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On Torsion of I-beam with a Web of Variable Height

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Synopsis

Problems of the torsion of I-beam with constant cross section have been analyzed.

In the present paper, the differential equation defining the torsion problem for the I-beam so that the web height is tapered in accordance with a linear or second order formula, has been introduced and solved.

As examples, two types of I-beams with the web height variation above mentioned which are fixed at one end and twisted at the other are given. The stress states of them were analyzed, and compared with the characteristics of the I-beams of the constant cross section.

1 Differential Equation and Solution for Torsion of I-beam in which Web Height is linearly variable

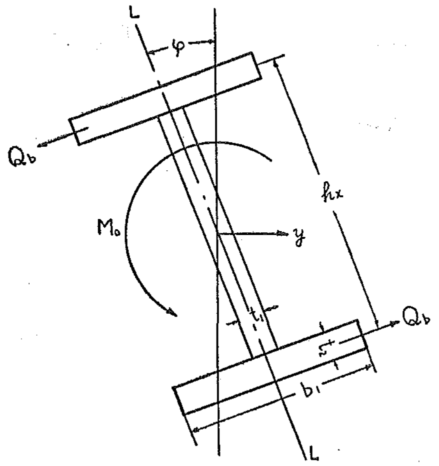
When an I-beam of constant cross section is fixed at one end and twisted at the other free end, it may generally be assumed that the distortion of cross section is restrained at the fixed end and free at the twisted end. The resultant twisting moment M_0 of external forces might be resisted by the Saint-Venant's torsional rigidity and the horizontal flexural rigidity of flanges in any cross section. For this condition the differential equation has been given by

$$EI_f \frac{h^2}{2} \frac{d^3\varphi}{dx^3} - GJ_d \frac{d\varphi}{dx} = -M_0 , \qquad (1)$$

where

EI_f : flexural rigidity of a flange through $L-L$ axis,
GJ_d: torsional rigidity,
M_0: resultant twisting moment of external forces,
h : distance between center of gravity of upper flange and that of lower flange,
φ : angle of twist,
x : coordinate distance along I-beam from free end.

Now, when the height of web is tapered in accordance with a linear formula and the origin of the coordinates is taken as shown in Fig. (1), X axis is



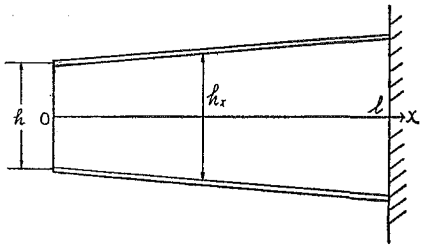


Fig. 1.

a torsional axis and passes the center of shearing force; h_x and J_d are denoted by the expressions:

$$h_x = ax + h \,, \tag{2}$$

$$\begin{aligned} J_d &= \frac{1}{3}\mu \sum_i s_i t_i = \frac{1}{3}\mu\left\{(ax+h)t_1^3 + 2b_1 t_2^3\right\} \\ &= \frac{1}{3}\mu t_1^3 \left\{(ax+h) + \frac{t_2^3}{t_1^3}\cdot 2b_1\right\}\,, \end{aligned} \tag{3}$$

where

μ : coefficient in cross section of I-beam,

s_i, t_i : breadth, thickness of strip.

Provided a twisting moment M_0 is applied at the free end of the variable I-beam, the twisting moment $\mathfrak{M}_d$ produced with the torsional rigidity in any cross section is

$$\begin{aligned} GJ_d \frac{d\varphi}{dx} &= G\frac{1}{3}\mu\left\{(ax+h)t_1^3 + 2b_1 t_2^3\right\}\frac{d\varphi}{dx} \\ &= \beta\left\{(ax+h) + \gamma\right\}\frac{d\varphi}{dx}\,, \end{aligned} \tag{4}$$

where

$$\beta = G\frac{1}{3}\mu t_1^3\,, \qquad \gamma = \frac{t_2^3}{t_1^3}\cdot 2b_1\,.$$

The bending moment $\mathfrak{M}_b$ and shearing force Q_b applied to the horizontal direction in any cross section of the upper flange are respectively given by

$$-EI_f \frac{d^2}{dx^2}\left(\frac{1}{2} h_x \cdot \varphi\right)$$

and

$$-EI_f \frac{d^3}{dx^3}\left(\frac{1}{2} h_x \cdot \varphi\right).$$

The torsional moment $Q_b \cdot h_x$ produced with the two opposite Q_b applied at the upper and lower flanges is expressed as follows:

$$-EI_f \cdot h_x \frac{d^3}{dx^3}\left(\frac{1}{2} h_x \cdot \varphi\right) = -\alpha \cdot h_x \left\{3a \frac{d^2\varphi}{dx^2} + h_x \frac{d^3\varphi}{dx^3}\right\}, \tag{5}$$

where

$$\alpha = \frac{1}{2} EI_f .$$

Hence, it may be assumed that the following equiribrium equation is satisfied in any cross section:

$$Q_b \cdot h_x + \mathfrak{M}_d = M_0 . \tag{6}$$

By substituting $\mathfrak{M}_d$ and $Q_b \cdot h_x$ into the equation (6), the differential equation for φ can be written by

$$\alpha(ax+h)^2 \frac{d^3\varphi}{dx^3} + 3a \cdot \alpha(ax+h) \frac{d^2\varphi}{dx^2} - \beta\{(ax+h)+\gamma\} \frac{d\varphi}{dx} = -M_0 . \tag{7}$$

Then, by replacing new variables with

$$X = ax+h , \quad \Phi = \frac{d\varphi}{dX} , \tag{8}$$

and arranging the constants, Eq. (7) can be transformed

$$\frac{d^2\Phi}{dX^2} + \frac{3}{X}\frac{d\Phi}{dX} + \left\{\left(\frac{1}{a}\sqrt{\frac{\beta}{\alpha}}\, i\right)^2 \frac{1}{X} + \left(\frac{1}{a}\sqrt{\frac{\beta\gamma}{\alpha}}\, i\right)^2 \frac{1}{X^2}\right\}\Phi = -\frac{M_0}{\alpha a^3 X^2}, \tag{9}$$

where i denotes $\sqrt{-1}$.

The solution Φ_0 of the homogeneous equation of Eq. (9) is given as follows:

$$\Phi_0 = X^{-1}\left[P \cdot J_\nu(wi \cdot X^{-\frac{1}{2}}) + Q \cdot Y_\nu(wi \cdot X^{-\frac{1}{2}})\right], \tag{10}$$

where

P, Q: arbitrary constants,

ν: $\quad 2\sqrt{1+\dfrac{\beta\gamma}{\alpha a^2}}$,

w: $\quad \dfrac{2}{a}\sqrt{\dfrac{\beta}{\alpha}}$,

$J_\nu(wi\cdot X^{-\frac{1}{2}}), Y_\nu(wi\cdot X^{-\frac{1}{2}})$: Bessel functions of the first kind of order ν and the second kind of order ν.

The particular solution Φ_1 of Eq. (9) can be introduced by the method of variation of parameters, namely

$$\begin{aligned}\Phi_1 &= X^{-1}\cdot J_\nu\cdot\int\frac{X^{-1}\,Y_\nu}{\{X^{-1}\cdot J_\nu\}'\{X^{-1}\cdot Y_\nu\}-\{X^{-1}\cdot Y_\nu\}'\{X^{-1}\cdot J_\nu\}}\cdot\frac{-M_0}{\alpha a^3X^2}\,dX\\ &\quad -X^{-1}\cdot Y_\nu\int\frac{X^{-1}J_\nu}{\{X^{-1}\cdot J_\nu\}'\{X^{-1}\cdot Y_\nu\}-\{X^{-1}\cdot Y_\nu\}'\{X^{-1}\cdot J_\nu\}}\cdot\frac{-M_0}{\alpha a^3X^2}\,dX\\ &= X^{-1}\Big[J_\nu\int Y_\nu dX-Y_\nu\int J_\nu dX\Big]\frac{\pi M_0}{\alpha a^3}. \end{aligned}\tag{11}$$

Therefore, the complete solution of Eq. (9) is given by

$$\begin{aligned}\Phi &= \Phi_0+\Phi_1\\ &= X^{-1}\Big[PJ_\nu+QY_\nu+J_\nu\int Y_\nu M_1 dX-Y_\nu\int J_\nu M_1 dX\Big],\end{aligned}\tag{12}$$

where

$$M_1=\frac{\pi M_0}{\alpha a^3}\,.$$

Integrating Eq. (12) to get φ,

$$\begin{aligned}\varphi &= P\int X^{-1}\cdot J_\nu dX+Q\int X^{-1}\cdot Y_\nu dX\\ &\quad +\int X^{-1}\cdot J_\nu\Big\{\int Y_\nu M_1 dX\Big\}dX-\int X^{-1}\cdot Y_\nu\Big\{\int J_\nu M_1 dX\Big\}dX+R\,,\end{aligned}\tag{13}$$

when ν is not an integer,

$$\begin{aligned}\varphi &= P\sum_{m=0}^{\infty}S_m^\nu\Big(\frac{w}{2}i\Big)^{\nu+2m}\cdot(ax+h)^{\frac{1}{2}(\nu+2m)}\cdot\frac{2}{\nu+2m}\\ &\quad +Q\sum_{m=0}^{\infty}S_m^{-\nu}\Big(\frac{w}{2}i\Big)^{-\nu+m}\cdot(ax+h)^{\frac{1}{2}(-\nu+2m)}\cdot\frac{2}{-\nu+2m}\\ &\quad +2M_1\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}S_m^\nu S_n^{-\nu}\Big(\frac{w}{2}i\Big)^{2(m+n)}\cdot(ax+h)^{m+n+1}\frac{1}{m+n+1}\\ &\quad \times\Big(\frac{1}{-\nu+2n+2}-\frac{1}{\nu+2m+2}\Big)+R\,,\end{aligned}\tag{14}$$

where

R : arbitrary constant,

S_m^ν : $\dfrac{(-1)^m}{\Gamma(m+1)\cdot\Gamma(m+\nu+1)}$,

$S_n^{-\nu}$: $\dfrac{(-1)^n}{\Gamma(n+1)\cdot\Gamma(n-\nu+1)}$.

The constants P, Q and R must be determined so as to satisfy the condition at the ends.

It may be assumed that both the angle of twist and the slope of each flange are zero at the fixed end and the bending moment $\mathfrak{M}_b$ applied at the each flange is zero at the twisted end, namely

$$[\varphi]_{x=l}=0\,, \tag{15}$$

$$\left[\frac{d}{dx}\left(\frac{1}{2}h_x\varphi\right)\right]_{x=l}=0\,, \tag{16}$$

$$\left[-EI_f\frac{d^2}{dx^2}\left(\frac{1}{2}h_x\varphi\right)\right]_{x=0}=0\,. \tag{17}$$

Accordingly,

$$P\cdot\left[\int X^{-1}\cdot J_\nu dX\right]^{al+h}+Q\cdot\left[\int X^{-1}\cdot Y_\nu dx\right]^{al+h}$$
$$+\left[\int X^{-1}\cdot J_\nu\left\{\int Y_\nu dX\right\}dX-\int X^{-1}\cdot Y_\nu\left\{\int J_\nu dX\right\}dX\right]^{al+h}\cdot M_1+R=0, \tag{18}$$

$$P\cdot\left[J_\nu\right]^{al+h}+Q\cdot\left[Y_\nu\right]^{al+h}$$
$$+\left[J_\nu\cdot\left\{\int Y_\nu dX\right\}-Y_\nu\cdot\left\{\int J_\nu dX\right\}\right]^{al+h}\cdot M_1=0\,, \tag{19}$$

$$P\cdot\left[X^{-1}\cdot J_\nu+J_\nu'\right]^h+Q\cdot\left[X^{-1}\cdot Y_\nu+Y_\nu'\right]^h$$
$$+\left[\left\{X^{-1}\cdot J_\nu+J_\nu'\right\}\cdot\int Y_\nu dX-\left\{X^{-1}\cdot Y_\nu+Y_\nu'\right\}\cdot\int J_\nu dX\right]^h\cdot M_1=0\,. \tag{20}$$

Solving the Eq. (18), (19) and (20), constants P, Q and R are given as follows:

the numerator of P

$$-\left[J_\nu\cdot\left\{\int Y_\nu dX\right\}-Y_\nu\cdot\left\{\int J_\nu dX\right\}\right]^{al+h}\cdot\left[X^{-1}\cdot Y_\nu+Y_\nu'\right]^h\cdot M_1$$
$$+\left[\left\{X^{-1}\cdot J_\nu+J_\nu'\right\}\cdot\int Y_\nu dX-\left\{X^{-1}\cdot Y_\nu+Y_\nu'\right\}\cdot\int J_\nu dX\right]^h\cdot\left[Y_\nu\right]^{al+h}\cdot M_1\,, \tag{21}$$

the numerator of Q

$$\Big[J_\nu\cdot\Big\{\int Y_\nu dX - Y_\nu\cdot\Big\{\int J_\nu dX\Big\}\Big]^{al+h}\Big[X^{-1}\cdot J_\nu + J_\nu'\Big]^h\cdot M_1$$
$$-\Big[\Big\{X^{-1}\cdot J_\nu + J_\nu'\Big\}\cdot\int Y_\nu dX - \Big\{X^{-1}\cdot Y_\nu + Y_\nu'\Big\}\cdot\int J_\nu dX\Big]^h\cdot\Big[J_\nu\Big]^{al+h}\cdot M_1, \quad (22)$$

the numerator of R

$$\Big[\int X^{-1}\cdot J_\nu\cdot\Big\{\int Y_\nu dX\Big\}dX - \int X^{-1}\cdot Y_\nu\cdot\Big\{\int J_\nu dX\Big\}dX\Big]^{al+h}$$
$$\times\Big\{\Big[X^{-1}\cdot J_\nu + J_\nu'\Big]^h\cdot\Big[Y_\nu\Big]^{al+h} - \Big[X^{-1}\cdot Y_\nu + Y_\nu'\Big]^h\cdot\Big[J_\nu\Big]^{al+h}\Big\}\cdot M_1$$
$$+\Big[J_\nu\cdot\Big\{\int Y_\nu dX\Big\} - Y_\nu\cdot\Big\{\int J_\nu dX\Big\}\Big]^{al+h}$$
$$\times\Big\{\Big[\int X^{-1}\cdot J_\nu dX\Big]^{al+h}\cdot\Big[X^{-1}\cdot Y_\nu + Y_\nu'\Big]^h - \Big[\int X^{-1}\cdot Y_\nu dX\Big]^{al+h}\cdot\Big[X^{-1}\cdot J_\nu + J_\nu'\Big]^h\Big\}M_1$$
$$+\Big[\Big\{X^{-1}\cdot J_\nu + J_\nu'\Big\}\cdot\int Y_\nu dX - \Big\{X^{-1}\cdot Y_\nu + Y_\nu'\Big\}\cdot\int J_\nu dX\Big]^h$$
$$\times\Big[J_\nu\cdot\int X^{-1}\cdot Y_\nu dX - Y_\nu\cdot\int X^{-1}\cdot J_\nu dX\Big]^{al+h}\cdot M_1, \quad (23)$$

the denominators of P, Q and R

$$[J_\nu]^{al+h}\cdot[X^{-1}\cdot Y_\nu + Y_\nu']^h - [Y_\nu]^{al+h}\cdot[X^{-1}\cdot J_\nu + J_\nu']^h. \quad (24)$$

By the determination of constants P, Q and R, φ to satisfy the conditions at the both ends may be obtained.

The descriptions of the stress state of the I-beam can be derived from φ as follows:

$$\mathfrak{M}_a = GJ_a\frac{d\varphi}{dx} = \beta\Big\{(ax+h)+\gamma\Big\}\frac{d\varphi}{dx}, \quad (25)$$

$$\mathfrak{M}_b = -EI_f\frac{d^2}{dx^2}\Big(\frac{1}{2}h_x\varphi\Big) = -\alpha\Big\{2a\frac{d\varphi}{dx} + (ax+h)\frac{d^2\varphi}{dx^2}\Big\}, \quad (26)$$

$$Q_b = -EI_f\frac{d^3}{dx^3}\Big(\frac{1}{2}h_x\varphi\Big) = -\alpha\Big\{3a\frac{d^2\varphi}{dx^2} + (ax+h)\frac{d^3\varphi}{dx^3}\Big\}. \quad (27)$$

2 Differential Equation and Solution for Torsion of I-beam in which Web Height varies in accordance with a second order Formula

When the height of web is tapered in accordance with a second order formula $h_x = ax^2 + bx + h$ as shown in Fig. (2), by repeating the preceding argument in the same way, the differential equation corresponding to Eq. (7) may be readily introduced.

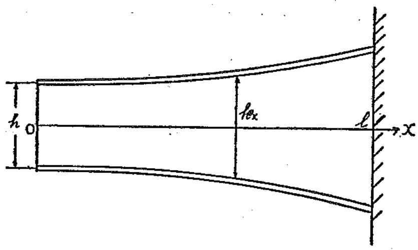


Fig. 2.

Namely,

$$GJ_d\frac{d\varphi}{dx}=G\frac{1}{3}\mu\left\{(ax^2+bx+h)t_1^3+2b_1t_2^3\right\}$$

$$=\beta\left\{(ax^2+bx+h)+\gamma\right\}, \tag{28}$$

$$-EI_fh_x\frac{d^3}{dx^3}\left(\frac{1}{2}h_x\varphi\right)$$

$$=-\alpha h_x\left\{6a\frac{d\varphi}{dx}+3(2ax+b)\frac{d^2\varphi}{dx^2}+(ax^2+bx+h)\frac{d^3\varphi}{dx^3}\right\}, \tag{29}$$

where

$$\alpha=\frac{1}{2}EI_f,\quad \beta=G\frac{1}{3}\mu t_1^3,\quad \gamma=\frac{t_2^3}{t_1^3}2b_1.$$

Accordingly, by substituting these into Eq. (6), the differential equation for φ can be written by

$$\alpha(ax^2+bx+h)^2\frac{d^3\varphi}{dx^3}+3\alpha(ax^2+bx+h)(2ax+b)\frac{d^2\varphi}{dx^2}$$

$$+\left\{(6a\alpha-\beta)(ax^2+bx+h)-\beta\gamma\right\}\frac{d\varphi}{dx}=-M_0. \tag{30}$$

Then, by putting $\frac{d\varphi}{dx}=\Phi$ and arranging, the above equation can be transformed as follows:

$$\frac{d^2\Phi}{dx^2}+3\frac{2ax+b}{ax^2+bx+h}\frac{d\Phi}{dx}+\left\{\frac{\left(6a-\frac{\beta}{\alpha}\right)}{ax^2+bx+h}-\frac{\frac{\beta\gamma}{\alpha}}{(ax^2+bx+h)^2}\right\}\Phi$$

$$=-\frac{M_0}{\alpha(ax^2+bx+h)^2}. \tag{31}$$

(a) Solution of Eq. (31) in the vicinity of an ordinary point $x=\sigma$

Assuming that the coefficients of $\dfrac{d\Phi}{dx}$, Φ and the right hand side of Eq. (31) can be expanded in a Taylor's series, $x=\sigma$ denotes any point in the complex plane which is ordinary for all the coefficients of Eq. (31); and a domain of σ can be constructed by taking all the points x in the plane, such that $|x-\sigma| < |\sigma_s-\sigma|$, where σ_s is the nearest to σ among all the singularities of all the coefficients.

$$3\,\frac{2ax+b}{ax^2+bx+h}=\sum_{n=0}^{\infty} A_n(x-\sigma)^n\,, \tag{32}$$

$$\left\{\frac{6a-\dfrac{\beta}{\alpha}}{ax^2+bx+h}-\frac{\dfrac{\beta\gamma}{\alpha}}{(ax^2+bx+h)^2}\right\}=\sum_{n=0}^{\infty} B_n(x-\sigma)^n\,, \tag{33}$$

$$\frac{-M_0}{\alpha(ax^2+bx+h)^2}=\sum_{n=0}^{\infty} C_n(x-\sigma)^n\,, \tag{34}$$

where for $\sigma=0$ specially

$$A_0=3\frac{b}{h}\,,\quad A_1=3\left(\frac{2a}{h}-\frac{b^2}{h^2}\right),\quad A_2=3\left(-\frac{3ab}{h^2}+\frac{b^3}{h^3}\right),$$

$$A_3=3\left(-\frac{2a^2}{h^2}+\frac{4ab^2}{h^3}-\frac{b^4}{h^4}\right),\quad A_4=3\left(\frac{5a^2b}{h^3}-\frac{5ab^3}{h^4}+\frac{b^5}{h^5}\right),$$

$$A_5=3\left(\frac{2a^3}{h^3}-\frac{9a^2b^2}{h^4}+\frac{6ab^4}{h^5}-\frac{b^6}{h^6}\right),\quad A_6=3\left(-\frac{7a^3b}{h^4}+\frac{14a^2b^3}{h^5}-\frac{7ab^5}{h^6}+\frac{b^7}{h^7}\right),\cdots$$

$$B_0=\frac{1}{h^2}\left\{\left(6a-\frac{\beta}{\alpha}\right)h-\frac{\beta\gamma}{\alpha}\right\},\quad B_1=\frac{1}{h^2}\left\{\left(6a-\frac{\beta}{\alpha}\right)(-b)+\frac{\beta\gamma}{\alpha}\cdot\frac{2b}{h}\right\},$$

$$B_2=\frac{1}{h^2}\left\{\left(6a-\frac{\beta}{\alpha}\right)\left(-a+\frac{b^2}{h}\right)-\frac{\beta\gamma}{\alpha}\left(3\frac{b^2}{h^2}-\frac{2a}{h}\right)\right\},$$

$$B_3=\frac{1}{h^2}\left\{\left(6a-\frac{\beta}{\alpha}\right)\left(\frac{2ab}{h}-\frac{b^3}{h^2}\right)+\frac{\beta\gamma}{\alpha}\left(4\frac{b^3}{h^3}-\frac{6ab}{h^2}\right)\right\},$$

$$B_4=\frac{1}{h^2}\left\{\left(6a-\frac{\beta}{\alpha}\right)\left(\frac{a^2}{h}-\frac{3ab^2}{h^2}+\frac{b^4}{h^3}\right)-\frac{\beta\gamma}{\alpha}\left(\frac{5b^4}{h^4}-\frac{12ab^2}{h^3}+\frac{3a^2}{h^2}\right)\right\},\cdots$$

$$C_0=\frac{-M_0}{\alpha h^2}\,,\quad C_1=\frac{-M_0}{\alpha h^2}\left(-\frac{2b}{h}\right),\quad C_2=\frac{-M_0}{\alpha h^2}\left(\frac{3b^2}{h^2}-\frac{2a}{h}\right),$$

$$C_3=\frac{-M_0}{\alpha h^2}\left(-\frac{4b^3}{h^3}+\frac{6ab}{h^2}\right),\quad C_4=\frac{-M_0}{\alpha h^2}\left(\frac{5b^4}{h^4}-\frac{12ab^2}{h^3}+\frac{3a^2}{h^2}\right),$$

$$C_5=\frac{-M_0}{\alpha h^2}\left(-\frac{6b^5}{h^5}+\frac{20ab^3}{h^4}-\frac{12a^2b}{h^3}\right),\cdots$$

The solution of Eq. (31) can be taken in the form of a power series $\sum_{n=0}^{\infty}D_n(x-\sigma)^n$. Substituting this series in Eq. (31) and equating the coefficients for each power of x to zero or $\sum_{n=0}^{\infty}C_n(x-\sigma)^n$, the coefficients of the solution Φ_0 of the homogeneous equation and these of the particular solution Φ_0 of Eq. (31) can be determined.

The complete solution Φ converges within the domain of σ, specially for $\sigma=0$ Φ_0 and Φ_1 are respectively denoted in the following form:

$$\Phi=\Phi_0+\Phi_1\,. \tag{35}$$

$$\begin{aligned}
\Phi_0=P\Big\{&x+x^2\left(-\frac{1}{2}A_0\right)+x^3\left(-\frac{1}{6}A_1-\frac{1}{6}B_0+\frac{1}{6}A_0^2\right)\\
&+x^4\left(-\frac{1}{12}A_2-\frac{1}{12}B_1+\frac{1}{12}A_0B_0+\frac{1}{8}A_0A_1-\frac{1}{24}A_0^3\right)\\
&+x^5\left(-\frac{1}{20}A_3-\frac{1}{20}B_2+\frac{1}{40}A_1^2+\frac{1}{120}B_0^2+\frac{1}{15}A_0A_2+\frac{1}{24}A_0B_1+\frac{1}{30}A_1B_0\right.\\
&\left.-\frac{1}{20}A_0^2A_1-\frac{1}{40}A_0^2B_0+\frac{1}{120}A_0^4\right)+x^6\left(-\frac{1}{30}A_4-\frac{1}{30}B_3+\frac{1}{24}A_0A_3+\frac{1}{40}A_0B_2\right.\\
&+\frac{1}{36}A_1A_2+\frac{7}{360}A_2B_0+\frac{1}{60}A_1B_1+\frac{1}{120}B_0B_1-\frac{1}{36}A_0^2A_2-\frac{1}{48}A_0A_1^2\\
&\left.-\frac{1}{80}A_0^2B_1-\frac{1}{240}A_0B_0^2-\frac{1}{48}A_0A_1B_0+\frac{1}{72}A_0^3A_1+\frac{1}{180}A_0^3B_0-\frac{1}{720}A_0^5\right)+\cdots\Big\}\\
+Q\Big\{&1+x^2\left(-\frac{1}{2}B_0\right)+x^3\left(-\frac{1}{6}B_1+\frac{1}{6}A_0B_0\right)\\
&+x^4\left(-\frac{1}{12}B_2+\frac{1}{24}A_0B_1-\frac{1}{24}A_0^2B_0+\frac{1}{12}A_1B_0+\frac{1}{24}B_0^2\right)\\
&+x^5\left(-\frac{1}{20}B_3+\frac{1}{60}A_0B_2+\frac{1}{20}A_2B_0+\frac{1}{40}A_1B_1+\frac{1}{30}B_0B_1-\frac{1}{120}A_0^2B_1\right.\\
&\left.-\frac{1}{60}A_0B_0^2-\frac{1}{24}A_0A_1B_0+\frac{1}{120}A_0^3B_0\right)+x^6\left(-\frac{1}{30}B_4+\frac{1}{30}A_3B_0+\frac{1}{60}A_2B_1\right.\\
&+\frac{1}{90}A_1B_2+\frac{1}{120}A_0B_3+\frac{7}{360}B_0B_2+\frac{1}{180}B_1^2-\frac{1}{360}A_0^2B_2-\frac{1}{80}A_0B_0B_1
\end{aligned}$$

$$-\frac{1}{40}A_0A_2B_0-\frac{1}{90}A_1^2B_0-\frac{1}{120}A_1B_0^2-\frac{1}{720}B_0^3-\frac{7}{720}A_0A_1B_1+\frac{1}{720}A_0^3B_1$$
$$\left.+\frac{1}{240}A_0^2B_0^2+\frac{1}{80}A_0^2A_1B_0-\frac{1}{720}A_0^4B_0\Big)+\cdots\right\} \qquad (36)$$

$$\Phi_1=x^2\left(\frac{1}{2}C_0\right)+x^3\left\{\frac{1}{6}(C_1-A_0C_0)\right\}$$
$$+x^4\left\{\frac{1}{12}\left(C_2-A_1C_0-\frac{1}{2}A_0C_1-\frac{1}{2}B_0C_0+\frac{1}{2}A_0^2C_0\right)\right\}$$
$$+x^5\left\{\frac{1}{20}\left(C_3-A_2C_0-\frac{1}{2}A_1C_1-\frac{1}{2}B_1C_0-\frac{1}{3}A_0C_2-\frac{1}{6}B_0C_1\right.\right.$$
$$\left.\left.+\frac{1}{6}A_0^2C_1+\frac{5}{6}A_0A_1C_0+\frac{1}{3}A_0B_0C_0-\frac{1}{6}A_0^3C_0\right)\right\}+\cdots \qquad (37)$$

Integrating these with respect to x, the solution of Eq. (30) is obtained:

$$\varphi=P\left[\frac{1}{2}x^2+\frac{1}{3}x^3\left(-\frac{1}{2}A_0\right)+\frac{1}{4}x^4\left(-\frac{1}{6}A_1-\frac{1}{6}B_0+\frac{1}{6}A_0^2\right)\right.$$
$$+\frac{1}{5}x^5\left(-\frac{1}{12}A_2-\frac{1}{12}B_1+\frac{1}{12}A_0B_0+\frac{1}{8}A_0A_1-\frac{1}{24}A_0^3\right)$$
$$\left.+\frac{1}{6}x^6(\cdots\qquad)+\frac{1}{7}x^7(\cdots\qquad)+\cdots\qquad\right]$$
$$+Q\left[x+\frac{1}{3}x^3\left(-\frac{1}{2}B_0\right)+\frac{1}{4}x^4\left(-\frac{1}{6}B_1+\frac{1}{6}A_0B_0\right)\right.$$
$$+\frac{1}{5}x^5\left(-\frac{1}{12}B_2+\frac{1}{24}A_0B_1-\frac{1}{24}A_0^2B_0+\frac{1}{12}A_1B_0+\frac{1}{24}B_0^2\right)$$
$$\left.+\frac{1}{6}x^6(\cdots\qquad)+\frac{1}{7}x^7(\cdots\qquad)+\cdots\qquad\right]$$
$$+\left[\frac{1}{3}x^3\left(\frac{1}{2}C_0\right)+\frac{1}{4}x^4\left\{\frac{1}{6}(C_1-A_0C_0)\right\}\right.$$
$$+\frac{1}{5}x^5\left\{\frac{1}{12}\left(C_2-A_1C_0-\frac{1}{2}A_0C_1-\frac{1}{2}B_0C_0+\frac{1}{2}A_0^2C_0\right)\right\}$$
$$\left.+\frac{1}{6}x^6\left\{\frac{1}{20}(\cdots\qquad)\right\}+\frac{1}{7}x^7\left\{\cdots\qquad\right\}+\cdots\qquad\right]+R, \qquad (38)$$

where P, Q and R are arbitraly constants.

(b) Solution of Eq. (31) in the vicinity of a regular singular point $x=\infty$

In order to investigate the singularity of Eq. (31) in the range denoting the large values of x, substituting the new variable $x=\frac{1}{\xi}$ in Eq. (31), the differential equation for Φ with respect to ξ may be obtained as follows:

$$\frac{d^2\Phi}{d\xi^2}+\left\{\frac{2h\xi^2-b\xi-4a}{\xi(h\xi^2+b\xi+a)}\right\}\frac{d\Phi}{d\xi}$$
$$+\left\{\frac{\left(6ah-\frac{\beta\gamma}{\alpha}-\frac{\beta h}{\alpha}\right)\xi^2+\left(6ab-\frac{\beta b}{\alpha}\right)\xi+\left(6a^2-\frac{\beta a}{\alpha}\right)}{\xi^2(h\xi^2+b\xi+a)^2}\right\}\Phi=\frac{-M_0}{\alpha(h\xi^2+b\xi+a)^2}. \tag{39}$$

$\xi=0$, namely $x=\infty$ denotes a regular singular point of Eq. (39), so that the regular sulution Φ can be obtained.

The coefficients of $\frac{d\Phi}{d\xi}$, Φ and the right hand side of Eq. (39) can be expanded in a power series

$$\frac{2h\xi^2-b\xi-4a}{\xi(h\xi^2+b\xi+a)}=\xi^{-1}\sum_{n=0}^{\infty}A_n\xi^n, \tag{40}$$

$$\frac{\left(6ah-\frac{\beta\gamma}{\alpha}-\frac{\beta h}{\alpha}\right)\xi^2+\left(6ab-\frac{\beta b}{\alpha}\right)\xi+\left(6a^2-\frac{\beta a}{\alpha}\right)}{\xi^2(h\xi^2+b\xi+a)^2}=\xi^{-2}\sum_{n=0}^{\infty}B_n\xi^n, \tag{41}$$

$$\frac{-M_0}{\alpha(h\xi^2+b\xi+a)^2}=\sum_{n=0}^{\infty}C_n\xi^n \tag{42}$$

where

$$A_0=-4,\quad A_1=\frac{1}{a}(3b),\quad A_2=\frac{1}{a}\left(6h-\frac{3b^2}{a}\right),\quad A_3=\frac{1}{a}\left(-\frac{9bh}{a}+\frac{3b^3}{a^2}\right),$$
$$A_4=\frac{1}{a}\left(-\frac{6h^2}{a}+\frac{12b^2h}{a^2}-\frac{3b^4}{a^3}\right),\quad A_5=\frac{1}{a}\left(\frac{15bh^2}{a^2}-\frac{15b^3h}{a^3}+\frac{3b^5}{a^4}\right),\cdots$$
$$B_0=\frac{1}{a}\left(6a-\frac{\beta}{\alpha}\right),\quad B_1=\frac{1}{a^2}\left(-6ab+\frac{b\beta}{\alpha}\right),$$
$$B_2=\frac{1}{a^2}\left(-6ah+6b^2-\frac{\beta\gamma}{\alpha}+\frac{\beta h}{\alpha}-\frac{b^2\beta}{\alpha a}\right),$$
$$B_3=\frac{1}{a^2}\left(12bh-\frac{6b^3}{a}+\frac{2b\beta\gamma}{\alpha a}-\frac{2b\beta h}{\alpha a}+\frac{b^3\beta}{\alpha a^2}\right),$$
$$B_4=\frac{1}{a^2}\left(6h^2-\frac{18b^2h}{a}+\frac{6b^4}{a^2}-\frac{h^2\beta}{\alpha a}-\frac{2h\beta\gamma}{\alpha a}-\frac{3b^2\beta\gamma}{\alpha a^2}+\frac{3b^2h\beta}{\alpha a^2}-\frac{b^4\beta}{\alpha a^3}\right),\cdots$$
$$C_0=\frac{-M_0}{\alpha a^2},\quad C_1=\frac{-M_0}{\alpha a^2}\left(-\frac{2b}{a}\right),$$
$$C_2=\frac{-M_0}{\alpha a^2}\left(\frac{3b^2}{a^2}-\frac{2h}{a}\right),\quad C_3=\frac{-M_0}{\alpha a^2}\left(-\frac{4b^3}{a^3}+\frac{6hb}{a^2}\right),$$

$$C_4=\frac{-M_0}{\alpha a^2}\left(\frac{5b^4}{a^4}-\frac{12hb^2}{a^3}+\frac{3h^2}{a^2}\right),\quad C_5=\frac{-M_0}{\alpha a^2}\left(-\frac{6b^5}{a^5}+\frac{20hb^3}{a^4}-\frac{12h^2b}{a^3}\right),\cdots$$

The solution of Eq. (39) can be taken in the form of a power series $\xi^\lambda \sum_{n=0}^{\infty} D_n\xi^n$. Substituting this series in Eq. (39) and equating the coefficients of each power of ξ to zero,

$$\begin{aligned}
0&=D_0f_0(\lambda)\,,\\
0&=D_0f_1(\lambda)+D_1f_0(\lambda+1)\,,\\
0&=D_0f_2(\lambda)+D_1f_1(\lambda+1)+D_2f_0(\lambda+2)\,,\\
0&=D_0f_3(\lambda)+D_1f_2(\lambda+1)+D_2f_1(\lambda+2)+D_3f_0(\lambda+3),\\
&\vdots
\end{aligned}$$

By these equations the coefficients of Φ_0 can be determined as follows:

$$\begin{aligned}
D_1&=D_0\frac{-f_1(\lambda)}{f_0(\lambda+1)}\,,\\
D_2&=D_0\frac{g_2(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)}\,,\\
D_3&=D_0\frac{g_3(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)\cdot f_0(\lambda+3)}\,,\\
&\vdots
\end{aligned}$$

where

$$\begin{aligned}
f_0(\lambda)&=\lambda(\lambda-1)+A_0\lambda+B_0\,,\\
f_0(\lambda+1)&=(\lambda+1)\lambda+A_0(\lambda+1)+B_0\,,\\
f_0(\lambda+2)&=(\lambda+2)(\lambda+1)+A_0(\lambda+2)+B_0\,,\\
&\vdots\\
f_1(\lambda)&=A_1\lambda+B_1\,,\\
f_2(\lambda)&=A_2\lambda+B_2\,,\\
f_3(\lambda)&=A_3\lambda+B_3\,,\\
&\vdots\\
f_1(\lambda+1)&=A_1(\lambda+1)+B_1\,,\\
f_2(\lambda+1)&=A_2(\lambda+1)+B_2\,,\\
f_3(\lambda+1)&=A_3(\lambda+1)+B_3\,,\\
&\vdots\\
g_2(\lambda)&=f_1(\lambda)\cdot f_1(\lambda+1)-f_2(\lambda)\cdot f_0(\lambda+1)\,,\\
g_3(\lambda)&=-g_2(\lambda)\cdot f_1(\lambda+2)+f_0(\lambda+2)\cdot f_1(\lambda)\cdot f_2(\lambda+1)-f_0(\lambda+1)\cdot f_0(\lambda+2)\cdot f_3(\lambda)\,,\\
g_4(\lambda)&=-g_3(\lambda)\cdot f_1(\lambda+3)-g_2(\lambda)\cdot f_2(\lambda+2)\cdot f_0(\lambda+3)\\
&\quad+f_1(\lambda)\cdot f_3(\lambda+1)\cdot f_0(\lambda+3)\cdot f_0(\lambda+2)-f_4(\lambda)\cdot f_0(\lambda+3)\cdot f_0(\lambda+2)\cdot f_0(\lambda+1),\\
&\vdots
\end{aligned}$$

However, $f_0(\lambda)=\lambda(\lambda-1)+A_0\lambda+B_0$ must equate to zero, so that two different

λ_1 and λ_2 are determined as follows:

$$\lambda_1=\frac{1-A_0+\sqrt{(1-A_0)^2-4B_0}}{2}\,, \tag{43}$$

$$\lambda_2=\frac{1-A_0-\sqrt{(1-A_0)^2-4B_0}}{2}\,. \tag{44}$$

1) *When λ_1 is not equal to λ_2 and the difference of both is not integer,*

$$\begin{aligned}\Phi_0=P\Big\{&\xi^{\lambda_1}+\frac{-f_1(\lambda_1)}{f_0(\lambda_1+1)}\xi^{\lambda_1+1}+\frac{g_2(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)}\xi^{\lambda_1+2}\\&+\frac{g_3(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)\cdot f_0(\lambda_1+3)}\xi^{\lambda_1+3}+\cdots\Big\}\\+Q\Big\{&\xi^{\lambda_2}+\frac{-f_1(\lambda_2)}{f_0(\lambda_2+1)}\xi^{\lambda_2+1}+\frac{g_2(\lambda_2)}{f_0(\lambda_2+1)\cdot f_0(\lambda_2+2)}\xi^{\lambda_2+2}\\&+\frac{g_3(\lambda_2)}{f_0(\lambda_2+1)\cdot f_0(\lambda_2+2)\cdot f_0(\lambda_2+3)}\xi^{\lambda_2+3}+\cdots\Big\}\,,\end{aligned} \tag{45}$$

where P and Q are arbitrary constants.

2) *When λ_1 is equal to λ_3,*

$$\begin{aligned}\Phi_0=P\Big\{&\xi^{\lambda_1}+\frac{-f_1(\lambda_1)}{f_0(\lambda_1+1)}\xi^{\lambda_1+1}+\frac{g_2(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)}\xi^{\lambda_1+2}\\&+\frac{g_3(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)\cdot f_0(\lambda_1+3)}\xi^{\lambda_1+3}+\cdots\cdots\cdots\cdots\Big\}\\+Q\Big[\Big\{&\xi^{\lambda_1}+\frac{-f_1(\lambda_1)}{f_0(\lambda_1+1)}\xi^{\lambda_1+1}+\frac{g_2(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)}\xi^{\lambda_1+2}\\&+\frac{g_3(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)\cdot f_0(\lambda_1+3)}\xi^{\lambda_1+3}+\cdots\cdots\cdots\cdots\Big\}\log\xi\\&+\Big\{\xi^{\lambda_1+1}\frac{\partial}{\partial\lambda}\left(\frac{-f_1(\lambda)}{f_0(\lambda+1)}\right)_{\lambda=\lambda_1}+\xi^{\lambda_1+2}\frac{\partial}{\partial\lambda}\left(\frac{g_2(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)}\right)_{\lambda=\lambda_1}\\&+\xi^{\lambda_1+3}\frac{\partial}{\partial\lambda}\left(\frac{g_3(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)\cdot f_0(\lambda+3)}\right)_{\lambda=\lambda_1}+\cdots\Big\}\Big]\,.\end{aligned} \tag{46}$$

3) *When $\lambda_1-s=\lambda_2$ and then s is a positive integer and not zero,*

$$\begin{aligned}\Phi_0=P\Big[\xi^{\lambda}\sum_{n=0}D_n\xi^n\Big]_{\lambda=\lambda_1}+Q\Big[&\xi^{\lambda_2}\log\xi\sum_{n=s}^{\infty}\{(\lambda-\lambda_2)D_n\}_{\lambda=\lambda_2}\cdot\xi^n\\&+\xi^{\lambda_2}\sum_{n=0}^{\infty}\left(\frac{d(\lambda-\lambda_2)D_n}{d\lambda}\right)_{\lambda=\lambda_2}\cdot\xi^n\Big]\,.\end{aligned} \tag{47}$$

Substituting $\Phi=\xi^{\lambda}\sum_{n=0} D_n\xi^n$ in Eq. (39) and equating the coefficients of each power of ξ to $\sum_{n=0} C_n\xi^n$,

$$\begin{aligned}
&\lambda=2\,,\\
&C_0=D_0\cdot f_0(2)\,,\\
&C_1=D_0\cdot f_1(2)+D_1\cdot f_0(3)\,,\\
&C_2=D_0\cdot f_2(2)+D_1\cdot f_1(3)+D_2\cdot f_0(4)\,,\\
&\ \vdots
\end{aligned}$$

By these equations the particular solution Φ_1 can be easily obtained as follows:

$$\begin{aligned}
\Phi_1=&\frac{C_0}{f_0(2)}\xi^2+\frac{C_2 f_0(2)-C_0 f_1(2)}{f_0(2)\cdot f_0(3)}\xi^3\\
&+\frac{C_0\{f_1(2)\cdot f_1(3)-f_2(2)\cdot f_0(3)\}-C_1 f_0(2)\cdot f_1(3)+C_2\cdot f_0(2)f_0(3)}{f_0(2)\cdot f_0(3)\cdot f_0(4)}\xi^4+\cdots\ .
\end{aligned}\tag{48}$$

By integrating $\Phi=\Phi_0+\Phi_1$ the complete solution for φ with respect to x can be obtained.

(c) Solution of Eq. (31) in the vicinity of a regular singular point σ_1 or σ_2

Defining that the two roots of $ax^2+bx+h=0$ are respectively σ_1 and σ_2, Eq. (31) is represented as follows:

$$\begin{aligned}
&\frac{d^2\Phi}{dx^2}+3\frac{2ax+b}{(x-\sigma_1)(x-\sigma_2)}\frac{d\Phi}{dx}+\left\{\frac{\left(6a-\dfrac{\beta}{\alpha}\right)}{(x-\sigma_1)(x-\sigma_2)}-\frac{\dfrac{\beta\gamma}{\alpha}}{(x-\sigma_1)^2(x-\sigma_2)^2}\right\}\Phi\\
&\qquad=-\frac{M_0}{\alpha(x-\sigma_1)^2(x-\sigma_2)^2}
\end{aligned}\tag{49}$$

To obtain the regular solution of Eq. (49) in the vicinity of a singular point $x=\sigma_1$, expanding the coefficients of $\dfrac{d\Phi}{dx}$, Φ and the right hand side of Eq. (49) in a Taylor's series,

$$\begin{aligned}
3\frac{2ax+b}{(x-\sigma_1)(x-\sigma_2)}&=(x-\sigma_1)^{-1}\sum_{n=0}A_n(x-\sigma_1)^n\\
&=(x-\sigma_1)^{-1}\left\{\frac{6a\sigma_1+3b}{\sigma_1-\sigma_2}+(-1)^n\sum_{n=1}\frac{6a\sigma_2+3b}{(\sigma_1-\sigma_2)^{n+1}}(x-\sigma_1)^n\right\}\,,
\end{aligned}\tag{50}$$

$$\frac{6a-\frac{\beta}{\alpha}}{(x-\sigma_1)(x-\sigma_2)}-\frac{\frac{\beta\gamma}{\alpha}}{(x-\sigma_1)^2(x-\sigma_2)^2}=(x-\sigma_1)^{-2}\sum_{n=0}B_n(x-\sigma_1)^n$$

$$=(x-\sigma_1)^{-2}\left\{-\frac{\frac{\beta\gamma}{\alpha}}{(\sigma_1-\sigma_2)^2}+(-1)^{n+1}\sum_{n=1}\frac{\left(6a-\frac{\beta}{\alpha}\right)(\sigma_1-\sigma_2)^2+(n+1)\frac{\beta\gamma}{\alpha}}{(\sigma_1-\sigma_2)^{n+2}}(x-\sigma_1)^n\right\}, \tag{51}$$

$$-\frac{M_0}{\alpha(x-\sigma_1)^2(x-\sigma_2)^2}=(x-\sigma_1)^{-2}\sum_{n=0}C_n(x-\sigma_1)^n$$

$$=(x-\sigma_1)^{-2}\left[\sum_{n=0}\frac{M_0}{\alpha}\cdot\frac{(-1)^{n+1}\cdot(n+1)}{(\sigma_1-\sigma_2)^{n+2}}(x-\sigma_1)^n\right]. \tag{52}$$

The regular solution in the vicinity of $x=\sigma_1$ can be taken in the form of a power series $(x-\sigma_1)^\lambda\sum_{n=0}D_n(x-\sigma_1)^n$, and converges for values of x such that $|x-\sigma_1|<|\sigma_1-\sigma_2|$.

Substituting this series in Eq. (49) and equating the coefficients of each power of $(x-\sigma_1)$ to zero, by using the same notations and method described in the preceding article (b), the coefficients of Φ_0 can be determined as follows:

$$D_1=D_0\frac{-f_1(\lambda)}{f_0(\lambda+1)},$$

$$D_2=D_0\frac{g_2(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)},$$

$$D_3=D_0\frac{g_3(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)\cdot f_0(\lambda+3)},$$

$$\vdots$$

However, $f_0(\lambda)=\lambda(\lambda-1)+A_0\lambda+B_0$ must equate to zero, so that two different roots λ_1 and λ_2 are determined as follows:

$$\lambda_1=\frac{1}{2}\left\{1-A_0+\sqrt{(1-A_0)^2-4B_0}\right\}, \tag{53}$$

$$\lambda_2=\frac{1}{2}\left\{1-A_0-\sqrt{(1-A_0)^2-4B_0}\right\}. \tag{54}$$

1) When λ_1 is not equal to λ_2 and the difference of both is not integer,

$$\Phi_0=P\left\{(x-\sigma_1)^{\lambda_1}+\frac{-f_1(\lambda_1)}{f_0(\lambda_1+1)}(x-\sigma_1)^{\lambda_1+1}+\frac{g_2(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda+2)}(x-\sigma_1)^{\lambda_1+2}\right.$$
$$\left.+\frac{g_3(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)\cdot f_0(\lambda_1+3)}(x-\sigma_1)^{\lambda_1+3}+\cdots\right\}$$

$$+Q\left\{(x-\sigma_1)^{\lambda_2}+\frac{-f_1(\lambda_2)}{f_0(\lambda_2+1)}(x-\sigma_1)^{\lambda_2+1}+\frac{g_2(\lambda_2)}{f_0(\lambda_2+1)\cdot f_0(\lambda_2+2)}(x-\sigma_1)^{\lambda_2+2}\right.$$
$$\left.+\frac{g_3(\lambda_2)}{f_0(\lambda_2+1)\cdot f_0(\lambda_2+2)\cdot f_0(\lambda_2+3)}(x-\sigma_1)^{\lambda_2+3}+\cdots\right\}, \tag{55}$$

where P and Q are arbitrary constants.

2) *When λ_1 is equal to λ_2,*

$$\Phi_0=P\left\{(x-\sigma_1)^{\lambda_1}+\frac{-f_1(\lambda_1)}{f_0(\lambda_1+1)}(x-\sigma_1)^{\lambda_1+1}+\frac{g_2(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)}(x-\sigma_1)^{\lambda_1+2}\right.$$
$$\left.+\frac{g_3(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)\cdot f_0(\lambda_1+3)}(x-\sigma_1)^{\lambda_1+3}+\cdots\right\}$$
$$+Q\left[\left\{(x-\sigma_1)^{\lambda_1}+\frac{-f_1(\lambda_1)}{f_0(\lambda_1+1)}(x-\sigma_1)^{\lambda_1+1}+\frac{g_2(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)}(x-\sigma_1)^{\lambda_1+2}\right.\right.$$
$$\left.+\frac{g_3(\lambda_1)}{f_0(\lambda_1+1)\cdot f_0(\lambda_1+2)\cdot f_0(\lambda_1+3)}(x-\sigma_1)^{\lambda_1+3}+\cdots\right\}\log(x-\sigma_1)$$
$$+\left\{(x-\sigma_1)^{\lambda_1+1}\frac{\partial}{\partial\lambda}\left(\frac{-f_1(\lambda)}{f_0(\lambda+1)}\right)_{\lambda=\lambda_1}+(x-\sigma_1)^{\lambda_1+2}\frac{\partial}{\partial\lambda}\left(\frac{g_2(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)}\right)_{\lambda=\lambda_1}\right.$$
$$\left.\left.+(x-\sigma_1)^{\lambda_1+3}\frac{\partial}{\partial\lambda}\left(\frac{g_3(\lambda)}{f_0(\lambda+1)\cdot f_0(\lambda+2)\cdot f_0(\lambda+3)}\right)_{\lambda=\lambda_1}+\cdots\right\}\right]. \tag{56}$$

3) *When $\lambda_1-s=\lambda$ and then s is a positive integer and not zero,*

$$\Phi_0=P\left[(x-\sigma_1)^{\lambda}\sum_{n=0}D_n(x-\sigma_1)^n\right]_{\lambda=\lambda_1}$$
$$+Q\left[(x-\sigma_1)^{\lambda_2}\log(x-\sigma_1)\sum_{n=s}^{\infty}\left\{(\lambda-\lambda_2)D_n\right\}_{\lambda=\lambda_2}(x-\sigma_1)^n\right.$$
$$\left.+(x-\sigma_1)^{\lambda_2}\sum_{n=0}\left(\frac{d(\lambda-\lambda_2)D_n}{d\lambda}\right)_{\lambda=\lambda_2}(x-\sigma_1)^n\right]. \tag{57}$$

Substituting $\Phi=(x-\sigma_1)^{\lambda}\sum_{n=0}D_n(x-\sigma_1)^n$ in Eq. (49) and equating the coefficients of each power of $(x-\sigma_1)$ to $(x-\sigma_1)^{-2}\sum_{n=0}C_n(x-\sigma_1)^n$, the particular solution Φ_1 can be easily obtained as follows:

$$\Phi_1=\frac{C_0}{f_0(0)}+\frac{C_2f_0(0)-C_0f_1(0)}{f_0(0)\cdot f_0(1)}(x-\sigma_1)$$
$$+\frac{C_0\{f_1(0)\cdot f_1(1)-f_2(0)\cdot f_0(1)\}-C_1f_0(0)\cdot f_1(1)+C_2f_0(0)\cdot f_0(1)}{f_0(0)\cdot f_0(1)\cdot f_0(2)}$$
$$\times(x-\sigma_1)^2+\cdots \quad . \tag{58}$$

By integrating $\Phi = \Phi_0 + \Phi_1$ the complete solution φ with respect to x can be obtained.

The constants P, Q and R must be determined so that they satisfy the conditions at the both ends. The necessary three end conditions are exactly equal to those of the preceding article, namely;

$$[\varphi]_{x=l} = 0 \, , \tag{59}$$

$$\left[\frac{d}{dx}\left(\frac{1}{2} h_x \varphi\right)\right]_{x=l} = 0 \, , \tag{60}$$

$$\left[\frac{d^2}{dx^2}\left(\frac{1}{2} h_x \varphi\right)\right]_{x=0} = 0 \, . \tag{61}$$

Substituting φ in the above Eqs. (59), (60) and (61), and by solving them, P, Q and R can be determined.

As described before, the descriptions of the stress state of the above mentioned I-beam can be derived from φ as follows:

$$\mathfrak{M}_a = GJ_a \frac{d\varphi}{dx} = \beta\left\{(ax^2 + bx + h) + \gamma\right\} \frac{d\varphi}{dx} \, , \tag{62}$$

$$\begin{aligned} \mathfrak{M}_b &= -EI_f \frac{d^2}{dx^2}\left(\frac{1}{2} h_x \varphi\right) \\ &= -\alpha\left\{2a\varphi + 2(2ax + b)\frac{d\varphi}{dx} + (ax^2 + bx + h)\frac{d^2\varphi}{dx^2}\right\} \, , \end{aligned} \tag{63}$$

$$\begin{aligned} Q_b &= -EI_f \frac{d^3}{dx^3}\left(\frac{1}{2} h_x \varphi\right) \\ &= -\alpha\left\{6a\frac{d\varphi}{dx} + 3(2ax + b)\frac{d^2\varphi}{dx^2} + (ax^2 + bx + h)\frac{d^3\varphi}{dx^3}\right\} \, . \end{aligned} \tag{64}$$

3 Examples

As to the application of torsion problems, as shown in Fig. (3), consider a riveted I-beam which is assembled with the same two flange plates 180×10 mm, four angles $80 \times 80 \times 10$ mm and a web plate of 1 cm in thickness, with a length of 400 cm in which the web height varies in accordance with $\frac{1}{5}x + 50$, from 50 cm at the twisted end to 130 cm at the fixed end.

Assuming that the twisting moment M_0 is applied at the free end, at first the values of P, Q and R may be calculated by using;

$$E = 2150 \text{ t/cm}^2 \, , \qquad G = 830 \text{ t/cm}^2, \qquad \mu = 1.30.$$

Accordingly, by using these constants, φ, $\mathfrak{M}_a$, $\mathfrak{M}_b$ and Q_b have been

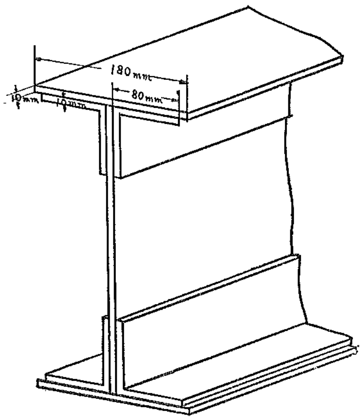


Fig. 3.

calculated and presented with the suffix 1 on the shoulder for convenience in Figs. (4), (5), (6) and (7).

Next, for the riveted I-beam the height of the web plate varies in accordance with $\frac{1}{4000}x^2+\frac{1}{10}x+50$ from 50 cm at the twisted end to 130 cm at the fixed end while the other dimensions are the same as in the above defined I-beam. The values of φ, $\mathfrak{M}_d$, $\mathfrak{M}_b$ and Q_b have been calculated and presented with, the suffix 2 on the shoulder in Figs. (4), (5), (6) and (7).

In order to compare the previous two riveted I-beams with I-beams of constant cross section, assume two different I-beams in which the height of the web plate is a constant 50 cm and 130 cm respectively while the other dimensions are the same as in the above mentioned variable I-beams.

Therefore, solving Eq. (1) so as to satisfy the end conditions:

$$[\varphi]_{x=l}=0\,,\quad \left[\frac{d\varphi}{dx}\right]_{x=l}=0\,,\quad \left[\frac{d^2\varphi}{dx^2}\right]_{x=0}=0\,,$$

$$\varphi=-\frac{M_0}{kGJ_d}\left[\frac{\sinh kx}{\cosh kl}-\frac{\sinh kl}{\cosh kl}+k(l-x)\right],$$

$$\mathfrak{M}_d=-M_0\left[\frac{\cosh kx}{\cosh kl}-1\right],$$

$$\mathfrak{M}_b=\frac{M_0}{kh}\left[\frac{\sinh kx}{\cosh kl}\right],$$

$$Q_b=\frac{M_0}{h}\left[\frac{\cosh kx}{\cosh kl}\right],$$

where

$$k=\frac{1}{h}\sqrt{2\frac{G}{E}\frac{J_d}{I_f}}\,.$$

By using these, two sets of φ, $\mathfrak{M}_b$, $\mathfrak{M}_d$ and Q_b for the two I-beams with a web height of 50 cm and 130 cm have been respectively shown by marking with suffixes 3 and 4 in Figs. (4), (5), (6) and (7).

It may be seen in Fig. (5) that $\mathfrak{M}_d^1$ and $\mathfrak{M}_d^2$ are larger than M_0 in the approach of the twisted end, while in contrast, $\mathfrak{M}_d^3$ and $\mathfrak{M}_d^4$ are always smaller than M_0.

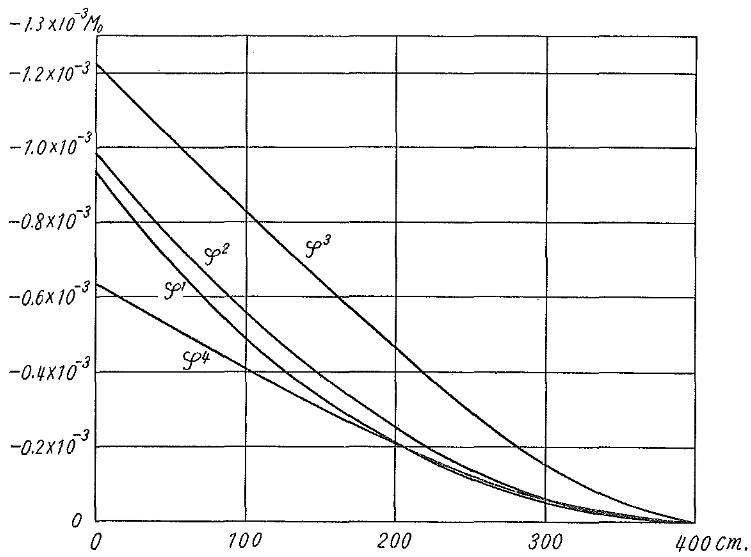


Fig. 4.

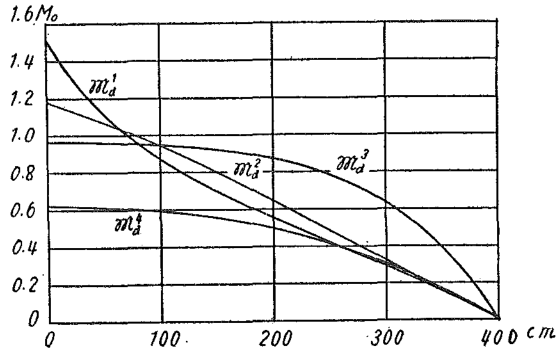


Fig. 5.

Though $\mathfrak{M}_b^3$ and $\mathfrak{M}_b^4$ are always plus values, $\mathfrak{M}_b^1$ and $\mathfrak{M}_b^2$ vary from plus values to minus values near the twisted end as shown in Fig. (6).

Q_b^1 and Q_b^2 being respectively the gradient of $\mathfrak{M}_b^1$ and $\mathfrak{M}_b^2$ must be varied from plus near the fixed end to minus near the twisted end. Further this is evident for the reason that Q_b^1 and Q_b^2 should be varied from plus to minus so as to satisfy $Q_b h_x + \mathfrak{M}_d = M_0$.

Q_b^3 and Q_b^4 are the gradients of $\mathfrak{M}_b^3$ and $\mathfrak{M}_b^4$, and must be satisfied by $Q_b h + \mathfrak{M}_d = M_0$, so that they always indicate plus values as shown in Fig. (7).

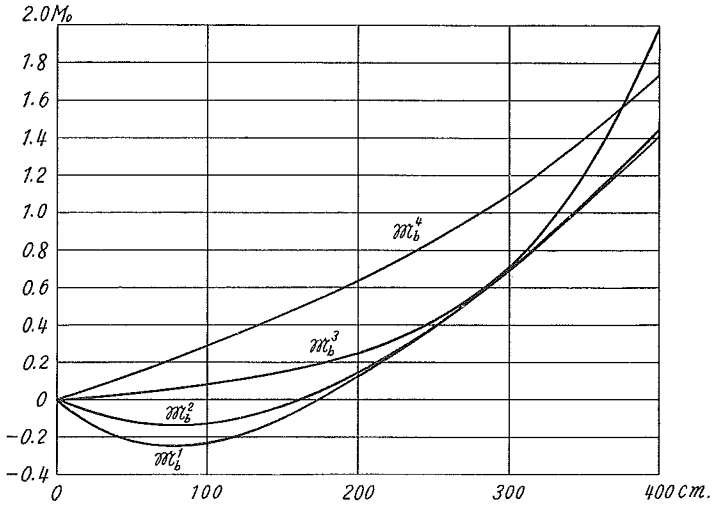


Fig. 6.

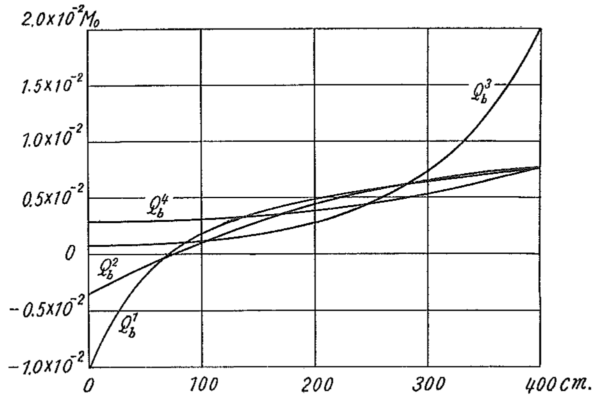


Fig. 7.

In general, the difference of the stress state between the I-beam with variable web height and that of constant web height may be sufficiently infered by the results shown in the above mentioned examples.

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