Microwave Amplification by Interaction between Electron Beam and Plasma

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Contents

Abstract

When an electron beam modulated by a microwave signal passes through plasma in which the plasma frequency is close to the frequency of this signal, then the electrons agitated thermally in the plasma oscillate with this frequency, and consequently the electron space-charge wave grows in the plasma. When this wave and the slow- and fast space-charge waves in the electron beam are interacted mutually, it may be expected that the microwave signal in the electron beam is amplified.

The authors analyzed this problem theoretically. First, the propagation constants and the characteristic impedance of the electron space-charge wave in the plasma were analyzed. Next it was confirmed that the space-charge wave in the electron beam can be amplified by mutual coupling with the electron space-charge wave in the plasma. Finally the amplification action by three modes coupling between the electron space-charge wave in the plasma and the slow- and fast electron space-charge waves in the electron beam was confirmed. As a result of a numerical example, the gain of this amplifier is considerably high (approximately 38 dB/cm), although its frequency band-width is very narrow (less than 60 kc/s).

I. Introduction

If the plasma and electron beam are modulated by a microwave signal at

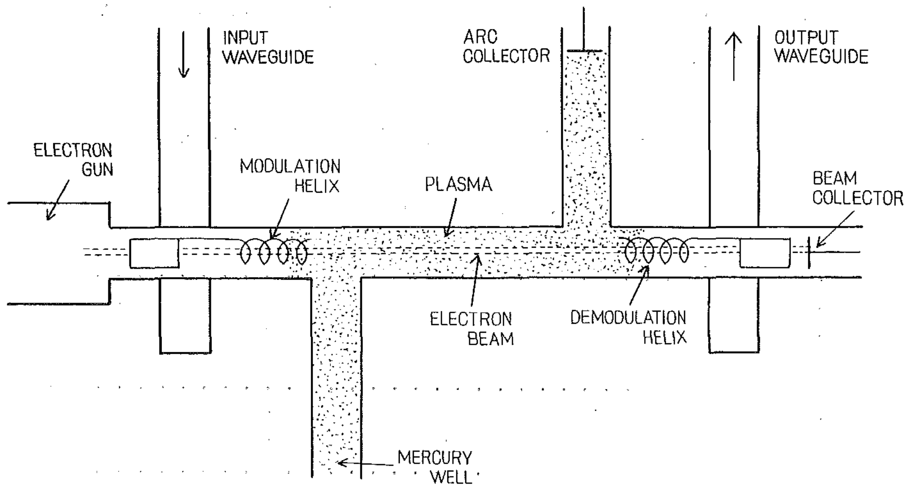

Fig. 1. Schematic drawing of plasma-beam tube

the input and the electron beam passes through the plasma as shown in Fig. 1, then the microwave signal in the electron beam is excited and amplified as a result of the interaction between the plasma and the electron beam. This fact has been analyzed by Bohm and Gross[1], and others, and has been confirmed experimentally by Gould and others[2].

The authors analyzed this problem by means of mode coupling of the electron space-charge wave in the plasma and the space-charge waves in the electron beam. From the results of this analysis, the physical meaning of the amplification became still more clear, and some interesting conclusions were newly obtained.

II. The Character of the Electron Space-Charge Wave in the Plasma

It is assumed that the ions are at rest and the electrons are agitated thermally in the plasma. When this plasma is excited by the microwave signal of O-mode at one end, the components of the a–c electron charge density and the a–c electron velocity with the same frequency as this signal are originated and propergate in the plasma.

Now $f_0\ (v_0)$ and $f\ (v_0)$ respectively being the d–c and a–c distribution function of the electron charge density in which the velocity interval is $v_0-(v_0+dv_0)$ per unit plasma cross section, then the d–c component of the electron charge density, ρ_0, is

$$\rho_0 = -e\int_{-\infty}^{+\infty} f_0(v_0)\ dv_0 \tag{1}$$

and its a-c component ρ is

$$\rho = -e\int_{-\infty}^{+\infty} f(v_0)\ dv_0 \tag{2}$$

the a-c component of the current density, i, is shown as follows

$$i = -e\int_{-\infty}^{+\infty}\left\{v_0 f(v_0) + v f_0(v_0)\right\} dv_0 \tag{3}$$

the equation of the current continuity is shown as follows

$$j\omega\rho - \Gamma i = 0 \tag{4}$$

where

$$\frac{\partial}{\partial z} = -\Gamma$$

From Eqs. (2), (3) and (4), the following equation can be obtained:

$$-e\int_{-\infty}^{+\infty}\left[j\omega f(v_0) - \Gamma\left\{v_0 f(v_0) + v f_0(v_0)\right\}\right] dv_0 = 0$$

$$\therefore\quad (j\omega - \Gamma v_0) f(v_0) = \Gamma v f_0(v) \tag{5}$$

Next from the Poisson's equation, the following equation can be introduced:

$$-\Gamma E_z = \rho/\varepsilon \tag{6}$$

and from the equation of motion, the following equation can be introduced:

$$j\omega v - \Gamma v_0 v = -\frac{e}{m} E_z \tag{7}$$

By eliminating v from Eqs. (5) and (7), then

$$f(v_0) = \frac{\Gamma f_0(v_0)}{(j\omega - \Gamma v_0)^2}\left(-\frac{e}{m}\right) E_z \tag{8}$$

From Eqs. (2), (6) and (8), the next equation can be obtained:

$$E_z = -\frac{(-e)}{\Gamma\varepsilon}\int_{-\infty}^{+\infty} f(v_0)\ dv_0 = -\frac{(-e)}{\Gamma\varepsilon}\left(-\frac{e}{m}\right) E_z \Gamma \int_{-\infty}^{+\infty}\frac{f_0(v_0) dv_0}{(j\omega - \Gamma v_0)}$$

$$\therefore\quad 1 = \frac{e}{m\varepsilon}(-e)\int_{-\infty}^{+\infty}\frac{f_0(v_0) dv_0}{(j\omega - \Gamma v_0)^2} \tag{9}$$

If the condition, $\Gamma v_0/\omega < 1$ is true, the integrand of Eq. (9) can be expanded in infinite series. Usually $\Gamma v_0/\omega$ is much smaller than one, so that this condition is substantiated virtually. Then

$$\int_{-\infty}^{+\infty}\frac{f_0(v_0)dv_0}{(j\omega-\Gamma v_0)^2}=-\frac{1}{\omega^2}\int_{-\infty}^{+\infty}\frac{f_0(v_0)dv_0}{\left(1+\frac{j\Gamma v_0}{\omega}\right)^2}$$

$$=-\frac{1}{\omega^2}\int_{-\infty}^{+\infty}\left(1-j\frac{2\Gamma v_0}{\omega}-\frac{3\Gamma^2 v_0^2}{\omega^2}-j\frac{4\Gamma^3 v_0^3}{\omega^3}-\cdots\right)f_0(v_0)\,dv_0$$

All terms of odd power in this series become zero after the integration, so that

$$\int_{-\infty}^{+\infty}\frac{f_0(v_0)dv_0}{(j\omega-\Gamma v_0)^2}=-\frac{1}{\omega^2}\int_{-\infty}^{+\infty}\left(1-\frac{3\Gamma^2 v_0^2}{\omega^2}-\cdots\right)f_0(v_0)\,dv_0$$

Therefore, by considering the condition $|\Gamma v_0/\omega|\ll 1$, Eq. (9) becomes as follows

$$1=\frac{\omega_p^2}{\omega^2}\left(1-\frac{3\Gamma^2\overline{v_0^2}}{\omega^2}\right)$$

where $-e\rho_0/m\varepsilon=\omega_p^2$, and $\overline{v_0^2}$ is the mean square thermal velocity. Finally from this equation, Γ can be obtained as follows

$$\Gamma=\pm\frac{\omega}{\bar{v}_0}\sqrt{1-\frac{\omega^2}{\omega_p^2}}=\pm j\beta \tag{10}$$

where $\bar{v}_0=\sqrt{3\overline{v^2}}$

It can be concluded from Eq. (10) that the electron space-charge wave can propergate in the plasma in the case of $\omega>\omega_p$ because Γ is imaginary quantity, but it can not propergate in the case of $\omega<\omega_p$ because Γ is real quantity. This conclusion shows that the plasma acts as a kind of high pass filter whose cut-off angular frequency is ω_p.

The phase velocity v_1 of this wave is shown as follows

$$v_1=\frac{\bar{v}_0}{\sqrt{\frac{\omega^2}{\omega_p^2}-1}} \tag{11}$$

so that varying ω_p by varying the electron charge density, the phase velocity can be controled freely and synchronized with that of the space-charge wave in the electron beam.

Now the kinetic voltage $U=-(m/e)v_0v$ can be obtained from Eqs. (6) and (7) as follows

$$U=-\frac{m}{e}v_0v=\left(-\frac{m}{e}\right)v_0\frac{-e/m}{j\omega-\Gamma v_0}E_z=\frac{v_0}{j\omega-\Gamma v_0}\frac{-\rho}{\Gamma\varepsilon}$$

$$=\frac{-v_0}{(j\omega-\Gamma v_0)}\frac{\Gamma i}{j\omega}=\frac{jv_0}{(j\omega-\Gamma v_0)\omega\varepsilon}i \tag{12}$$

Moreover by using Eq. (5),

$$U=-\frac{m}{e}v_0v=-\frac{mv_0}{e}\frac{(j\omega-\Gamma v_0)f(v_0)}{\Gamma f_0(v_0)}$$
$$=\frac{-mv_0}{e}\frac{(j\omega-\Gamma v_0)}{\Gamma}\frac{f(v_0)}{f_0(v_0)} \tag{13}$$

As the signs of ρ and ρ_0 are the opposite of each other, so $f(v_0)/f_0(v_0)<0$. Now if it can be assumed that the variation of $f(v_0)$ is similar to that of $f_0(v_0)$ in respect of v_0 (i.e. $f(v_0)/f_0(v_0)$ is constant with respect to v_0), then it can be shown from Eqs. (1) and (2) that $f(v_0)/f_0(v_0)=-\rho/\rho_0$. Using this relationship, Eq. (13) becomes as follows

$$U=-\frac{v_0(j\omega-\Gamma v_0)E_z}{e\rho_0/m\varepsilon}=\frac{-jv_0(j\omega-\Gamma v_0)}{\omega_p^2\omega\varepsilon}i \tag{14}$$

From Eqs. (12) and (14), the following equation ban be obtained:

$$U=\frac{v_0}{\omega_p\omega\varepsilon}i \tag{15}$$

And the characteristic impedance Z_0 is

$$Z_0=\frac{U}{i}=\frac{v_0}{\omega_p\omega\varepsilon} \tag{16}$$

Therefore it can be seen from this equation that countless numbers of Z_0 exist corresponding to countless numbers of v_0. It is convenient, for avoiding the complexity, to define one effective value v_{0e} instead of considering all of v_0. Then only one effective impedance $\bar{Z}_0$ can be defined corresponding to v_{0e} as follows

$$\bar{Z}_0=\frac{v_{0e}}{\omega_p\omega\varepsilon} \tag{17}$$

III. The Mutual Interaction between the Electron Space-Charge Wave in the Plasma and the Slow Space-Charge Wave in the Electron Beam

When the electron beam modulated by a microwave signal passes through the plasma, two electron space-charge waves (one is the forward wave and the other the backward wave) grow in the plasma, and two electron space-charge waves (one is the slow wave and the other the fast wave) grow in the electron beam. Now in this section it is, for simplicity, assumed that the coupling exists between only two waves, i.e. the forward wave in the plasma and the slow wave in the electron beam. Then the coupling equations for these two waves can be written as follows.

$$\left.\begin{aligned} \frac{da_1}{dz} &= -j\beta_1 a_1 + C_{12} a_2 \\ \frac{da_2}{dz} &= C_{21} a_2 - j\beta_2 a_2 \end{aligned}\right\} \tag{18}$$

where all quantity for the forward wave are shown by subscript 1, and those for the slow wave by subscript 2. In Eq. (18), a_1 and a_2 are respectively normalized amplitudes of kinetic power, and C_{12} and C_{21} are respectively coupling coefficients. As the signs of the kinetic powers a_1 and a_2 are the opposite of each other, so

$$C_{21} = C_{12}^*$$

Now putting $a_1 = A_1 \exp(-\Gamma z)$ and $a_2 = A_2 \exp(-\Gamma z)$, then Eq. (18) can be rewritten as follows

$$\left.\begin{aligned} &(\Gamma - j\beta_1)\ A_1 e^{-\Gamma z} + C_{12} A_2 e^{-\Gamma z} = 0 \\ &C_{12}^* A_1 e^{-\Gamma z} + (\Gamma - j\beta_2)\ A_2 e^{-\Gamma z} = 0 \end{aligned}\right\} \tag{18'}$$

If Eq. (18') possesses non-trivial solutions, the next determinantal equation must be satisfied:

$$\begin{vmatrix} \Gamma - j\beta_1 & C_{12} \\ C_{12}^* & \Gamma - j\beta_2 \end{vmatrix} = 0$$

This equation can be rewritten as follows

$$(\Gamma - j\beta_1)\ (\Gamma - j\beta_2) - |C_{12}|^2 = 0$$

$$\therefore\quad \Gamma = \frac{1}{2}\left[j(\beta_1 + \beta_2) \pm j\sqrt{(\beta_1 - \beta_2)^2 - 4|C_{12}|^2}\right] \tag{19}$$

$\beta_1 \simeq \beta_2$, so that

$$\Gamma = j\beta \pm |C_{12}| \tag{19'}$$

where

$$\beta_1 = \beta_2 = \beta$$

From Eq. (19') it is seen that the power gain is $|C_{12}|$ neper/m. C_{12} can be obtained by the following techniques:

$$\frac{d}{dz}(a_1 a_1^*) = a_1 \frac{da_1^*}{dz} + a_1^* \frac{da_1}{dz} = a_1 C_{12}^* a_2^* + a_1^* C_{12} a_2 - 2j\beta a_1 a_1^*$$

The real part of this equation is

$$R_e\left[\frac{d}{dz}(a_1a_1^*)\right] = a_1C_{12}^*a_2^* + a_1^*C_{12}a_2 \tag{20}$$

Two terms of the right-hand side of Eq. (20) are the conjugate complex number. The left-hand term of Eq. (20) shows the electric power, so that it can be shown by the currents and the electric fields as follows

$$R_e\left[\frac{d}{dz}(a_1a_1^*)\right] = -\frac{1}{4}(i_1^*E_2 + i_1E_2^*) \tag{21}$$

Two terms of the right-hand side of Eq. (21) are also the conjugate complex number, so that from Eqs. (20) and (21) the following equation can be obtained:

$$a_1^*C_{12}a_2 = -\frac{1}{4}i_1^*E_2 \tag{22}$$

As a_1 and a_2 are normalized, the following relation comes into existence:

$$a_1^*a_2 = \frac{\sqrt{\bar{Z}_{01}\bar{Z}_{02}}\,i_1^*i_2}{2} \qquad E_2 = \frac{ji_2}{\omega\varepsilon}$$

From this equation and Eq. (22),

$$\frac{1}{2}\sqrt{\bar{Z}_{01}\bar{Z}_{02}}\;i_1^*i_2C_{12} = -\frac{1}{4}\frac{ji_1^*i_2}{\omega\varepsilon}$$

so that

$$C_{12} = -j\frac{1}{2\sqrt{\bar{Z}_{01}\bar{Z}_{02}}\;\omega\varepsilon} \tag{23}$$

where

$$\bar{Z}_{01} = \frac{v_{0e}}{\omega_p\omega\varepsilon} \qquad \bar{Z}_{02} = \frac{v_b}{\omega_b\omega\varepsilon}$$

and v_b and ω_b are respectively the d–c velocity of the electron beam and its plasma angular frequency. Eq. (23) can be rewritten as follows

$$C_{12} = -\frac{j}{2}\sqrt{\frac{\omega_p\omega_b}{v_{0e}v_b}} \tag{24}$$

The value of v_{0e} is small because it is the effective mean value of the electron thermal velocity in the plasma, so that it can be expected that $|C_{12}|$ and consequently the power gain can lead to large values.

There is a common point for the amplification action of some microwave amplifiers, i.e. the traveling wave tube, the double stream amplifier, the amplifier using the plasma which is considered in this paper and so on. In other words, the mutual interaction between the space-charge waves in the electron beam and some waves (for example, the circuit wave traveling on the helix for the traveling wave tube, the space-charge waves in the other electron beam for the double stream amplifier, and the electron space-charge wave in the plasma for this amplifier using the plasma) has been utilized for all these amplifier.

In the case of the traveling wave tube, the phase velocity of the circuit wave propergating on the helix and the impedance parameter of the helix are almost constant over a wide range of frequency, although they decrease slightly as the frequency increases. Consequently, the dimensions must be varied if it is required to vary the frequency range.

In the case of the double stream amplifier, the phase velocities of the space-charge waves in the other electron beam which correspond to the circuit wave on the helix are $\omega/(\beta_e \pm \beta_b) = v_b/(1 \pm \omega_b/\omega)$, so that the velocity v_b or the plasma angular frequency ω_b of the electron beam must be varied if it is required to vary the phase velocities of these waves or the frequency range of this amplifier. The characteristic impedance of this wave is $v_b/\omega_b \omega \varepsilon$, so that it decreases as the frequency increases.

Now in the case of the amplifier using the plasma, the phase velocity of the electron space-charge wave in the plasma can be varied by varying ω_p, as shown in Eq. (11). But the wave can propergate only in the case of $\omega > \omega_p$ and it can not in the case of $\omega < \omega_p$. The group velocity of the wave propergating in the plasma can be obtained from Eq. (10) as follows

$$v_g = \frac{\partial \omega}{\partial \beta} = \frac{\bar{v}_0 \sqrt{\frac{\omega^2}{\omega_p^2} - 1}}{2\frac{\omega^2}{\omega_p^2} - 1} \tag{25}$$

From this equation it can be seen that $v_g = 0$ in the case of $\omega = \omega_p$, i.e. the cut-off frequency. The formula of the characteristic impedance of the electron space-charge wave in the plasma is similar to that in the electron beam, but the value of the former is much smaller than that of the latter because of the small value of v_{0e}. Finally it can be seen by considering Eq. (11) that the band-width of this amplifier is very narrow. The reason for this is as follows. The amplification action for this amplifier occurs when the phase velocity of the electron space-charge wave in the plasma is nearly synchronous with the slow space-charge wave in the electron beam. Now this synchronization occurs when $\omega \simeq \omega_p$ i.e. the frequency near the cut-off frequency, because the numerator $\bar{v}_0$ in Eq. (11) is much smaller than the phase velocity of the slow wave in the electron beam so that the denominator in Eq. (11) must become nearly zero for this synchronization and so ω must be nearly equal to ω_p. In conclusion this amplifier using the plasma possesses the properties of high gain and a very narrow band-width of frequency.

IV. The Mutual Interaction between the Electron Space-Charge Wave in the Plasma and the Slow- and Fast Space-Charge Waves in the Electron Beam

In the previous section the effect of the fast electron space-charge wave in the electron beam was neglected. In this section we will consider a more rigorous analysis not neglecting this wave. The coupling equations in this case can be written as follows

$$\left.\begin{aligned} \frac{da_1}{dz} &= -j\beta_1 a_1 + C_{12}a_2 + C_{13}a_3 \\ \frac{da_2}{dz} &= C_{21}a_1 - j\beta_2 a_2 \\ \frac{da_3}{dz} &= C_{31}a_1 - j\beta_3 a_3 \end{aligned}\right\} \tag{26}$$

where all quantity for the forward wave in the plasma are shown by subscript 1, those for the slow wave in the electron beam by subscript 2, and those for the fast wave in the electron beam by subscript 3. In Eq. (26) a_1, a_2 and a_3 are respectively normalized amplitudes of kinetic power, and C_{12}, C_{13}, C_{21} and C_{31} are coupling coefficients respectively. There is no interaction between the slow- and fast waves in the electron beam, hence $C_{23}=C_{32}=0$. The signs of the kinetic powers a_1 and a_2 are the opposite of each other, and those of a_1 and a_3 are the same as each other, hence $C_{21}=C_{12}^*$ and $C_{31}=-C_{13}^*$.

The values of the coupling coefficients can be obtained by the following techniques. In other words C_{12} can be obtained from the analysis of the mutual interaction between the forward wave and the slow wave, and C_{13} from the analysis of the mutual interaction between the forward wave and the fast wave. Accordingly C_{12} is the same value as that obtained in the previous section, i.e.

$$C_{12} = -\frac{j}{2}\sqrt{\frac{\omega_p\omega_b}{\bar{v}_{0e}v_b}}$$

C_{13} can be obtained as follows

$$\frac{d}{dz}(a_1a_1^*) = a_1C_{13}^*a_3^* + a_1^*C_{13}a_3 - 2j\beta a_1a_1^*$$

The real part of this equation is

$$R_e\left[\frac{d}{dz}(a_1a_1^*)\right]=a_1C_{13}^*a_3^*+a_1^*C_{13}a_3 \tag{27}$$

The left-hand term of Eq. (27) can be rewritten in the same manner with Eq. (21) as follows

$$R_e\left[\frac{d}{dz}(a_1a_1^*)\right]=\frac{1}{4}(i_1^*E_3+i_1E_3^*) \tag{28}$$

Two terms of the right-hand side in Eq. (27) and Eq. (28) are the conjugate complex number, hence

$$a_1^*C_{13}a_3=\frac{1}{4}i_1^*E_3$$

As a_1 and a_3 are normalized, the next relation comes into existence:

$$a_1^*a_3=\frac{\sqrt{\bar{Z}_{01}\bar{Z}_{02}}\,i_1^*i_3}{2} \qquad E_3=\frac{ji_3}{\omega\varepsilon}$$

From those two equations

$$\frac{\sqrt{\bar{Z}_{01}\bar{Z}_{02}}\,i_1^*i_3C_{13}}{2}=\frac{ji_1^*i_3}{4\omega\varepsilon}$$

thus

$$C_{13}=j\frac{1}{2\sqrt{\bar{Z}_{01}\bar{Z}_{02}}\,\omega\varepsilon} \tag{29}$$

where

$$\bar{Z}_{01}=\frac{v_{0e}}{\omega_p\omega\varepsilon} \qquad \bar{Z}_{02}=\frac{v_b}{\omega_b\omega\varepsilon}$$

Eq. (29) can be rewritten as follows

$$C_{13}=\frac{j}{2}\sqrt{\frac{\omega_p\omega_b}{v_{0e}v_b}} \tag{30}$$

Now putting $a_1=A_1\exp(-\Gamma z)$, $a_2=-A_2\exp(-\Gamma z)$ and $a_3=A_3\exp(-\Gamma z)$, then Eq. (26) can be rewritten as follows

$$\left.\begin{aligned}&(\Gamma-j\beta_1)A_1+C_{12}A_2+C_{13}A_3=0\\&C_{12}^*A_1+(\Gamma-j\beta_2)A_2=0\\&-C_{13}^*A_1+(\Gamma-j\beta_3)A_3=0\end{aligned}\right\} \tag{31}$$

If Eq. (31) possesses non-trivial solutions, the next determinantal equation must be satisfied:

$$\begin{vmatrix} \Gamma-j\beta_1 & C_{12} & C_{13} \\ C_{12}^* & \Gamma-j\beta_2 & 0 \\ -C_{13}^* & 0 & \Gamma-j\beta_3 \end{vmatrix} = 0 \tag{32}$$

This equation can be rewritten by putting $C_{12}=-C_{13}=k$ as follows

$$\begin{aligned}\Gamma^3-j\Gamma^2(\beta_1+\beta_2+\beta_3)-\Gamma(\beta_1\beta_2+\beta_1\beta_3+\beta_2\beta_3)\\ +j\beta_1\beta_2\beta_3+j|k|^2(\beta_3-\beta_2)=0\end{aligned} \tag{33}$$

Putting

$$\Gamma=j\beta_e(1+x) \qquad \beta_1=\beta_e(1+\delta_1) \qquad \beta_2=\beta_e(1+\delta_2)$$

$\beta_3=\beta_e(1-\delta_2) \qquad \beta_e=\omega/v_b$: phase velocity of electron beam

then Eq. (33) can be rewritten as follows

$$x^3-\delta_1x^2-\delta_2^2x+\delta_1\delta_2^2+A=0 \tag{34}$$

where

$$A=\frac{\omega_p\omega_b\delta_2(1+\delta_1)}{2\omega^2\sqrt{\omega^2/\omega_p^2-1}}$$

Now putting

$$\omega=\omega_p+\Delta\omega \qquad \xi=\sqrt{\frac{\Delta\omega}{\omega_p}}\ll 1$$

$$p=\frac{\sqrt{2}v_b}{v_{0e}}\gg 1 \qquad 1>q=\frac{\omega_b}{\omega_p}>0$$

then

$$\delta_1=\frac{v_b}{v_{0e}}\sqrt{\frac{\omega^2}{\omega_p^2}-1}\ -1\simeq p\xi-1$$

$$\delta_2=\frac{\omega_b}{\omega}\simeq q\,(1-\xi^2)$$

$$A\simeq\frac{pq^2(1-\xi^2)^3}{2\sqrt{2}}$$

$$\delta_1\delta_2^2+A\simeq\frac{q^2}{2\sqrt{2}}(1-2\xi^2)\left\{p\,(1+2\sqrt{2}\,\xi-\xi^2)-2\sqrt{2}\right\}=B$$

Eq. (34) can be rewritten by using those relationships as follows

$$y^3+ay+b=0 \tag{35}$$

where

$$y = x - \frac{\delta_1}{3}$$

$$a = -\delta_2^2 - \frac{\delta_1^2}{3} = -q^2(1-2\xi^2) - \frac{1}{3}(1-p\xi)^2$$

$$b = -\frac{2}{27}\delta_1^3 - \frac{1}{3}\delta_1\delta_2^2 + B = -\frac{2}{27}(p\xi-1)^3 - \frac{1}{3}(p\xi-1)\,q^2(1-2\xi^2)$$
$$+ \frac{1}{2\sqrt{2}}q^2(1-2\xi^2)\left\{p\,(1+2\sqrt{2}\,\xi-\xi^2)-2\sqrt{2}\right\}$$

If $R=(b/2)^2+(a/3)^3>0$, then two of the three roots of Eq. (34) are complex numbers (the remaining one is real), so that the exponential growing wave should exist. Now calculating the value of R, then

$$\frac{27R}{q^2} = 2\xi^6 + \xi^5\left(-8p + \frac{3}{2\sqrt{2}}p^4\right) + \xi^4\left\{-1 + 8p^2 - \frac{9}{2\sqrt{2}}p^3\right.$$
$$\left. + 2\,(1-3q^2) + 2\,(1-q^2)\right\} + \xi^3\left\{4p - \frac{1}{2\sqrt{2}}p^4 - 4p\,(1-q^2)\right.$$
$$\left. - 4p\,(1-3q^2) + \frac{9}{2\sqrt{2}}p^2 - 15\,q^2\frac{3}{2\sqrt{2}}p^2\right\} + \xi^2\left\{-2\,(1-q^2)\right.$$
$$+ \frac{3}{2\sqrt{2}}p^3 + 2\,(1-3q^2)\,(1-q^2) - \frac{3}{2\sqrt{2}}p$$
$$\left. + \frac{45}{2\sqrt{2}}pq^2 - \frac{3\times 27}{16}p^2q^2\right\} + \xi\left\{4p\,(1-q^2) - \frac{3}{2\sqrt{2}}p^2\right.$$
$$\left. + \frac{9}{2\sqrt{2}}p^2q^2\right\} + \left\{-(1-q^2)^2 + \frac{p}{2\sqrt{2}} - 9q^2\frac{p}{2\sqrt{2}} + \frac{27}{16}pq^2\times\frac{p}{2}\right\}$$
$$\simeq 2\xi^6 + \frac{3}{2\sqrt{2}}p^4\xi^5 - \frac{9}{2\sqrt{2}}p^3\xi^4 - \frac{p^4\xi^3}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}p^3\xi^2 - \frac{3}{2\sqrt{2}}p^2\xi$$
$$+ \frac{p}{2\sqrt{2}} + \frac{27}{32}p^2q^2 \simeq -\frac{1}{2\sqrt{2}}p\,(p\xi)^3 + \frac{3p}{2\sqrt{2}}(p\xi)^2 - \frac{3p}{2\sqrt{2}}(p\xi)$$
$$+ \frac{p}{2\sqrt{2}} + \frac{27}{32}p^2q^2 = \frac{p}{2\sqrt{2}}(1-p\xi)^3 + \frac{27}{32}p^2q^2 \tag{36}$$

where the next conditions have been used for approximation:

$$p\xi \simeq 1 \qquad 0<\xi\ll 1 \qquad q\simeq 0.1$$

From Eq. (36) the condition of $R>0$ is

$$-(1-p\xi)^3 < \frac{27\sqrt{2}}{16}pq^2 \tag{37}$$

From Eq. (37) the frequency band-width of this amplifier $\xi^2 = \Delta\omega/\omega_p \simeq \Delta\omega/\omega$ is

$$0<\frac{\Delta\omega}{\omega}<\frac{1}{p^2}\left(1+\frac{3}{2}\sqrt[3]{\frac{pq^2}{\sqrt{2}}}\right)^2 \tag{38}$$

Next it is neccessary to calculate the value of b for the sake of obtaining numerically the roots $y_i (i=1, 2, 3)$

$$\begin{aligned} b &= \frac{2}{27}(1-p\xi)^3-\frac{2}{3}q^2(1-2\xi^2)(1-p\xi)+\frac{q^2}{2\sqrt{2}}(1-2\xi^2)\,p\,(1-\xi)^2 \\ &= \xi^4\left(\frac{pq^2}{\sqrt{2}}\right)+\xi^3\left(-\frac{2}{27}p^3-\frac{4}{3}pq^2\right)+\xi^2\left(\frac{2}{9}p^2+\frac{4}{3}q^2-\frac{1}{\sqrt{2}}pq^2\right. \\ &\quad \left.-\frac{1}{2\sqrt{2}}pq^2\right)+\xi\left(-\frac{2}{9}p+\frac{2}{3}pq^2\right)+\left(\frac{2}{27}-\frac{2}{3}q^2+\frac{1}{2\sqrt{2}}pq^2\right) \\ &\simeq \frac{q^2}{\sqrt{2}}p\xi^4-\frac{2}{27}(p\xi)^3+\frac{6}{27}(p\xi)^2-\frac{6}{27}(p\xi)+\frac{2}{27}+\frac{1}{2\sqrt{2}}pq^2 \\ &\simeq \frac{2}{27}(1-p\xi)^3+\frac{1}{2\sqrt{2}}pq^2 \end{aligned} \tag{39}$$

In the last equation of Eq. (39), the term $(1/\sqrt{2})\,pq^2\xi^4$ was ignored because of the very small value.

Now the three roots of Eq. (35) y_1, y_2 and y_3 are

$$\left.\begin{aligned} y_1 &= C+D \\ y_2 &= \omega_1 C+\omega_2 D \\ y_3 &= \omega_1 D+\omega_2 C \end{aligned}\right\} \tag{40}$$

where

$$\begin{aligned} C&=\left(-\frac{b}{2}+\sqrt{R}\right)^{\frac{1}{3}}=\left\{-\frac{1}{27}(1-p\xi)^3-\frac{1}{4\sqrt{2}}pq^2\right. \\ &\quad \left.+\frac{q}{3}\sqrt{\frac{1}{3}\left\{\frac{p}{2\sqrt{2}}(1-p\xi)^3+\frac{27}{32}p^2q^2\right\}}\right\}^{\frac{1}{3}} \end{aligned} \tag{41}$$

$$\begin{aligned} D&=\left(\frac{b}{2}+\sqrt{R}\right)^{\frac{1}{3}}=\left\{\frac{1}{27}(1-p\xi)^3+\frac{1}{4\sqrt{2}}pq^2\right. \\ &\quad \left.+\frac{q}{3}\sqrt{\frac{1}{3}\left\{\frac{p}{2\sqrt{2}}(1-p\xi)^3+\frac{27}{32}p^2q^2\right\}}\right\}^{\frac{1}{3}} \end{aligned} \tag{42}$$

$$\omega_1 = (1+j\sqrt{3})/2 \tag{43}$$

$$\omega_2 = (-1-j\sqrt{3})/2 \tag{44}$$

and x_1, x_2 and x_3 are

$$\left.\begin{aligned} x_1 &= \frac{\delta_1}{3} + y_1 \\ x_2 &= \frac{\delta_1}{3} + y_2 \\ x_3 &= \frac{\delta_1}{3} + y_3 \end{aligned}\right\} \tag{45}$$

Introducing the following numerical example for the calculation of the frequency band-width and the exponential gain of this amplifier,

the electron beam currents $= 10$ mA
the radius of this beam $= 1$ mm
the current density of this beam, $J = 0.32$ A/cm^2
the accelerating voltage of this beam, $V = 1000$ V
$\omega_b = 1.85 \times 10^{10}\sqrt{J/V^{\frac{1}{2}}} = 1.86 \times 10^9$ rad/sec
the plasma frequency of the main discharge ($\simeq$ the input frequency)
$f_p = 4000$ Mc
$\omega_p = 2\pi f_p = 2.52 \times 10^{10}$ rad/sec
$q = \omega_b/\omega_p = 0.0738$
the beam velocity $v_b = \sqrt{2eV/m} = 5.93 \times 10^7$ m/s
$v_{0e} = 10^5$ m/s
$p = \sqrt{2}\, v_b/v_{0e} = 8.39 \times 10^2$

In the above case, from Eq. (38) the frequency band-width is

$$0 < \frac{\Delta\omega}{\omega} < 1.48 \times 10^{-5}$$

hence

$$0 < \Delta\omega < 3.73 \times 10^5$$
$$0 < \Delta f < 60 \text{ kc/s}$$

The maximum gain occurs when $\xi = 0$. In this case, the values of C, D and the imaginary part of x are respectively 0, 1.19 and 1.03 so that the value of maximum gain G is

$$\begin{aligned} G = \beta_e I_m(x) &= 4.25 \times 10^2 \times 1.03 = 438 \text{ Nep/m} \\ &= 38 \text{ dB/cm} \end{aligned}$$

As may be seen from this numerical example, this amplifier has the very high gain (about 37.8 dB/cm), while being very narrow with respect to the frequency band-width (smaller than 60 kc/s).

The numerical value of the gain when the two modes are considered is 121 dB/cm, so the above value of the gain is slightly smaller than one third of that obtained in the two modes. This fact shows that the fast wave in the electron beam acts as an impeding force against the exponential grow of the signal wave.

V. Conclusion

In this paper the amplification action in the presence of the electron beam modulated by a microwave signal passing through the plasma was analyzed. The processes of this analysis are as follows. First the propagation constant and the characteristic impedance of the electron space-charge wave in the plasma excited by a microwave signal was obtained. Next it was analyzed that the amplification action occurs by the mutual interaction of two modes (the electron space-charge wave in the plasma and the slow space-charge wave in the electron beam). At the same time the coupling coefficients between those two waves were obtained. Finally the amplification phenomena as a result of the interaction of three modes (the electron space-charge wave in the plasma and the slow- and fast wave in the electron beam) was analyzed. At the same time a numerical example was shown. The physical meaning of this amplification phenomena may be interpreted reasonably by the two mode theory, but the three mode theory must be used for the sake of obtaining accurate values of the gain and the frequency band-width. According to this theory, though the angular frequency ω of the microwave signal is very close to the plasma angular frequency ω_p, ω must be larger than ω_p. With respect to this point, this paper greatly differ from the paper written by Bohm and Gross[1], and others in which the amplification action occurs when $\omega<\omega_p$ in spite of the fact that ω is very close to ω_p.

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