left(\frac{b}{l}\right)^3 \quad \text{for fixed beam ,}$$

$$\alpha = \frac{1}{3}\left(\frac{a}{l}\right)^3 \quad \text{for cantilever beam.}$$

Therefore,

$$T = \sqrt{T_1^2+T_2^2} = \frac{2\pi}{(\beta_1 l)^2}\sqrt{1+\alpha(\beta_1 l)^4\frac{P}{W}}\sqrt{\frac{Wl^3}{gEI}},$$

in which

$$W = wl$$

and then,

for simple beam $T = \dfrac{2}{\pi}\sqrt{1+\dfrac{\pi^4}{3}\left(\dfrac{a}{l}\right)^2\left(\dfrac{b}{l}\right)^2\dfrac{P}{W}}\sqrt{\dfrac{Wl^3}{gEI}}$,

for fixed beam

$$T = \frac{2\pi}{(4.73004)^2}\sqrt{1+\frac{(4.73004)^4}{3}\left(\frac{a}{l}\right)^3\left(\frac{b}{l}\right)^3\frac{P}{W}}\sqrt{\frac{Wl^3}{gEI}},$$

for cantilever beam

$$T = \frac{2\pi}{(1.87510)^2}\sqrt{1+\frac{(1.87510)^4}{3}\left(\frac{a}{l}\right)^3}\sqrt{\frac{Wl^3}{gEI}}.$$

Generally, one can express that

$$T = K\frac{2\pi}{(\beta_1 l)^2}\sqrt{\frac{Wl^3}{gEI}},$$

in which

$$K=\sqrt{1+\alpha(\beta_1 l)^4\frac{P}{W}}\ .$$

When a beam is subjected to a concentrated load and a uniform load simultaneously, the approximate value of the smallest root of the frequency equation of the approximate value of βl for the fundamental mode of vibration becomes:

$$\beta l=\frac{\beta_1 l}{\sqrt{K}}\ .$$

Therefore, in the case of simple beam

$$\beta l=\frac{\pi}{\sqrt[4]{1+\frac{\pi^4}{3}\left(\frac{a}{l}\right)^2\left(\frac{b}{l}\right)^2\frac{P}{W}}}\ ,$$

in the case of fixed beam

$$\beta l=\frac{4.73004}{\sqrt[4]{1+\frac{(4.73004)^4}{3}\left(\frac{a}{l}\right)^3\left(\frac{b}{l}\right)^3\frac{P}{W}}}\ ,$$

and in the case of cantilever beam

$$\beta l=\frac{1.87510}{\sqrt[4]{1+\frac{(1.87510)^4}{3}\left(\frac{a}{l}\right)^3\frac{P}{W}}}\ .$$

III. Tables of the Exact Values of the Free Lateral Vibration Period of Beams

In the case of simple beam, when a concentrated load P, which has various ratio to total uniform load W, is acting at the position where $a/l=0.1$, 0.2, 0.3, 0.4 and 0.5, the period of the free lateral vibration of beam is expressed as the following forms:

for $\quad P\geqq W \qquad T=K_s\frac{\pi}{\sqrt{12}}\sqrt{\frac{Pl^3}{gEI}}\ ,$

for $\quad P\leqq W \qquad T=K_s\frac{2}{\pi}\sqrt{\frac{Wl^3}{gEI}}\ ,$

in which, $\frac{2}{\pi}\sqrt{\frac{Wl^3}{gEI}}$ is the period of the free vibration due to uniform load, and $\frac{\pi}{\sqrt{12}}\sqrt{\frac{Pl^3}{gEI}}$ is the period when a concentrated load P is acting at the middle of beam and uniform load is zero. Therefore, K_s in these expressions means a rivisional coefficient for another load condition.

Similarly, for the case of a fixed beam,

$$\text{for} \qquad P \geqq W \qquad T = K_f \frac{\pi}{2\sqrt{12}} \sqrt{\frac{Pl^3}{gEI}} \,,$$

$$\text{for} \qquad P \leqq W \qquad T = K_f \frac{2\pi}{(4.78004)^2} \sqrt{\frac{Wl^3}{gEI}} \,.$$

In these formulae, $\frac{\pi}{2\sqrt{12}} \sqrt{\frac{Pl^3}{gEI}}$ is the period of the free vibration when a concentrated load P is acting at the middle of beam, and $\frac{2\pi}{(4.78004)^2} \sqrt{\frac{Wl^3}{gEI}}$ is the period due to only uniform load W. Further, in the case of cantilever beam, the formulae for the period of the free vibration due to only uniform load W is $\frac{2\pi}{(1.87310)^2} \sqrt{\frac{Wl^3}{gEI}}$, and the period of the beam acted only by a concentrated load P at the free end is $\frac{2\pi}{\sqrt{3}} \sqrt{\frac{Pl^3}{gEI}}$. Therefore, if K_c is considered as the revisional coefficient, the period can be expressed by the following forms:

$$\text{for} \qquad P \geqq W \qquad T = K_c \frac{2\pi}{\sqrt{3}} \sqrt{\frac{Pl^3}{gEI}} \,,$$

$$\text{for} \qquad P \leqq W \qquad T = K_c \frac{2\pi}{(1.87510)^2} \sqrt{\frac{Wl^3}{gEI}} \,.$$

The exact values of six revisional coefficients are shown in tables (1), (2) and (3), and the diagrams obtained from these values are diagrams (1), (2) and (3). By the use of these tables or diagrams, one can easily obtain the period of the free lateral vibration of beams acted by a concentrated load P and a uniform load W simultaneously, so far as the value of P/W and the position of P are given.

TABLE 1. (a) K_s for $P \geqq W$

W/P	a/l					
	0.0	0.1	0.2	0.3	0.4	0.5
1.0	0.7020	0.7701	0.9294	1.0834	1.1850	1.2199
0.9	0.6659	0.7379	0.9038	1.0612	1.1643	1.1996
7.8	0.6279	0.7042	0.8771	1.0385	1.1433	1.1791
0.7	0.5873	0.6689	0.8500	1.0154	1.1219	1.1581
0.6	0.5437	0.6318	0.8222	0.9919	1.1001	1.1369
0.5	0.4964	0.5928	0.7936	0.9679	1.0779	1.1152
0.4	0.4440	0.5479	0.7644	0.9434	1.0553	1.0931
0.3	0.3845	0.5074	0.7343	0.9184	1.0322	1.0705
0.2	0.3139	0.4605	0.7036	0.8928	1.0087	1.0475
0.1	0.2220	0.4109	0.6721	0.8667	0.9846	1.0241
0.05	0.1570	0.3854	0.6562	0.8534	0.9724	1.0122
0.01	0.0702	0.3651	0.6433	0.8429	0.9627	1.0024
0.0	0.0	0.3600	0.6400	0.8400	0.9600	1.0000

TABLE 1. (b) K_s for $P \leqq W$

P/W	a/l					
	0.0	0.1	0.2	0.3	0.4	0.5
0.0	1.0	1.0	1.0	1.0	1.0	1.0
0.05	1.0	1.0048	1.0173	1.0324	1.0444	1.0489
0.1	1.0	1.0096	1.0344	1.0642	1.0872	1.0957
0.2	1.0	1.0192	1.0686	1.1258	1.1684	1.1840
0.3	1.0	1.0289	1.1023	1.1848	1.2448	1.2662
0.4	1.0	1.0385	1.1356	1.2416	1.3169	1.3436
0.5	1.0	1.0483	1.1683	1.2963	1.3855	1.4168
0.6	1.0	1.0580	1.2006	1.3490	1.4509	1.4865
0.7	1.0	1.0678	1.2322	1.3998	1.5136	1.5531
0.8	1.0	1.0775	1.2634	1.4492	1.5740	1.6170
0.9	1.0	10.873	1.2940	1.4970	1.6319	1.6785
1.0	1.0	1.0971	1.3240	1.5434	1.6880	1.7378

TABLE 2. (a) K_f for $P \geqq W$

W/P	a/l					
	0.0	0.1	0.2	0.3	0.4	0.5
1.0	0.6193	0.6324	0.7557	0.9621	1.1170	0.1725
0.9	0.5876	0.6003	0.7321	0.9438	1.1005	1.1563
0.8	0.5540	0.5677	0.7031	0.9252	1.0836	1.1400
0.7	0.5182	0.5330	0.6837	0.9064	1.0666	1.1233
0.6	0.4797	0.4961	0.6590	0.8875	1.0494	1.1064
0.5	0.4379	0.4566	0.6340	0.8683	1.0318	1.0394
0.4	0.3917	0.4131	0.6090	0.8490	1.0149	1.0720
0.3	0.3392	0.3656	0.5839	0.8294	0.9961	1.0545
0.2	0.2770	0.3126	0.5609	0.8097	0.9779	1.0366
0.1	0.1959	0.2586	0.5352	0.7889	0.9594	1.0185
0.05	0.1385	0.2339	0.5242	0.7799	0.9500	1.0091
0.01	0.0619	0.2192	0.5191	0.7713	0.9417	1.0012
0.0	0.0000	0.2160	0.5120	0.7699	0.9406	1.0000

TABLE 2. (b) K_f for $P \leqq W$

P/W	a/l					
	0.0	0.1	0.2	0.3	0.4	0.5
0.0	1.0	1.0	1.0	1.0	1.0	1.0
0.05	1.0	1.0009	1.0097	1.0301	1.0521	1.0614
0.1	1.0	1.0018	1.0196	1.0604	1.1025	1.1198
0.2	1.0	1.0036	1.0401	1.1206	1.1983	1.2289
0.3	1.0	1.0055	1.0612	1.1799	1.2882	1.3296
0.4	1.0	1.0074	1.0829	1.2379	1.3729	1.4235
0.5	1.0	1.0093	1.1051	1.2943	1.4531	1.5117
0.6	1.0	1.0113	1.1277	1.3491	1.5293	1.5952
0.7	1.0	1.0133	1.1506	1.4024	1.6021	1.6746
0.8	1.0	1.0153	1.1737	1.4542	1.6719	1.7504
0.9	1.0	1.0173	1.1969	1.5045	1.7390	1.8232
1.0	1.0	1.0194	1.2202	1.5534	1.8036	1.8932

TABLE 3. (a) K_c for $P \geqq W$

P/W	a/l					
	0.0	0.2	0.4	0.6	0.8	1.0
1.0	0.4926	0.4967	0.5444	0.6735	0.8685	1.1122
0.9	0.4673	0.4717	0.5218	0.6555	0.8527	1.1015
0.8	0.4406	0.4452	0.4983	0.6369	0.8402	1.0900
0.7	0.4122	0.4171	0.4736	0.6178	0.8256	1.0797
0.6	0.3816	0.3869	0.4477	0.5981	0.8108	1.0686
0.5	0.3483	0.3542	0.4203	0.5779	0.7957	1.0575
0.4	0.3116	0.3181	0.3913	0.5569	0.7804	1.0462
0.3	0.2698	0.2775	0.3602	0.5352	0.7647	1.0348
0.2	0.2203	0.2298	0.3269	0.5126	0.7487	1.0233
0.1	0.1558	0.1699	0.2910	0.4892	0.7323	1.0117
0.05	0.1102	0.1316	0.2722	0.4771	0.7240	1.0059
0.01	0.0493	0.0969	0.2569	0.4673	0.7174	1.0012
0.0	0.0000	0.0894	0.2530	0.4648	0.7155	1.0000

TABLE 3. (b) K_c for $P \leqq W$

W/P	a/l					
	0.0	0.2	0.4	0.6	0.8	1.0
0.0	1.0	1.0	1.0	1.0	1.0	1.0
0.05	1.0	1.0004	1.0053	1.0211	1.0513	1.0959
0.1	1.0	1.0008	1.0106	1.0418	1.1003	1.1847
0.2	1.0	1.0017	1.0211	1.0822	1.1922	1.3457
0.3	1.0	1.0025	1.0317	1.1214	1.2775	1.4900
0.4	1.0	1.0033	1.0423	1.1594	1.3576	1.6218
0.5	1.0	1.0041	1.0528	1.1964	1.4332	1.7438
0.6	1.0	1.0049	1.0633	1.2323	1.5049	1.8579
0.7	1.0	1.0058	1.0737	1.2673	1.5735	1.9654
0.8	1.0	1.0066	1.0842	1.3013	1.6391	2.0674
0.9	1.0	1.0074	1.0946	1.3346	1.7022	2.1647
1.0	1.0	1.0083	1.1050	1.3672	1.7631	2.2578

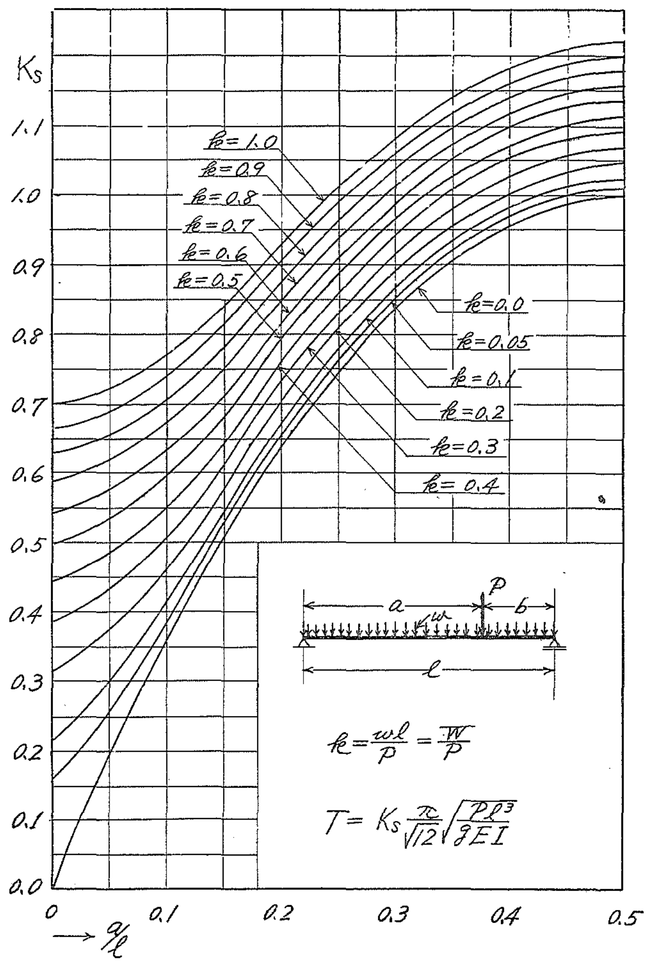

Diagram 1. (a) K_s for $P \geqq W$.

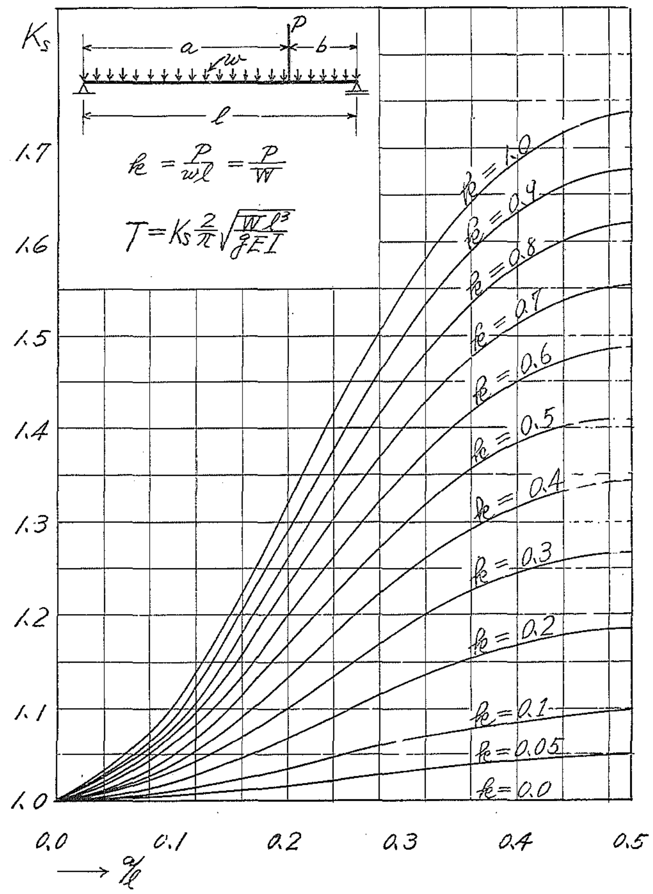

Diagram 1. (b) K_s for $P \leqq W$.

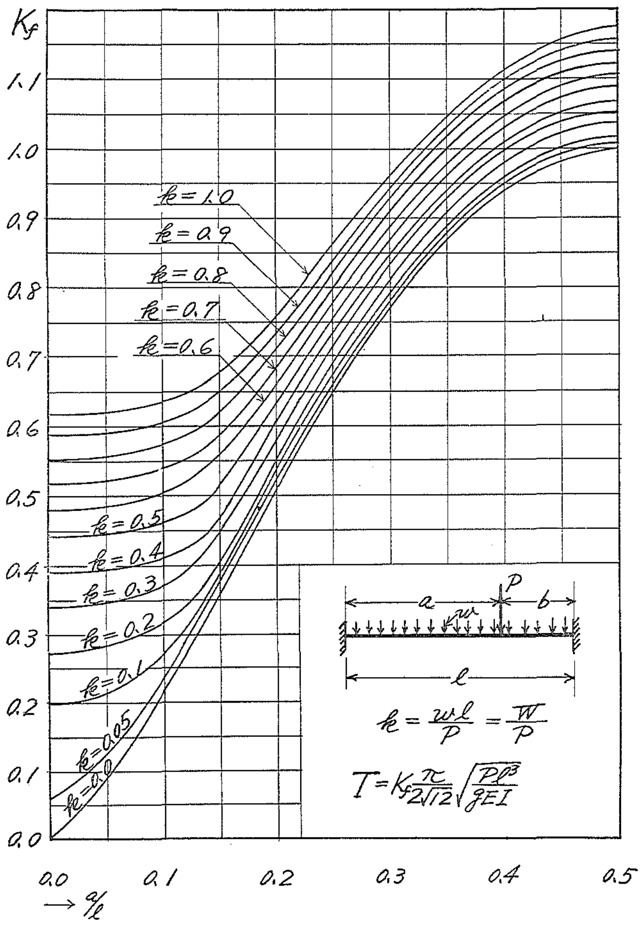

Diagram 2. (a) K_f for $P \geqq W$.

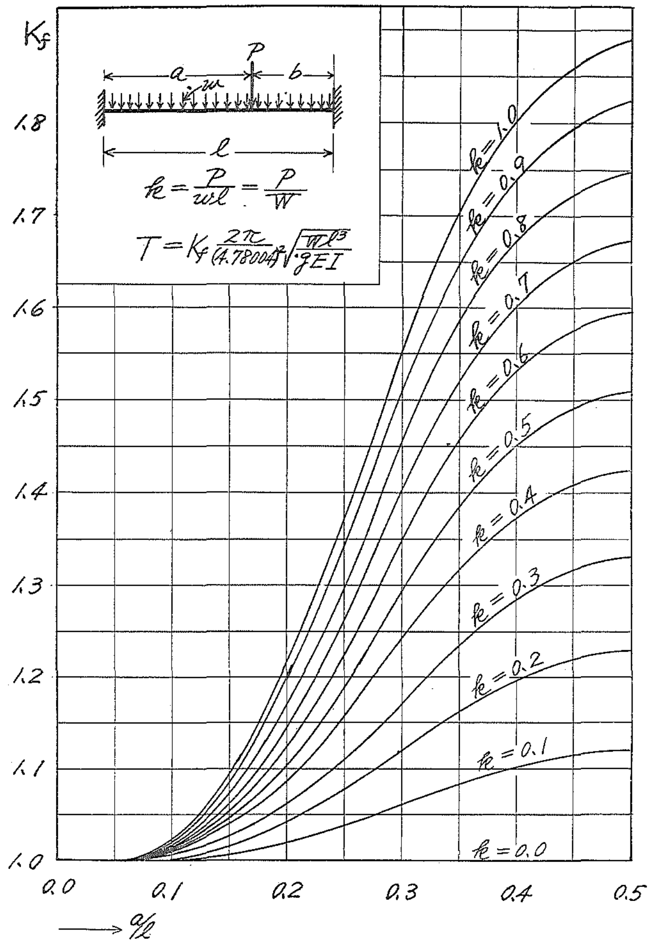

Diagram 2. (b) K_f for $P \leqq W$.

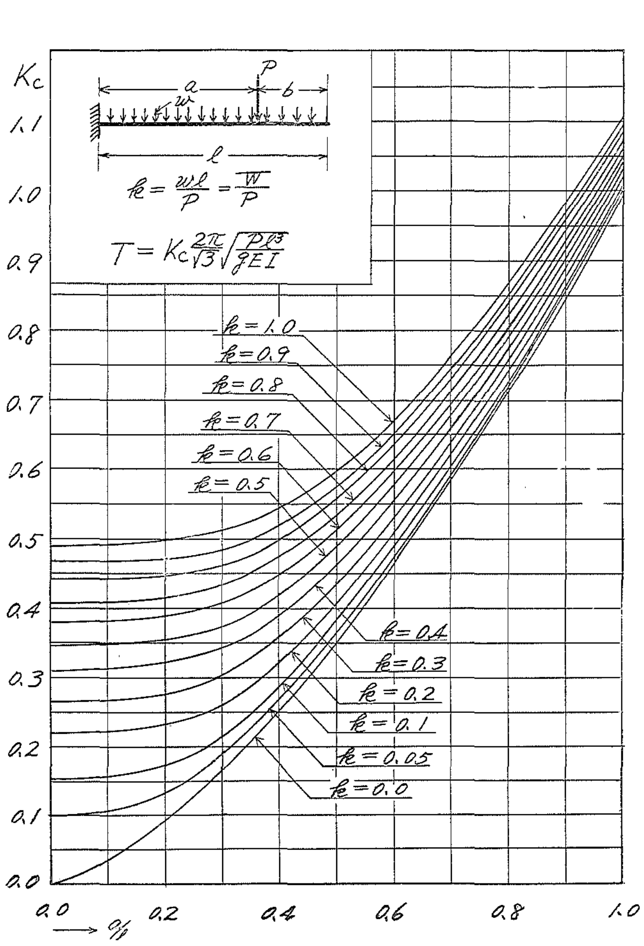

Diagram 3. (a) K_c for $P \geqq W$.

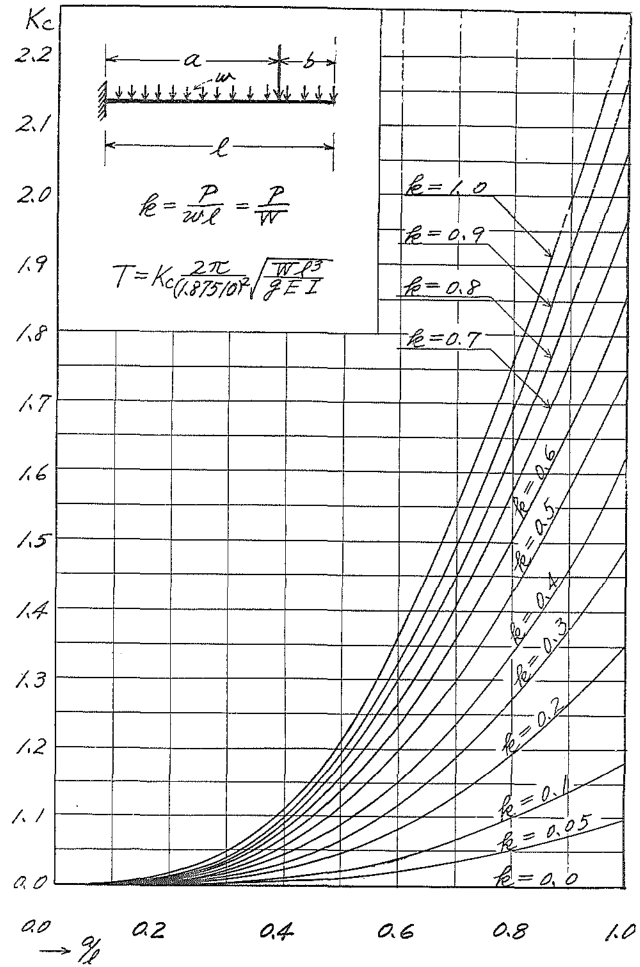

Diagram 3. (b) K_c for $P \leqq W$.