A Method of Analysis for Feed-back Systems Containing a Transfer Element with a Gain Constant Varying Periodically

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Contents

	Abstracts	511
1.	Introduction	511
2.	Derivation of Method	512
3.	Example	522
4.	Conclusions	525
	Reference	525

Abstracts

The theory of sampled-data system with finite pulse width which has been developed by G. Farmanfarma and E. I. Jury is regarded as a way to provide a method of analysis for feed-back systems containing a transfer element with a gain constant varying as periodic pulse train.

In this paper, it will be shown that the analysis of feed-back systems containing a transfer element with a gain constant varying freely in shape and periodically is quite possible by some extension of their method using P transform technique.

This new method is, for example, applicable to the analysis of variable coefficient electric circuits containing a mechanical chopper and feed-back control systems with a gain constant or a time constant varying periodically etc., and thereby we can derive the response for arbitrary input, the transfer function and the stability condition of these systems.

1. Introduction

The theory of a sampled-data system with finite pulse width which has been developed by G. Farmanfarma and E. I. Jury is regarded as a way to provide a method of analysis for feed-back systems containing a transfer element with a gain constant varying as periodic pulse train[1].

Therefore, if the varying gain constant of transfer element is approximated by an adequate staircase function, it will be presumed that the general method of analysis for feed-back systems containing a transfer element with a gain constant varying periodically is derived by some extention of their method using P transform technique.

In this paper especially, the general method of analysis using P transform technique for typical time varying gain system shown in Fig. (1) is derived, and the analysis of a typical chopper circuit shown in Fig. (7) is described as an application example of this method.

2. Derivation of method

In Fig. (1), each symbol is as follows ;

$R(s)$: Laplace transform of the input $r(t)$,
$C(s)$: Laplace tronsform of the out put $c(t)$,
$K(t)$: Gain varying periodically and in free shape,
$G(s)$: Arbitrary transfer function as shown equation (1),

$$G(s)=\frac{N(s)}{D(s)}=\frac{b_q s^q+b_{q-1}s^{q-1}+\cdots+b_0}{s^q+a_{q-1}s^{q-1}+\cdots+a_0} \quad (1)$$

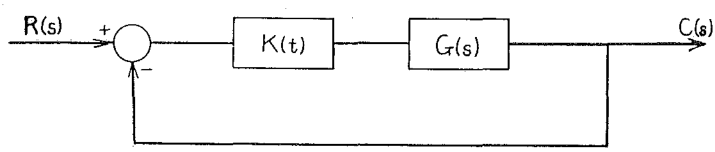


Fig. 1. Typical feed-back system containing an element with a gain constant varying periodically.

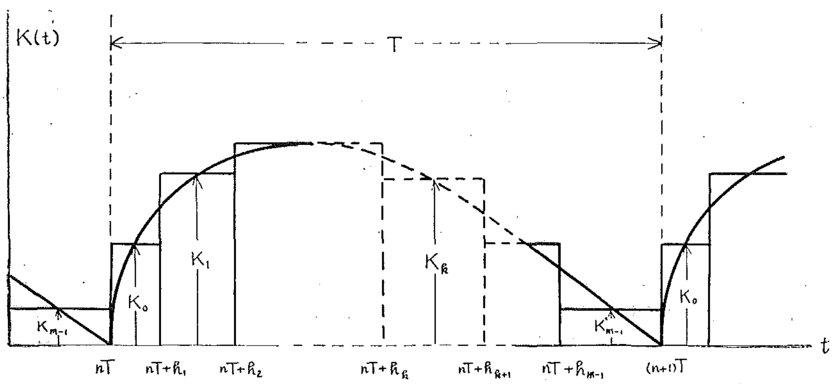


Fig. 2. Approximation of $K(t)$ by a staircase function.

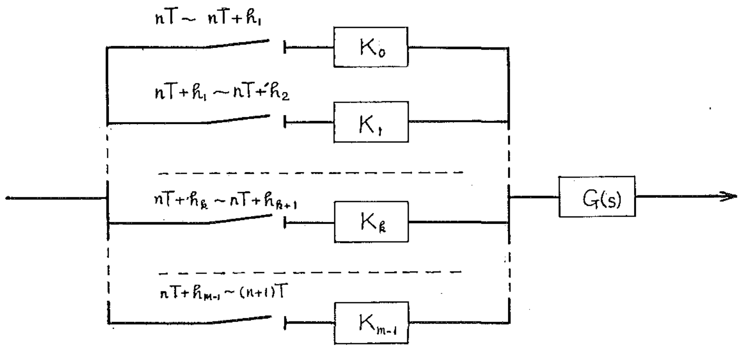


Fig. 3. Approximate expression of feed-forward path.

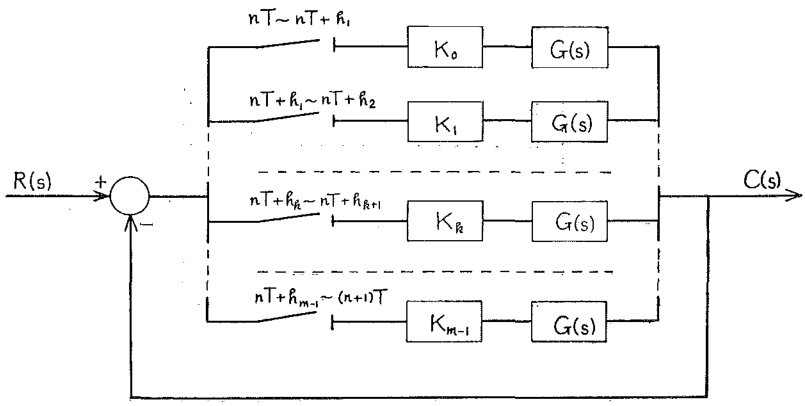


Fig. 4. Approximate expression of system.

Now, if $K(t)$ is approximated by an adequate staircase function as shown in Fig. (2), the forward path of this system may be represented as shown in Fig. (3). Therefore, using the superposition principle, the system shown in Fig. (1) is represented as shown in Fig. (4) and finally represented as shown in Fig. (5). In Fig. (3), Fig. (4) and Fig. (5), the symbols at each sampler indicate the interval for which each sampler closes. Therefore, for example, the system response $C(s)$ for the interval $nT+h_k \leq t < nT+h_{k+1}$ is given by the summation of $C_{00}(s)$, $C_{01}(s)$, $\cdots$, $C_{0,m-1}(s)$, $C_{10}(s)$, $C_{11}(s)$, $\cdots$, $C_{1,m-1}(s)$, $\cdots$, $C_{n0}(s)$, $C_{n1}(s)$, $\cdots$, $C_{nk}(s)$, as shown in Fig. (6).

Referring to Fig. (5) and Fig. (6), the next relation is derived,

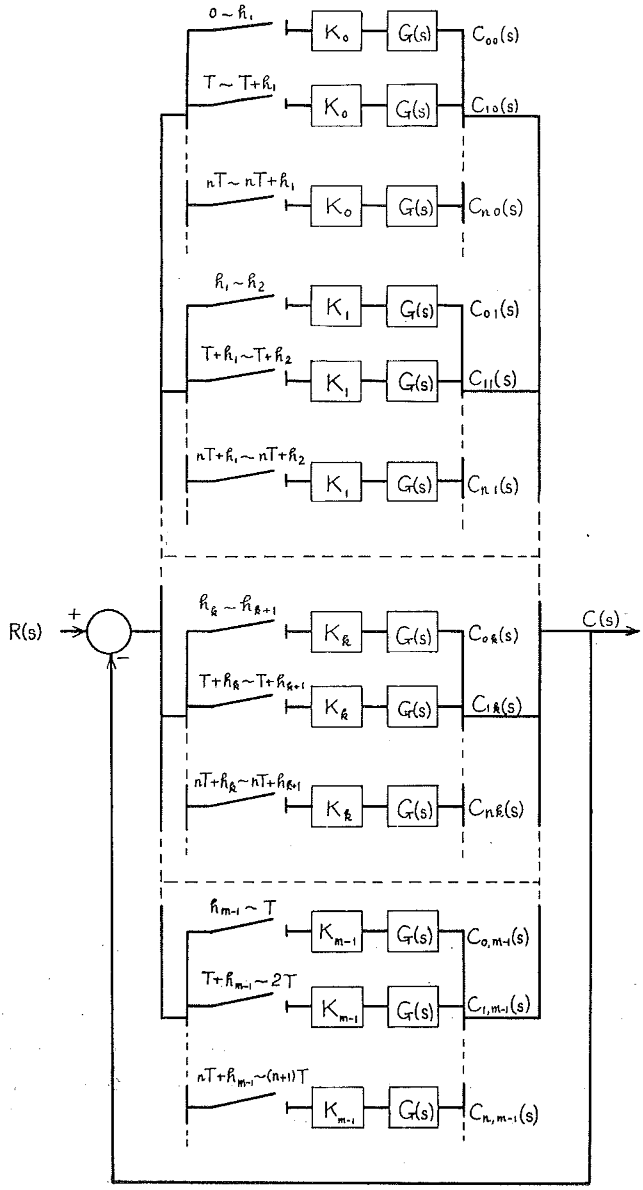


Fig. 5. Decomposition of sampler.

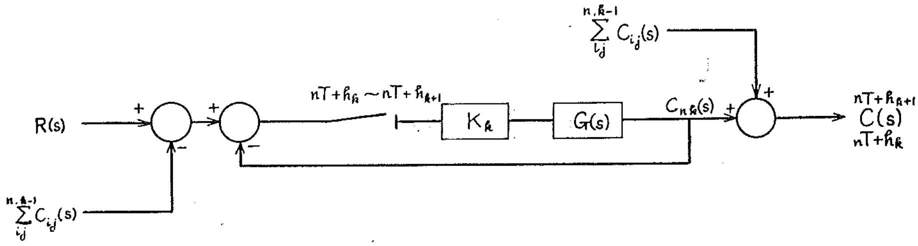
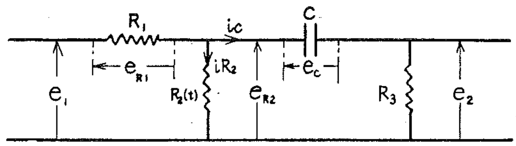
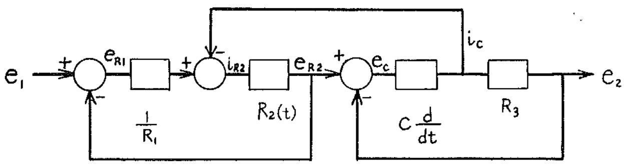


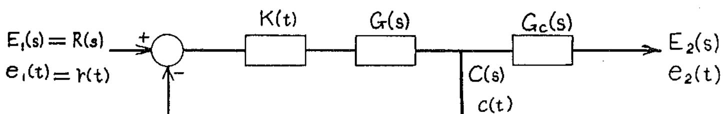
Fig. 6. Estimation of $C_{nk}(s)$ and $\overset{nT+h_{k+1}}{\underset{nT+h_k}{C}}(s)$



(a)



(b)



$$K(t)=\frac{R_2(t)}{R_1+R_2(t)}\ ,\quad G(s)=\frac{cR_1s}{1+cR_3s}\ ,\quad G_c(s)=\frac{R_3}{R_1}$$

(c)

Fig. 7. Example.

$$\left.\begin{aligned}
C_{n0}(s) &= e^{-nTs}\overset{h_1}{\underset{0}{P}}\left[\frac{e^{nTs}\overset{\infty}{\underset{nT}{P}}\left\{R(s)-\sum_{ij}^{n-1,m-1}C_{ij}(s)\right\}}{1+K_0G(s)}\right]K_0G(s)\\
C_{n1}(s) &= e^{-(nT+h_1)s}\overset{h_2-h_1}{\underset{0}{P}}\left[\frac{e^{(nT+h_1)s}\overset{\infty}{\underset{nT+h_1}{P}}\left\{R(s)-\sum_{ij}^{n,0}C_{ij}(s)\right\}}{1+K_1G(s)}\right]K_1G(s)\\
&\vdots\\
C_{nk}(s) &= e^{-(nT+h_k)s}\overset{h_{k+1}-h_k}{\underset{0}{P}}\left[\frac{e^{(nT+h_k)s}\overset{\infty}{\underset{nT+h_k}{P}}\left\{R(s)-\sum_{ij}^{n,k-1}C_{ij}(s)\right\}}{1+K_kG(s)}\right]K_kG(s)\\
&\vdots\\
C_{n,m-1}(s) &= e^{-(nT+h_{m-1})s}\overset{T-h_{m-1}}{\underset{0}{P}}\left[\frac{e^{(nT+h_{m-1})s}\overset{\infty}{\underset{nT+h_{m-1}}{P}}\left\{R(s)-\sum_{ij}^{n,m-2}C_{ij}(s)\right\}}{1+K_{m-1}G(s)}\right]K_{m-1}G(s)\\
C_{(n+1),0}(s) &= e^{-(n+1)Ts}\overset{h_1}{\underset{0}{P}}\left[\frac{e^{(n+1)Ts}\overset{\infty}{\underset{(n+1)T}{P}}\left\{R(s)-\sum_{ij}^{n,m-1}C_{ij}(s)\right\}}{1+K_0G(s)}\right]K_0G(s)
\end{aligned}\right\} \tag{2}$$

Where,

$$\left.\begin{aligned}
\sum_{ij}^{n,k-1}C_{ij}(s) &= C_{00}(s)+C_{01}(s)+\cdots+C_{0,m-1}(s)\\
&\quad+C_{10}(s)+C_{11}(s)+\cdots+C_{1,m-1}(s)\\
&\quad\vdots\\
&\quad+C_{n,0}(s)+C_{n1}(s)+\cdots+C_{n,k-1}(s)
\end{aligned}\right\} \tag{3}$$

Therefore, the inverse Laplace transform of $\overset{nT+h_1}{\underset{nT}{C}}(s)$, $\overset{nT+h_2}{\underset{nT+h_1}{C}}(s)$, $\cdots$, $\overset{nT+h_{k+1}}{\underset{nT+h_k}{C}}(s)$, $\cdots$, $\overset{(n+1)T}{\underset{nT+h_{m-1}}{C}}(s)$, $\overset{(n+1)T+h_1}{\underset{(n+1)T}{C}}(s)$ given by the next relations coincide with the system respon se for the interval $nT\leq t<nT+h_1$, $nT+h_1\leq t<nT+h_2$, $\cdots$, $nT+h_k\leq t< nT+h_{k+1}$, $\cdots$, $nT+h_{m-1}\leq t<(n+1)\,T$, $(n+1)\,T\leq t<(n+1)\,T+h_1$,

$$\left.\begin{aligned}
\overset{nT+h_1}{\underset{nT}{C}}(s) &= \overset{\infty}{\underset{nT}{P}}\left[\sum_{ij}^{n-1,m-1}C_{ij}(s)\right]+C_{n0}(s)\\
\overset{nT+h_2}{\underset{nT+h_1}{C}}(s) &= \overset{\infty}{\underset{nT+h_1}{P}}\left[\sum_{ij}^{n,0}C_{ij}(s)\right]+C_{n1}(s)\\
&\vdots\\
\overset{nT+h_{k+1}}{\underset{nT+h_k}{C}}(s) &= \overset{\infty}{\underset{nT+h_k}{P}}\left[\sum_{ij}^{n,k-1}C_{ij}(s)\right]+C_{nk}(s)\\
&\vdots\\
\overset{(n+1)T}{\underset{nT+h_{m-1}}{C}}(s) &= \overset{\infty}{\underset{nT+h_{m-1}}{P}}\left[\sum_{ij}^{n,m-2}C_{ij}(s)\right]+C_{n,m-1}(s)\\
\overset{(n+1)T+h_1}{\underset{(n+1)T}{C}}(s) &= \overset{\infty}{\underset{(n+1)T}{P}}\left[\sum_{ij}^{n,m-1}C_{ij}(s)\right]+C_{n+1,0}(s)
\end{aligned}\right\} \tag{4}$$

Referring to Fig. (5), the first term of each right side in equation (4) represents the summation of out put from each $G(s)$ at the time when input to each $G(s)$ is zero already.

Therefore each term should be given as shown in equation (5).
Here, $\overset{(0)}{A}_{q-1}(nT)$, $\overset{(0)}{A}_{q-2}(nT)$, $\cdots$, $\overset{(0)}{A}_{j}(nT)$, $\cdots$, $\overset{(k)}{A}_{q-1}(nT)$, $\overset{(k)}{A}_{q-2}(nT)$, $\cdots$, $\overset{(k)}{A}_{0}(nT)$, $\cdots$, $\overset{(m-1)}{A}_{q-1}(nT)$, $\overset{(m-1)}{A}_{q-2}(nT)$, $\cdots$, $\overset{(m-1)}{A}_{0}(nT)$, are constants determined by nT.

$$\left.\begin{aligned}
&\underset{nT}{\overset{\infty}{P}}\left[\sum_{ij}^{n-1,m-1} C_{ij}(s)\right] = e^{-nTs}\frac{\overset{(0)}{A}_{q-1}(nT)s^{q-1}+\overset{(0)}{A}_{q-2}(nT)s^{q-2}+\cdots+\overset{(0)}{A}_{0}(nT)}{D(s)}\\
&\underset{nT+h_1}{\overset{\infty}{P}}\left[\sum_{ij}^{n,0} C_{ij}(s)\right] = e^{-(nT+h_1)s}\frac{\overset{(1)}{A}_{q-1}(nT)s^{q-1}+\overset{(1)}{A}_{q-2}(nT)s^{q-2}+\cdots+\overset{(1)}{A}_{0}(nT)}{D(s)}\\
&\qquad\vdots\\
&\underset{nT+h_k}{\overset{\infty}{P}}\left[\sum_{ij}^{n,k-1} C_{ij}(s)\right] = e^{-(nT+h_k)s}\frac{\overset{(k)}{A}_{q-1}(nT)s^{q-1}+\overset{(k)}{A}_{q-2}(nT)s^{q-2}+\cdots+\overset{(k)}{A}_{0}(nT)}{D(s)}\\
&\qquad\vdots\\
&\underset{nT+h_{m-1}}{\overset{\infty}{P}}\left[\sum_{ij}^{n,m-2} C_{ij}(s)\right] = e^{-(nT+h_{m-1})s}\frac{\overset{(m-1)}{A}_{q-1}(nT)s^{q-1}+\overset{(m-1)}{A}_{q-2}(nT)s^{q-2}+\cdots+\overset{(m-1)}{A}_{0}(nT)}{D(s)}\\
&\underset{(n+1)T}{\overset{\infty}{P}}\left[\sum_{ij}^{n,m-1} C_{ij}(s)\right]\\
&\qquad = e^{-(n+1)Ts}\frac{\overset{(0)}{A}_{q-1}(n+1)Ts^{q-1}+\overset{(0)}{A}_{q-2}(n+1)Ts^{q-2}+\cdots+\overset{(0)}{A}_{0}(n+1)T}{D(s)}
\end{aligned}\right\} \tag{5}$$

Hence, from equations (2) (4) and (5), the next relation is derived,

$$\left.\begin{aligned}
&\underset{nT}{\overset{nT+h_1}{C}}(s) = e^{-nTs}\left\{\frac{\overset{(0)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(0)}{A}_{0}(nT)}{D(s)}\right.\\
&\qquad\left.+\underset{0}{\overset{h_1}{P}}\left[\frac{e^{nTs}\underset{nT}{\overset{\infty}{P}}\{R(s)\}-\dfrac{\overset{(0)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(0)}{A}_{0}(nT)}{D(s)}}{1+K_0G(s)}\right]K_0G(s)\right\}\\
&\underset{nT+h_1}{\overset{nT+h_2}{C}}(s) = e^{-(nT+h_1)s}\left\{\frac{\overset{(1)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(1)}{A}_{0}(nT)}{D(s)}\right.\\
&\qquad\vdots\\
&\qquad\left.+\underset{0}{\overset{h_2-h_1}{P}}\left[\frac{e^{(nT+h_1)s}\underset{nT+h_1}{\overset{\infty}{P}}\{R(s)\}-\dfrac{\overset{(1)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(1)}{A}_{0}(nT)}{D(s)}}{1+K_1G(s)}\right]K_1G(s)\right\}
\end{aligned}\right\}$$

$$\overset{nT+h_{k+1}}{\underset{nT+h_k}{C}}(s) = e^{-(nT+h_k)s}\left\{\frac{\overset{(k)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(k)}{A}_0(nT)}{D(s)}\right.$$

$$\left.+\overset{h_{k+1}-h_k}{\underset{0}{P}}\left[\frac{e^{(nT+h_k)s}\overset{\infty}{\underset{nT+h_k}{P}}\left\{R(s)\right\}-\dfrac{\overset{(k)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(k)}{A}_0(nT)}{D(s)}}{1+K_kG(s)}\right]K_kG(s)\right\}$$

$$\overset{(n+1)T}{\underset{nT+h_{m-1}}{C}}(s) = e^{-(nT+h_{m-1})s}\left\{\frac{\overset{(m-1)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(m-1)}{A}_0(nT)}{D(s)}\right.$$

$$\left.+\overset{T-h_{m-1}}{\underset{0}{P}}\left[\frac{e^{(nT+h_{m-1})s}\overset{\infty}{\underset{nT+h_{m-1}}{P}}\left\{R(s)\right\}-\dfrac{\overset{(m-1)}{A}_{q-1}(nT)s^{q-1}+\cdots+\overset{(m-1)}{A}_0(nT)}{D(s)}}{1+K_{m-1}G(s)}\right]K_{m-1}G(s)\right\}$$

$$\overset{(n+1)T+h_1}{\underset{(n+1)T}{C}}(s) = e^{-(n+1)Ts}\left\{\frac{\overset{(0)}{A}_{q-1}(n+1)Ts^{q-1}+\cdots+\overset{(0)}{A}_0(n+1)T}{D(s)}\right.$$

$$\left.+\overset{h_1}{\underset{0}{P}}\left[\frac{e^{(n+1)Ts}\overset{\infty}{\underset{(n+1)T}{P}}\left\{R(s)\right\}-\dfrac{\overset{(0)}{A}_{q-1}(n+1)Ts^{q-1}+\cdots+\overset{(0)}{A}_0(n+1)T}{D(s)}}{1+K_0G(s)}\right]K_0G(s)\right\} \tag{6}$$

Then, in view of Fig. (5), Fig. (6) and equations (2) (6), the next relation is derived,

$$\overset{\infty}{\underset{nT+h_1}{P}}\left[\overset{nT+h_1}{\underset{nT}{C}}(s)\right] = e^{(-nT+h_1)s}\frac{\overset{(1)}{A}_{q-1}(nT)s^{q-1}+\overset{(1)}{A}_{q-2}(nT)s^{q-2}+\cdots+\overset{(1)}{A}_0(nT)}{D(s)}$$

$$\overset{\infty}{\underset{nT+h_l}{P}}\left[\overset{nT+h_k}{\underset{nT+h_{k-1}}{C}}(s)\right] = e^{-(nT+h_k)s}\frac{\overset{(k)}{A}_{q-1}(nT)s^{q-1}+\overset{(k)}{A}_{q-2}(nT)s^{q-2}+\cdots+\overset{(k)}{A}_0(nT)}{D(s)}$$

$$\overset{\infty}{\underset{(n+1)T}{P}}\left[\overset{(n+1)T}{\underset{nT+h_{m-1}}{C}}(s)\right] = e^{-(n+1)Ts}\frac{\overset{(0)}{A}_{q-1}(n+1)Ts^{q-1}+\overset{(0)}{A}_{q-2}(n+1)Ts^{q-2}+\cdots+\overset{(0)}{A}(n+1)T}{D(s)} \tag{7}$$

In both sides of equation (7), if the coefficients of same order terms for S is put equally, the next relation is derived,

$$
\left.\begin{aligned}
\overset{(1)}{A}_{q-1}(nT) &= \overset{(1)}{K}{}^{q-1}_{q-1}\overset{(0)}{A}_{q-1}(nT)+\overset{(1)}{K}{}^{q-1}_{q-2}\overset{(0)}{A}_{q-2}(nT)+\cdots\\
&\quad+\overset{(1)}{K}{}^{q-1}_{0}\overset{(0)}{A}_{0}(nT)+\overset{(1)}{K}{}^{q-1}_{R1}\overset{(0)}{R}_{1}(nT)+\overset{(1)}{K}{}^{q-1}_{R2}\overset{(0)}{R}_{2}(nT)+\cdots\\
&\ \ \vdots\\
\overset{(1)}{A}_{0}(nT) &= \overset{(1)}{K}{}^{0}_{q-1}\overset{(0)}{A}_{q-1}(nT)+\overset{(1)}{K}{}^{0}_{q-2}\overset{(0)}{A}_{q-2}(nT)+\cdots\\
&\quad+\overset{(1)}{K}{}^{0}_{0}\overset{(0)}{A}_{0}(nT)+\overset{(1)}{K}{}^{0}_{R1}\overset{(0)}{R}_{1}(nT)+\overset{(1)}{K}{}^{0}_{R2}\overset{(0)}{R}_{2}(nT)+\cdots\\
&\ \ \vdots\\
\overset{(k)}{A}_{q-1}(nT) &= \overset{(k)}{K}{}^{q-1}_{q-1}\overset{(k-1)}{A}_{q-1}(nT)+\overset{(k)}{K}{}^{q-1}_{q-2}\overset{(k-1)}{A}_{q-2}(nT)+\cdots\\
&\quad+\overset{(k)}{K}{}^{q-1}_{0}\overset{(k-1)}{A}_{0}(nT)+\overset{(k)}{K}{}^{q-1}_{R1}\overset{(k-1)}{R}_{1}(nT)+\overset{(k)}{K}{}^{q-1}_{R2}\overset{(k-1)}{R}_{2}(nT)+\cdots\\
&\ \ \vdots\\
\overset{(k)}{A}_{0}(nT) &= \overset{(k)}{K}{}^{0}_{q-1}\overset{(k-1)}{A}_{q-1}(nT)+\overset{(k)}{K}{}^{0}_{q-2}\overset{(k-1)}{A}_{q-2}(nT)+\cdots\\
&\quad+\overset{(k)}{K}{}^{0}_{0}\overset{(k-1)}{A}_{0}(nT)+\overset{(k)}{K}{}^{0}_{R1}\overset{(k-1)}{R}_{1}(nT)+\overset{(k)}{K}{}^{0}_{R2}\overset{(k-1)}{R}_{2}(nT)+\cdots\\
&\ \ \vdots\\
\overset{(0)}{A}_{q-1}(n+1)T &= \overset{(0)}{K}{}^{q-1}_{q-1}\overset{(m-)}{A}_{q-1}(nT)+\overset{(0)}{K}{}^{q-1}_{q-2}\overset{(m-1)}{A}_{q-2}(nT)+\cdots\\
&\quad+\overset{(0)}{K}{}^{q-1}_{0}\overset{(m-1)}{A}_{0}(nT)+\overset{(0)}{K}{}^{q-1}_{R1}\overset{(m-1)}{R}_{1}(nT)+\overset{(0)}{K}{}^{q-1}_{R2}\overset{(m-1)}{R}_{2}(nT)+\cdots\\
&\ \ \vdots\\
\overset{(0)}{A}_{0}(n+1)T &= \overset{(0)}{K}{}^{0}_{q-1}\overset{(m-1)}{A}_{q-1}(nT)+\overset{(0)}{K}{}^{0}_{q-2}\overset{(m-1)}{A}_{q-2}(nT)+\cdots\\
&\quad+\overset{(0)}{K}{}^{0}_{0}\overset{(m-1)}{A}_{0}(nT)+\overset{(0)}{K}{}^{0}_{R1}\overset{(m-1)}{R}_{1}(nT)+\overset{(0)}{K}{}^{0}_{R2}\overset{(m-1)}{R}_{2}(nT)+\cdots
\end{aligned}\right\}\quad(8)
$$

Where $\overset{(\gamma)}{K}{}^{q-\beta}_{q-\alpha}$ are constants dependent only on $K(t)$ and $G(s)$. The constants $\overset{(\gamma)}{K}{}^{q-\beta}_{R\alpha}$, the functions $\overset{(\beta)}{R}_{\alpha}(nT)$, and the number of them is dependent on type and order of the input function.

From equation (8), the next relation is derived,

$$
\left.\begin{aligned}
\overset{(0)}{A}_{q-1}(n+1)T &= K^{q-1}_{q-1}\overset{(0)}{A}_{q-1}(nT)+K^{q-1}_{q-2}\overset{(0)}{A}_{q-2}(nT)+\cdots\\
&\quad+K^{q-1}_{0}\overset{(0)}{A}_{0}(nT)+K^{q-1}_{R1}R_{1}(nT)+K^{q-1}_{R2}R_{2}(nT)+\cdots\\
\overset{(0)}{A}_{q-2}(n+1)T &= K^{q-2}_{q-1}\overset{(0)}{A}_{q-1}(nT)+K^{q-2}_{q-2}\overset{(0)}{A}_{q-2}(nT)+\cdots\\
&\ \ \vdots\\
&\quad+K^{q-2}_{0}\overset{(0)}{A}_{0}(nT)+K^{q-2}_{R1}R_{1}(nT)+K^{q-2}_{R2}R_{2}(nT)+\cdots\\
\overset{(0)}{A}_{0}(n+1)T &= K^{0}_{q-1}\overset{(0)}{A}_{q-1}(nT)+K^{0}_{q-2}\overset{(0)}{A}_{q-2}(nT)+\cdots\\
&\quad+K^{0}_{0}\overset{(0)}{A}_{0}(nT)+K^{0}_{R1}R_{1}(nT)+K^{0}_{R2}R_{2}(nT)+\cdots
\end{aligned}\right\}\quad(9)
$$

Where $K^{q-\beta}_{q-\alpha}$ are constants dependent only on $K(t)$ and $G(s)$. The constants $K_{R\alpha}^{q-\beta}$, the functions $R_\alpha(nT)$, and the number of them is dependent on type and order of input function.

Equation (9) is solved easily using the Z transform method. With zero initial conditions all the terms $\overset{(0)}{A}_{q-1}(0)$, $\overset{(0)}{A}_{q-2}(0)$, $\cdots$, $\overset{(0)}{A}_0(0)$ are zero. Therefore, taking the Z transform of equation (9), the next relation is derived,

$$\left.\begin{aligned}
&(Z-K_{q-1}^{q-1})\overset{(0)}{A}{}^*_{q-1}(Z)-K_{q-2}^{q-1}\overset{(0)}{A}{}^*_{q-2}(Z)-\cdots-K_0^{q-1}\overset{(0)}{A}{}^*_0(Z)\\
&\qquad\qquad = K_{R1}^{q-1}R_1^*(Z)+K_{R2}^{q-1}R_2^*(Z)+\cdots\\
&-K_{q-1}^{q-2}\overset{(0)}{A}{}^*_{q-1}(Z)+(Z-K_{q-2}^{q-2})\overset{(0)}{A}{}^*_{q-2}(Z)-\cdots-K_0^{q-2}\overset{(0)}{A}{}^*_0(Z)\\
&\qquad\qquad = K_{R1}^{q-2}R_1^*(Z)+K_{R2}^{q-2}R_2^*(Z)+\cdots\\
&\qquad\vdots\\
&-K_{q-1}^{0}\overset{(0)}{A}{}^*_{q-1}(Z)-K_{q-2}^{0}\overset{(0)}{A}{}^*_{q-2}(Z)-\cdots+(Z-K_0^0)\overset{(0)}{A}{}^*_0(Z)\\
&\qquad\qquad = K_{R1}^{0}R_1^*(Z)+K_{R2}^{0}R_2^*(Z)+\cdots
\end{aligned}\right\} \tag{10}$$

From equation (10), for example, $\overset{(0)}{A}{}^*_{q-1}(Z)$ is given as follows ;

$$\overset{(0)}{A}{}^*_{q-1}(Z)=\frac{\begin{vmatrix} K_{R1}^{q-1}R_1^*(Z)+K_{R2}^{q-1}R_2^*(Z)+\cdots, & -K_{q-2}^{q-1}, & \cdots, & -K_0^{q-1}\\ K_{R1}^{q-2}R_1^*(Z)+K_{R2}^{q-2}R_2^*(Z)+\cdots, & (Z-K_{q-2}^{q-2}), & \cdots, & -K_0^{q-2}\\ \vdots & & & \\ K_{R1}^{0}R_1^*(Z)+K_{R2}^{0}R_2^*(Z)+\cdots, & -K_{q-2}^{0}, & \cdots, & (Z-K_0^0)\end{vmatrix}}{\Delta(Z)} \tag{11}$$

Where,

$$\Delta(Z)=\begin{vmatrix} (Z-K_{q-1}^{q-1}), & -K_{q-2}^{q-1}, & \cdots, & -K_0^{q-1}\\ -K_{q-1}^{q-2}, & (Z-K_{q-2}^{q-2}), & \cdots, & -K_0^{q-2}\\ \vdots & & & \\ -K_{q-1}^{0}, & -K_{q-2}^{0}, & \cdots, & (Z-K_0^0)\end{vmatrix} \tag{12}$$

In the same manner, $\overset{(0)}{A}{}^*_{q-2}(Z)$, $\overset{(0)}{A}{}^*_{q-3}(Z)$, $\cdots$, $\overset{(0)}{A}{}^*_0(Z)$ are derived from equation (10).

Therefore $\overset{(1)}{A}{}^*_{q-1}(Z)$, $\cdots$, $\overset{(1)}{A}{}^*_0(Z)$, $\overset{(k)}{A}{}^*_{q-1}(Z)$, $\cdots$, $\overset{(k)}{A}{}^*_0(Z)$, $\cdots$, $\overset{(m-1)}{A}{}^*_{q-1}(Z)$, $\cdots$, $\overset{(m-1)}{A}{}^*_0(Z)$, are given by the Z transform of equation (8) and $\overset{(0)}{A}{}^*_{q-1}(Z)$, $\overset{(0)}{A}{}^*_{q-2}(Z)$, $\cdots$, $\overset{(0)}{A}{}^*_0(Z)$. Hence, by the inverse Z transform of these functions and the inverse Laplace transform of equation (6), the system out put at arbitrary instants is given.

In the next place, let us consider about the stability of system and the transfer function. Using initial value theorem in equation (6), the system response at instants $nT, \cdots, nT+h_k, \cdots, nT+h_{m-1}$, $C(nT), \cdots, C(nT+h_k), \cdots, C(nT+h_{m-1})$ are derived as follows ;

$$\left.\begin{aligned}
C(nT) &= \lim_{s\to\infty} se^{nTs} \overset{nT+h_1}{\underset{nT}{C}}(s) \\
&= \overset{(0)}{A}_{q-1}(nT) + \left\{ r(nT) - \overset{(0)}{A}_{q-1}(nT) \right\} \frac{K_0 b_q}{1+K_0 b_q} \\
C(nT+h_k) &= \lim_{s\to\infty} se^{(nT+h_k)s} \overset{nT+h_{k+1}}{\underset{nT+h_k}{C}}(s) \\
&= \overset{(k)}{A}_{q-1}(nT) + \left\{ r(nT+h_k) - \overset{(k)}{A}_{q-1}(nT) \right\} \frac{K_k b_q}{1+K_k b_q} \\
C(nT+h_{m-1}) &= \lim_{s\to\infty} se^{(nT+h_{m-1})s} \overset{(n+1)T}{\underset{nT+h_{m-1}}{C}}(s) \\
&= \overset{(m-1)}{A}_{q-1}(nT) + \left\{ r(nT+h_{m-1}) - \overset{(m-1)}{A}_{q-1}(nT) \right\} \frac{K_{m-1} b_q}{1+K_{m-1} b_q}
\end{aligned}\right\} \quad (13)$$

Where $r(nT), \cdots, r(nT+h_k), \cdots, r(nT+h_{m-1})$ are inputs of the system at instants $nT, \cdots, nT+h_k, \cdots, nT+h_{m-1}$.

Therefore it will be understood that for the stable system $\overset{(0)}{A}_{q-1}(nT)$, $\overset{(1)}{A}_{q-1}(nT), \cdots, \overset{(m-1)}{A}_{q-1}(nT)$ must be bounded at $n\to\infty$. Hence, in view of equations (11), (12) and (8), it will be understood that for the stable system the zeros of $\Delta(Z)$ must be within the unit circle.

Taking the Z transform of equation (13), the next relation is derived,

$$\left.\begin{aligned}
C_0^*(Z) &= \frac{1}{1+K_0 b_q} \left\{ \overset{(0)}{A}{}^*_{q-1}(Z) + K_0 b_q \overset{(0)}{R}{}^*(Z) \right\} \\
C_k^*(Z) &= \frac{1}{1+K_k b_q} \left\{ \overset{(k)}{A}{}^*_{q-1}(Z) + K_k b_q \overset{(k)}{R}{}^*(Z) \right\} \\
C_{m-1}^*(Z) &= \frac{1}{1+K_{m-1} b_q} \left\{ \overset{(m-1)}{A}{}^*_{q-1}(Z) + K_{m-1} b_q \overset{(m-1)}{R}{}^*(Z) \right\}
\end{aligned}\right\} \quad (14)$$

where $C_0^*(Z), \cdots, C_k^*(Z), \cdots, C_{m-1}^*(Z)$ are the Z transform of $C(nT), \cdots, C(nT+h_k), \cdots, C(nT+h_{m-1})$. Also $\overset{(0)}{R}{}^*(Z), \cdots, \overset{(k)}{R}{}^*(Z), \cdots, \overset{(m-1)}{R}{}^*(Z)$ are the Z transform of $r(nT), \cdots, r(nT+h_k), \cdots, r(nT+h_{m-1})$. Therefore $\overset{(0)}{R}{}^*(Z)$ is identical with $R^*(Z)$, that is, the Z transform of $R(s)$, and $\overset{(1)}{R}{}^*(Z), \cdots, \overset{(m-1)}{R}{}^*(Z)$ are given by the advanced Z transform of $R(s)$.

Therefore the approximate transfer function of this system is derived by applying $C_0^*(Z), \cdots, C_{m-1}^*(Z)$ to an adequate hold circuit and by making up the ratio of the out put from hold circuit to the input of the system.

3. Example

Let us consider the analysis of typical chopper circuit shown in Fig. (7 a) using this new method. In Fig. (7 a), $R_2(t)$ is a resistance varying periodically. Fig. (7 a) may be rewritten as shown in Fig. (7 b) using block diagram method, and may be rearranged as shown in Fig. (7 c).

Now, let us consider the variation of $R_2(t)$ as follows;

$$\left.\begin{aligned} R_2(t) &= \infty & nT \leq t < nT + h_1 \\ &= \delta R_1 & nT + h_1 \leq t < (n+1)T \end{aligned}\right\} \tag{15}$$

If the input to this circuit is a step voltage with amplitude U, each symbol at equation (6) is as follows;

$$\left.\begin{aligned} R(s) &= E_1(s) = \frac{U}{S} \\ m &= 2 \\ K_0 &= 1 \\ K_1 &= \frac{\delta}{1+\delta} \\ G(s) &= \frac{CR_1S}{1+CR_3S} = \frac{\dfrac{R_1}{R_3}S}{S+\dfrac{1}{CR_3}} \\ D(s) &= S + \frac{1}{CR_3} \\ q &= 1 \\ b_1 &= \frac{R_1}{R_3} \end{aligned}\right\} \tag{16}$$

Therefore the equation (6) becomes as shown in equation (17) and (18).

$$\overset{nT+h_1}{\underset{nT}{C}}(s) = e^{-nTs}\left\{\frac{\overset{(0)}{A_0}(nT)}{S+\dfrac{1}{CR_3}} + \overset{h_1}{\underset{0}{P}}\left[\frac{\left(\dfrac{U}{S} - \dfrac{\overset{(0)}{A_0}(nT)}{S+\dfrac{1}{CR_3}}\right)}{1+\dfrac{\dfrac{R_1}{R_3}S}{S+\dfrac{1}{CR_3}}}\right]\frac{\dfrac{R_1}{R_3}S}{S+\dfrac{1}{CR_3}}\right\}$$

$$= e^{-nTs}\left\{\frac{\overset{(0)}{A_0}(nT)}{S+\dfrac{1}{CR_3}}+\left(\frac{1-e^{-h_1 s}}{S}U-\frac{R_3\overset{(0)}{A_0}(nT)+R_1U}{R_1+R_3}\right.\right.$$

$$\left.\left.\times\frac{1-e^{-h_1 s}e^{\frac{-h_1}{(R_1+R_3)C}}}{S+\dfrac{1}{(R_1+R_3)C}}\right)\frac{\dfrac{R_1}{R_3}S}{S+\dfrac{1}{CR_3}}\right\} \tag{17}$$

$$\overset{(n+1)T}{\underset{nT+h_1}{C}}(s)=e^{-(nT+h_1)s}\left\{\frac{\overset{(1)}{A_0}(nT)}{S+\dfrac{1}{CR_3}}+\overset{T-h_1}{\underset{0}{P}}\left(\frac{\dfrac{U}{S}-\dfrac{\overset{(1)}{A_0}(nT)}{S+\dfrac{1}{CR_3}}}{1+\dfrac{\delta}{1+\delta}\dfrac{\dfrac{R_1}{R_3}S}{S+\dfrac{1}{CR_3}}}\right)\frac{\delta}{1+\delta}\frac{\dfrac{R_1}{R_3}S}{S+\dfrac{1}{CR_3}}\right\}$$

$$=e^{-(nT+h_1)s}\left\{\frac{\overset{(1)}{A_0}(nT)}{S+\dfrac{1}{CR_3}}+\left(\frac{1-e^{-(T-h_1)s}}{S}U-\frac{R_3\overset{(1)}{A_0}(nT)+\dfrac{\delta}{1+\delta}RU}{\dfrac{\delta}{1+\delta}R_1+R_3}\right.\right.$$

$$\left.\left.\times\frac{1-e^{-(T-h_1)s}e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}}{S+\dfrac{1}{\left(\dfrac{\delta}{1+\delta}R_1+R_3\right)C}}\right)\frac{\delta}{1+\delta}\frac{\dfrac{R_1}{R_3}S}{S+\dfrac{1}{CR_3}}\right\} \tag{18}$$

Hence, taking P transform of equation (17) and (18).

$$\underset{nT+h_1}{\overset{\infty}{P}}\left[\overset{nT+h_1}{\underset{nT}{C}}(s)\right]=e^{-(nT+h_1)s}\frac{e^{-\frac{h_1}{(R_1+R_3)C}}\overset{(0)}{A_0}(nT)-\dfrac{R_1}{R_3}\left\{e^{-\frac{h_1}{(R_1+R_3)C}}-1\right\}U}{S+\dfrac{1}{CR_3}} \tag{19}$$

$$\underset{(n+1)T}{\overset{\infty}{P}}\left[\overset{(n+1)T}{\underset{nT+h_1}{C}}(s)\right]$$

$$=e^{-(n+1)Ts}\frac{e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}\overset{(1)}{A_0}(nT)-\dfrac{R_1}{R_3}\dfrac{\delta}{1+\delta}\left\{e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}-1\right\}U}{S+\dfrac{1}{CR_3}} \tag{20}$$

Then, referring to equation (7) and (8), the next relations are derived,

$$\overset{(1)}{A}_0(nT) = e^{-\frac{h_1}{(R_1+R_3)C}} \overset{(0)}{A}_0(nT) + \frac{R_1}{R_3}\left\{e^{-\frac{h_1}{(R_1+R_3)C}} - 1\right\} U \tag{21}$$

$$\overset{(0)}{A}_0(n+1)T = e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}} \overset{(1)}{A}_0(nT) + \frac{R_1}{R_3}\frac{\delta}{1+\delta}\left\{e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}} - 1\right\} U \tag{22}$$

On the assumption with zero initial condition, taking Z transform of equation (21) and (22), $\overset{(0)}{A}{}_0^*(Z)$ and $\overset{(1)}{A}{}_0^*(Z)$ are derived as follows ;

$$\overset{(0)}{A}{}_0^*(Z) = \frac{K_R^0}{Z-K_0^0}\frac{Z}{Z-1} U \tag{23}$$

$$\overset{(1)}{A}{}_0^*(Z) = \frac{L_1 Z + L_0}{Z-K_0^0}\frac{Z}{Z-1} U \tag{24}$$

where,

$$\left.\begin{aligned} K_0^0 &= e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}} e^{-\frac{h_1}{(R_1+R_3)C}} \\ K_R^0 &= \frac{R_1}{R_3}\left[e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}\left\{e^{-\frac{h_1}{(R_1+R_3)C}} - 1\right\} \right. \\ &\quad \left. + \frac{\delta}{1+\delta}\left\{e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}} - 1\right\}\right] \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} L_1 &= \frac{R_1}{R_3}\left\{e^{-\frac{h_1}{(R_1+R_3)C}} - 1\right\} \\ L_0 &= \frac{\delta}{1+\delta}\frac{R_1}{R_3} e^{-\frac{h_1}{(R_1+R_3)C}}\left\{e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}} - 1\right\} \end{aligned}\right\} \tag{26}$$

Referring to equation (14), $C_0^*(Z)$ and $C_1^*(Z)$ are derived as follows ;

$$\left.\begin{aligned} C_1^*(Z) &= \left(\frac{R_3}{R_1'+R_3}\frac{K_R^0}{Z-K_0^0} + \frac{R_1}{R_1+R_3}\right)\frac{Z}{Z-1} U \\ C_1^*(Z) &= \left(\frac{R}{\frac{\delta}{1+\delta}R_1+R_3}\frac{L_1 Z + L_0}{Z-K_0^0} + \frac{\frac{\delta}{1+\delta}R_1}{\frac{\delta}{1+\delta}R_1+R_3}\right)\frac{Z}{Z-1} U \end{aligned}\right\} \tag{27}$$

Therefore, referring to Fig. (7 c), the out put of this circuit at instants nT and $nT+h_1$, $e_2(nT)$ and $e_2(nT+h_1)$ are given by the next equation,

$$\left.\begin{aligned} E_{20}^{*}(Z) &= \frac{R_3}{R_1} C_0^{*}(Z) \\ E_{21}^{*}(Z) &= \frac{R_3}{R_1} C_1^{*}(Z) \end{aligned}\right\} \qquad (28)$$

where $E_{20}^{*}(Z)$, $E_{21}^{*}(Z)$ are the Z transform of $e_2(nT)$, $e_2(nT+h_1)$.

Therefore the approximate transfer fundction of this circuit is derived by holding (0 order) $E_{20}^{*}(Z)$, $E_{21}^{*}(Z)$, and by making up the ratio of this to U/S.

$$G_0(s) = \frac{R_3}{R_1+R_3} \frac{1+\left[\dfrac{\delta}{1+\delta}\left\{e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}-1\right\}-e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}\right]e^{-sT}}{1-\left[e^{-\frac{h_1}{(R_1+R_3)C}}\,e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}\right]e^{-sT}} \qquad (29)$$

$$G_1(s) = \frac{R_3}{\dfrac{\delta}{1+\delta}R_1+R_3} \frac{\left\{e^{-\frac{h_1}{(R_1+R_3)C}}-\dfrac{1}{1+\delta}\right\}-\left\{\dfrac{\delta}{1+\delta}\,e^{-\frac{T-h_1}{(R_1+R_3)C}}\right\}e^{-sT}}{1-\left[e^{-\frac{h_1}{(R_1+R_3)C}}\,e^{-\frac{T-h_1}{\left(\frac{\delta}{1+\delta}R_1+R_3\right)C}}\right]e^{-sT}} \qquad (30)$$

In this case, referring to (23), ΔZ is given as shown in the next equation,

$$\Delta(Z) = Z - K_0^0 \qquad (31)$$

Referring to equation (25), it will be understood that K_0^0 is always a positive number smaller than 1. Therefore this circuit must be stable.

4. Conclusions

In this paper, a general method of analysis for feed-back systems containing a transfer element which has a gain constant varying periodically and in fre e shape is derived.

This method is based on the P transform and the superposition principle. But it is remarkable that the generalization was obtained by the approximate expression of time varying gain $K(t)$. Especially, this method is very useful for the derivation of system transfer function and the determination of system stability.

In this paper, as an example using this method, only the analysis of a chopper circuit was described. But the author believes that this method is powerful when applied to the analysis of many other parametric systems.

Reference

1) E. I. Jury: Sampled Data Control Systems, 1958.