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A Numerical Method for Stability Analysis of Time-Varying Linear Systems With Periodic Coefficients

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Introduction

In various engineering fields, we have many problems involving analysis of linear systems described by ordinary differential equations with periodic coefficients. For instance, the following problems come under this category:

(1) Analysis and synthesis of sampled data control systems.

(2) Analysis and synthesis of control systems having time-varying elements which characters are expressed by continuous periodic functions.

(3) Analysis of phenomena described by the Mathieu or the Hill equation.

(4) Analysis of electric circuits with periodically operated switches.

(5) Analysis of amplitude modulation and demodulation systems.

(6) Analysis of parametric amplifiers and oscillators

(7) Analysis of periodic solutions in nonlinear systems.

In these problems, the stability analysis of systems is especially considered to be one of the most basic and common subjects, although these problems should be studied from various standpoints according to the circumstances.

Hill's method[1,2] and the perturbation method[3,4] are usually available for the stability analysis of second order linear systems with periodic coefficients. It is, however, considered that the methods are difficult to be applied to systems where the order is third or more. Also other methods[5,6,7] which have been developed in recent years are not always to be satisfied from the viewpoints of accuracy of results and the universality of the methods. From this reason, a numerical method has been developed by the author, which is based on the thoerem derived by R. E. Kalman and J. E. Bertram[8], the Schur-Cohn criterion[9] and the numerical methods for initial value problems of ordinary differential equations[10,11,12,24]. The method is not only applicable to almost all linear systems with periodic coefficients, but also makes possible the obtaining of results of sufficiently high accuracy in accordance with the requirements, although in all cases the use of digital computer is assumed.

In this paper, the theorem derived by R. E. Kalman and J. E. Bertram is first reconsidered from a more general point of view. The relation between the theorem and other fundamental theorem[13] is also examined. Next the numerical method denoted in the title is derived. It is also indicated that the method is applicable to the systems with delay elements by the use of appropriate approximations. The paper is concluded with some examples to explain the method and some practical techniques treating the problems.

The relation between the stability and the transition matrix in linear systems with periodic coefficients

In order to apply the theorem derived by R. E. Kalman and J. E. Bertram to the stability analysis of more general linear systems with periodic coefficients, and not only in sampled data systems, it is required that the conditions which the system under consideration must satisfy in the application of the theorem be made clear; and that the theorem is still valid even if the state of the system between sampling instants is taken into consideration. In the following, the conditions and the validity of the theorem mentioned above are first examined.

Now it is assumed that the behavior of the system under consideration can be described by equation (1), where x is the m-dimensional state vector, u is the r-dimensional forcing vector, $A(t)$ is the $m\times m$ time-varying matrix and $B(t)$ is the $m\times r$ time-varying matrix. Equation (2) representing the periodic character of the system is also assumed.

$$\dot{x}=A(t)x+B(t)u \qquad (1)$$

$$A(t+T)=A(t) \qquad (2)$$

The stability of the system is fundamentally governed by that of equation (3). From this point of view, let us define that the system is stable if and only if the response of the system expressed by equation (3) with a freely selected, but not zero, initial state converges to the origin.

$$\dot{x}=A(t)x \qquad (3)$$

As far as the existence and uniqueness of solutions are guaranteed, the response of the system described by equation (3) at the instant $t=t_0+\tau$ should be represented by a linear combination of the initial state $x(t_0)$ of the system, because the system is linear, and the sum of optional solutions should be also a solution of the system. From this point of view, equation (4) is derived as follows :

$$x(t_0+\tau)=k(\tau, t_0)x(t_0) \qquad (4)$$

where $k(\tau, t_0)$ is the $m\times m$ transition matrix[8,14,15] of the system, and which is fixed by the matrix $A(t)$, the time origin t_0 and the time interval τ. Also, from equation (2), the next relation is derived.

$$k(\tau,\ t_0+nT_0)=k(\tau,\ t_0) \tag{5}$$

$$T_0=qT,\qquad q=1, 2, \cdots \tag{6}$$

Therefore, if τ is fixed as in equation (7), the solutions of equation (3) must also satisfy the difference equation with constant coefficients which is expressed by equation (8).

$$\tau=T_0 \tag{7}$$

$$x\left(t_0+(n+1)T_0\right)=k(T_0,\ t_0)\,x(t_0+nT_0) \tag{8}$$

Equation (8) can be solved easily using the Z transform method[9,16,17] as follows:

Taking the Z transform of equation (8) and rearranging the derived equation, we have

$$\left(ZE-k(T_0,\ t_0)\right)X(Z)=Zx(t_0) \tag{9}$$

where E is the unit matrix, and

$$X(Z)=z\left(x(t_0+nT_0)\right) \tag{10}$$

Therefore, the individual element of $X(Z)$ can be written as follows:

$$X_i(Z)=\frac{D_i}{D} \tag{11}$$

where

$$D=\det\left(ZE-k(T_0,\ t_0)\right) \tag{12}$$

also D_i is the determinant of the matrix which is derived substituting the right side term of equation (9) for the i-th column of the matrix $(ZE-k(T_0,\ t_0))$. Therefore, the individual element of the system response $x(t_0+nT_0)$ is derived as follows:

$$x_i(t_0+nT_0)=z^{-1}\left(X_i(Z)\right) \tag{13}$$

Thus, if only the states of the system at the instants of $t=t_0+nT_0$ are taken into consideration, the next statement will be valid from the nature of Z transformation[9,16,17,23]. Namely, if and only if all roots of equation (14), the characteristic roots of the transition matrix $k(T_0,\ t_0)$, are in the unit circle in the complex plane, the system described by equation (1) is stable.

$$D=0 \tag{14}$$

Moreover, for the optional time origin t_0 in the time interval $(n+1)T_0\geq t\geq nT_0$, the next relation is derived.

$$k\left(T_0+(t_0-nT_0),\ nT_0\right)$$
$$=\left(k\left(t_0-nT_0,\ (n+1)T_0\right)k\left(T_0-(t_0-nT_0),\ t_0\right)\right)k(t_0-nT_0,\ nT_0)$$
$$=k\left(t_0-nT_0,\ (n+1)T_0\right)\left(k\left(T_0-(t_0-nT_0),\ t_0\right)k(t_0-nT_0,\ nT_0)\right) \tag{15}$$

From equations (5) and (15),

$$k(T_0,\ t_0)=k(t_0-nT_0,\ 0)\,k(T_0,\ 0)\,k^{-1}(t_0-nT_0,\ 0) \tag{16}$$

From equation (16), it will be understood that the transition matrices $k(T_0, t_0)$ are, regardless of the values of t_0, all similar[18] and have the same characteristic roots. Therefore, related to the stability of linear systems with periodic coefficients, the following theorem is derived.

Theorem 1. A linear system with periodic coefficients in which the existence and uniqueness of solutions are guaranteed is stable, if and only if the characteristic roots of transition matrix $k(T_0, t_0)$ of the system are all in the unit circle in the complex plane. Here, T_0 is a common period of periodic coefficients, which is taken as shown in equation (6), also t_0 is a time origin which can be selected arbitrarily.

Next, let us consider the relation between the theorem and the other fundamental theorem connected with the stability of linear systems with periodic coefficients. According to the book of Solomon Lefschetz[13], the system described by equation (3) is asymptotically stable at the origin, if the characteristic exponents are all <1, in absolute value. Here, the characteristic exponents mean the characteristic roots of the constant, the nonsingular matrix C, which is defined by equation (18). Also $X(t)$ is a nonsingular solution of equation (17) which is the matrix equation corresponding to equation (3).

$$\dot{X}=A(t)X \tag{17}$$

$$X(t+T)=X(t)\,C \tag{18}$$

The matrix K satisfying equation (19) which is derived by substituting $t=t_0$ for equation (18) is obtained as equation (20).

$$X(t_0+T)=X(t_0)\,C=KX(t_0) \tag{19}$$

$$K=X(t_0)\,CX^{-1}(t_0) \tag{20}$$

Since the matrices C and K are similar and the matrix K corresponds to the transition matrix $k(T, t_0)$, it is clear that both theorems coincide completely. It is, however, remarkable that the former is obtained assuming only the existence and uniqueness of the solutions although the latter is derived by assuming the continuity of coefficients $A(t)$.

The process of treatments for the stability analysis

In time varying systems, it is usually difficult to obtain the transition matrix $k(T_0, t_0)$ by analytical treatments, but easy to derive it using adequate numerical methods. Namely, since the transition matrix $k(T_0, t_0)$ is defined by substituting $\tau = T_0$ in equation (4), if the system responses at the instant $t = t_0 + T_0$ with various initial states as shown in equation (21) are obtained by the use of numerical methods, the transition matrix $k(T_0, t_0)$ can be fixed as shown in equation (22).

$$x_1(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad x_2(t_0) = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad x_m(t_0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{21}$$

$$k(T_0, t_0) = \big(x_1(t_0 + T_0), x_2(t_0 + T_0), \cdots, x_m(t_0 + T_0)\big) \tag{22}$$

Therefore, for the stability analysis of the systems described by equation (1) in which the existence and uniqueness of solutions are guaranteed, the following process of treatments is always available.

(1) Referring to Theorem 1, the time interval T_0 and the time origin t_0 of the transition matrix $k(T_0, t_0)$ are adequately fixed.

(2) Using an adequate numerical method, the transition matrix $k(T_0, t_0)$ is derived.

(3) Applying the Schur-Cohn criterion or the bilinear transformation method[9] to the characteristic equation (14) in which roots are equal to the characteristic roots of the transition matrix $k(T_0, t_0)$ the region in which all of the roots exist is examined. If and only if the decision that the roots are all in the unit circle in the complex plane is obtained, then the system is decided to be stable.

Approximate treatments for the systems having delay elements

Usually the behavior of systems having delay elements can not be described by equation (1), but can be expressed by differential difference equations. Therefore, in order to derive the state transition equation corresponding to equation (8), the use of an infinite number of state variables are required.

Consequently, if the numerical method for stability analysis is required to apply for this type of systems, then it is necessary to derive an approximate expression using a finite number of state variables instead of the exact state transition equation corresponding to equation (8). For this purpose, the next two methods are available.

(1) The first method in which an adequate Padè approximation is used instead of the transfer function e^{-SL} of a delay element.

(2) The second method in which the values at $A_1, A_2, \cdots, A_R$ in the decomposed delay element shown in Fig. 1 are taken as the new state variables added to the usual state variables.

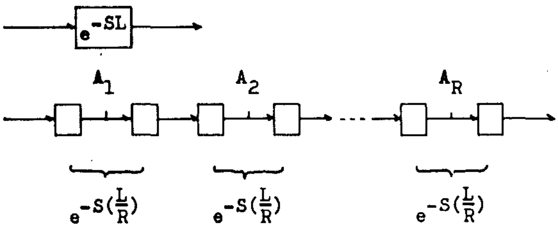

Fig. 1. Decomposition of delay element with time lag L.

Also, in the second method, the following two methods approximating the initial function which is memorized in a delay element are usually available.

(1) The method approximating the initial function by the staircase function as shown in Fig. 2.

(2) The method approximating the initial function by the piecewise linear function as shown in Fig. 3.

Here, it is remarkable that the coefficients of the new state variables for delay elements i. e., the new elements of the transition matrix $k(T_0, t_0)$, repre-

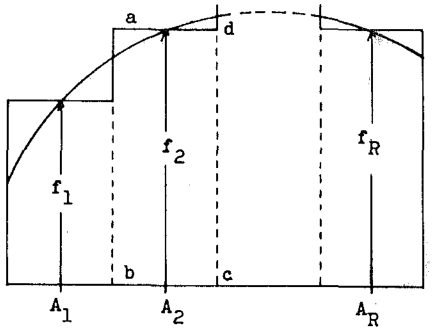

Fig. 2. Staircase function approximation for initial function.

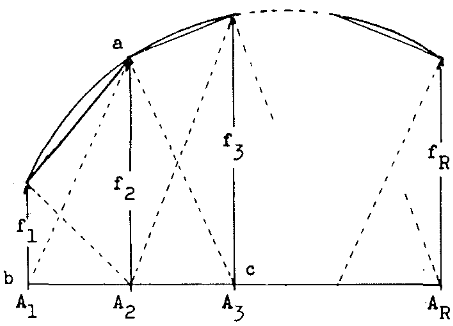

Fig. 3. Piecewise linear function approximation for initial function.

sent the contribution of the individual initial function as in the a b c d in Fig. 2 or the a b c in Fig. 3.

Examples and some remarks

Now, let us apply the method to the stability analysis of some typical systems and give some remarks on the practical use of the method.

Example 1. As the first example, let us consider the problem deciding the stable region of the third order system ($\omega_0=2$ rad/sec or $\omega_0=10$ rad/sec) shown in Fig. 4, which has been treated by W. W. Cooley, R. N. Clark and R. C. Buckner[5].

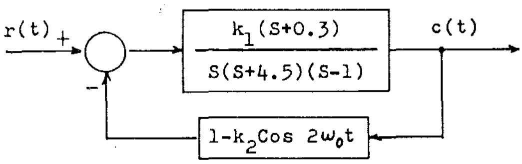

Fig. 4. Third order system for first example.

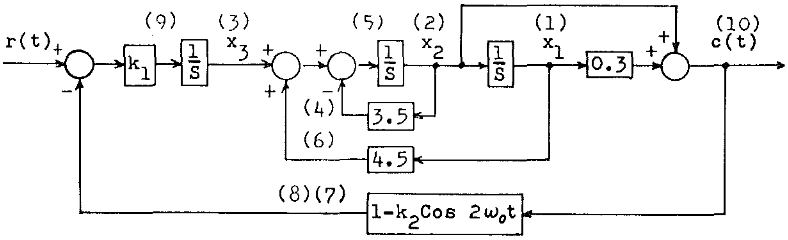

Fig. 5. Block diagram equivalent to that in Fig. 4.

From the block diagram shown in Fig. 4, time interval T_0 is fixed as equation (23). Also the time origin t_0 is fixed as equation (24).

$$T_0=\begin{cases}\pi/2\ \text{sec (when } \omega_0=2\ \text{rad/sec)}\\ \pi/10\ \text{sec (when } \omega_0=10\ \text{rad/sec)}\end{cases} \tag{23}$$

$$t_0=0 \tag{24}$$

Since the system is of the third order, the transition matrix $k(T_0, t_0)$ is described as follows :

$$k(T_0, t_0)=k(T_0, 0)=\begin{pmatrix} k_{11} & k_{12} & k_{13}\\ k_{21} & k_{22} & k_{23}\\ k_{31} & k_{32} & k_{33}\end{pmatrix} \tag{25}$$

TABLE 1. Computer program

```
  #  STABILITY ANALYSIS OF 3RD ORDER SYSTEM(1)
     DIMENSION START(10),INDEX(50),CONST(5,5)
     DIMENSION A(5),SMR(25),SPA(25),COEF(10)
6000 READ 0,L,M,N,LL,LM,ML
  READ 1,(START(K),K=1,L),DELT1,DELT2,OMEG,B
     READ 2,T
     COEF(2)=START(2)
     DO 1000 II=1,LL
     COEF(1)=START(1)
     DO 2000 IJ=1,LM
     GO TO SUBROUTINE 1100
     GO TO SUBROUTINE 1200
     INDEX(IJ)=IDEC
2000 COEF(1)=COEF(1)+DELT1
     TYPE 5000,(INDEX(IJ),IJ=1,LM)
1000 COEF(2)=COEF(2)+DELT2
     TYPE 7000,
5000 FORMAT(40I2)
7000 FORMAT(/////////)
     PAUSE
     GO TO 6000


1100 SUBROUTINE
     DO 335 I=1,M
     DO 25 J=1,N
25   SPA(J)=0.
     SPA(I)=1.
     TIME=0.
     SPA(10)=SPA(2)+SPA(1)*0.3
     SPA(7)=(1.-COEF(2))*SPA(10)
     SPA(8)=SPA(7)
     SPA(9)=-SPA(8)*COEF(1)
     SPA(4)=SPA(2)*3.5
     SPA(6)=SPA(1)*4.5
     SPA(5)=SPA(3)+SPA(6)-SPA(4)
     DO 225 JI=1,ML
     TIME=TIME+T
     SMR(8)=SPA(7)
     SMR(9)=-(2.*SMR(8)-SPA(8))*COEF(1)
     SMR(3)=SPA(3)+(SMR(9)+SPA(9))*T/2.
     SMR(4)=SPA(2)*3.5
     SMR(6)=SPA(1)*4.5
SMR(5)=SMR(3)+(2.*SMR(6)-SPA(6))-(2.*SMR(4)-SPA(4))
     SMR(2)=SPA(2)+(SMR(5)+SPA(5))*T/2.
     SMR(1)=SPA(1)+(SMR(2)+SPA(2))*T/2.
     SMR(10)=SMR(1)*0.3+SMR(2)
     ANG=2.*OMEG*TIME
     SMR(7)=(1.-COEF(2)*COSF(ANG)*SMR(10)
     DO 15 JJ=1,10
15   SPA(JJ)=SMR(JJ)
225  CONTINUE
     CONST(1,I)=SMR(1)
     CONST(2,I)=SMR(2)
335  CONST(3,I)=SMR(3)
     RETURN
```

for example 1.

```
1200 SUBROUTINE
     A(1)=B**3
     A(2)=(-CONST(1,1)-CONST(2,2)-CONST(3,3)*B**2
     A(3)=(CONST(1,1)*CONST(2,2)+CONST(2,2)*CONST(3,3)@
       +CONST(3,3)*CONST(1,1)-CONST(1,2)*CONST(2,1)@
       -CONST(2,3)*CONST(3,2)-CONST(3,1)*CONST(1,3)*B
     A(4)= CONST(1,1)*CONST(2,3)*CONST(3,2)@
          +CONST(2,2)*CONST(1,3)*CONST(3,1)@
          +CONST(3,3)*CONST(1,2)*CONST(2,1)@
          -CONST(1,1)*CONST(2,2)*CONST(3,3)@
          -CONST(1,3)*CONST(3,2)*CONST(2,1)@
          -CONST(1,2)*CONST(2,3)*CONST(3,1)
     IF(A(1)+A(2)+A(3)+A(4)) 55,55,56
56   IF(3.*(A(1)-A(4))+A(2)-A(3)) 55,55,57
57   IF(3.*(A(1)+A(4))-A(2)-A(3)) 55,55,58
58   IF(A(1)-A(2)+A(3)-A(4)) 55,55,59
59   IF(A(1)**2-A(4)**2-A(1)*A(3)+A(2)*A(4)) 55,55,60
60   IDEC=0
     GO TO 70
55   IDEC=1
70   RETURN
     END
```

```
2 3 3 21 21 100 0. 10. 1. -0.5 2. 1. 1.570796327E-2
2 3 3 21 21 100 0. 40. 1. -2. 10. 1. 3.141592654E-3
```

<u>Symbols</u>:

L: Number of parameters.
M: Order of system.
N: Number of state variables.
LL: Number of rows in Table 2 or 3.
LM: Number of columns in Table 2 or 3.
ML: Number of repeated computations of recurrence formulas.
START(i): Initial value of COEF(i).
DELT1: Variation of COEF(1).
DELT2: Variation of COEF(2).
OMEG: ω_0.
B: Stabilizing coefficient.
T: Sampling period for computation of system response.
TIME: t.
COEF(i): k_i.
SMR(i): Value of signal about point (i) at present.
SPA(i): Value of signal about point (i) at one sampling period past.
CONST(i,j): k_{ij}.
A(i): A_i.

where,

$$\left.\begin{aligned} x_1(T_0) &= k_{11}x_1(0)+k_{12}x_2(0)+k_{13}x_3(0) \\ x_2(T_0) &= k_{21}x_1(0)+k_{22}x_2(0)+k_{23}x_3(0) \\ x_3(T_0) &= k_{31}x_1(0)+k_{32}x_2(0)+k_{33}x_3(0) \end{aligned}\right\} \tag{26}$$

and where, x_1, x_2, x_3 are the state variables which are shown in Fig. 5. Also the characteristic equation corresponding to equation (14) is as follows:

$$D=\begin{vmatrix} Z-k_{11} & -k_{12} & -k_{13} \\ -k_{21} & Z-k_{22} & -k_{23} \\ -k_{31} & -k_{32} & Z-k_{33} \end{vmatrix} = A_1Z^3+A_2Z^2+A_3Z+A_4=0 \tag{27}$$

where,

$$\left.\begin{aligned} A_1 &= 1 \\ A_2 &= -k_{11}-k_{22}-k_{33} \\ A_3 &= k_{11}k_{22}+k_{22}k_{33}+k_{33}k_{11}-k_{12}k_{21}-k_{23}k_{32}-k_{31}k_{13} \\ A_4 &= k_{11}k_{23}k_{32}+k_{22}k_{13}k_{31}+k_{33}k_{12}k_{21}-k_{11}k_{22}k_{33}-k_{13}k_{32}k_{21}-k_{12}k_{23}k_{31} \end{aligned}\right\} \tag{28}$$

Using the Schur-Cohn criterion or the bilinear transformation method, the necessary and sufficient conditions that the roots of equation (27) are all in the unit circle in the complex plane are obtained as follows:

$$\left.\begin{aligned} & A_1+A_2+A_3+A_4>0 \\ & 3(A_1-A_4)+A_2-A_3>0 \\ & 3(A_1+A_4)-A_2-A_3>0 \\ & A_1-A_2+A_3-A_4>0 \\ & A_1^2-A_4^2-A_1A_3+A_2A_4>0 \end{aligned}\right\} \tag{29}$$

Now, if the parameters k_1 and k_2 are given, then all elements of the transition matrix $k(T_0, 0)$ are fixed from the system responses $x_1(T_0)$, $x_2(T_0)$, $x_3(T_0)$ for the three kinds of initial states ($x_1(0)=1$, $x_2(0)=0$, $x_3(0)=0$), ($x_1(0)=0$, $x_2(0)=1$, $x_3(0)=0$), ($x_1(0)=0$, $x_2(0)=0$, $x_3(0)=1$). Therefore, using the condition (29), the stability of the system for given k_1 and k_2 is decided.

The program for a digital computer by which the stable region of the system is found automatically changing the parameters k_1 and k_2 is shown in Table 1, although the FORTRAN type of language ALCON[19] which has been developed at the Hokkaido University Computer Center, is used. The 1100 SUBROUTINE is the program for finding the transition matrix $k(T_0, 0)$ in which the author's method[12] is used. Also, by the 1200 SUBROUTINE, the condition (29) is examined.

TABLE 2. The results of stability analysis for the system expressed in Fig. 4, $\omega_0=2$ rad/sec.

k_2																					
10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1
7	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1
5	1	1	1	0	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1
	1	1	1	0	1	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1
4	1	1	1	0	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1
	1	1	1	1	0	1	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1
3	1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1
	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1
2	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
k_1	0					5					10					15					20

TABLE 3. The results of stability analysis for the system expressed in Fig. 4, $\omega_0=10$ rad/sec.

k_2																					
40	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
30	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1
20	1	1	1	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1
	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1
	1	1	1	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
10	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
k_1	0					5					10					15					20

The results of computation by the program, represented by 0 and 1 corresponding to the decision of stable and unstable, are shown in Table 2 and Table 3. The results coincide with the results by analogue computer in *litterateur*[5] fairly well.

Example 2. As the next example, let us consider the problem of finding the stable region of the Mathieu equation shown in equation (30).

$$\frac{dx^2}{dt^2}+(k_1+k_2\cos t)\,x=0 \tag{30}$$

In this case, the decision for the stability is especially easy to be affected by the computation error of system responses, because all of the system responses in the stable region are of critical mode. Therefore, in order to avoid this difficulty, the use of stabilizing factor a is desirable, although in principle the problem can be solved by the same method used in Example 1. Namely, equation (33) which is derived by substituting equation (32) into the characteristic equation of the system shown in equation (31) is used as the characteristic equation of the system. By using a which is just a little larger than $+1$, the tendency of the decision "unstable" due to the computation error can be adequately compensated.

$$f(Z)=0 \tag{31}$$

$$Z=aW \tag{32}$$

$$G(W)=0 \tag{33}$$

For $a=1.05$, the results of the decision by the method are shown in Table 4. This shows a good coincidence with the results given in *litterateur*[4,20,21,22]. Also, for $a=1$, the decision is always unstable. Here, the step of computation is $2\pi/100$, also T_0 and t_0 are as follows:

$$T_0=2\pi \tag{34}$$

$$t_0=0 \tag{35}$$

Example 3. As the next example, let us consider the stability of the system with a delay element shown in Fig. 6.

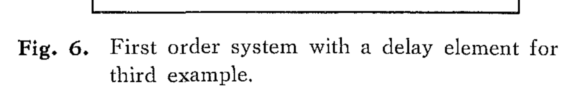

Fig. 6. First order system with a delay element for third example.

TABLE 4. The results of stability analysis for Equ. (30).

k_2

3.5-	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
3. -	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
2.5-	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2. -	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.5-	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	1	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1. -	1	1	1	1	1	1	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	1	1	1	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0.5-	1	1	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0. -	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0.					0.5					1.					1.5					2.					2.5					3.					3.5				

k_1

The results shown in Table 5 and Table 6 have been obtained respectively by the Padè approximation method and the staircase approximation method mentioned before. Also Table 7 shows the results which have been decided by the author from the computation of the system response over 10 periods of time varying coefficients.

From a comparison of the results, it is considered that the method approximating the initial function is of high accuracy compared with the method of the Padè approximation.

TABLE 5. The results of stability analysis for the system expressed in Fig. 6, which have been derived using the pade approximation, $e^{-SL}=\dfrac{6-2LS}{6+4LS+L^2S^2}$.

k_2													
10-	1	1	0	0	1	1	1	1	1	1	1	1	1
	1	1	0	0	1	1	1	1	1	1	1	1	1
8-	1	1	1	0	0	1	1	1	1	1	1	1	1
	1	1	1	0	0	1	1	1	1	1	1	1	1
6-	1	1	1	0	0	0	1	1	1	1	1	1	1
	1	1	1	1	0	0	1	1	1	1	1	1	1
4-	1	1	1	1	0	0	0	1	1	1	1	1	1
	1	1	1	1	0	0	0	0	1	1	1	1	1
2-	1	1	1	1	0	0	0	0	0	0	1	1	1
	1	1	1	1	0	0	0	0	0	0	1	1	1
0-	1	1	1	1	0	0	0	0	0	0	1	1	1
k_1	-5		-3		-1	0	1		3		5		7

TABLE 6. The results of stability analysis for the system expressed in Fig. 6, which have been derived using two steps staircase function approximation for initial functions.

k_2													
10-	1	1	1	0	0	0	1	1	1	1	1	1	1
	1	1	1	0	0	0	1	1	1	1	1	1	1
8-	1	1	0	0	0	0	1	1	1	1	1	1	1
	1	1	1	0	0	0	1	1	1	1	1	1	1
6-	1	1	1	1	0	0	1	1	1	1	1	1	1
	1	1	1	1	0	0	0	1	1	1	1	1	1
4-	1	1	1	1	0	0	0	1	1	1	1	1	1
	1	1	1	1	0	0	0	0	1	1	1	1	1
2-	1	1	1	1	0	0	0	0	1	0	1	1	1
	1	1	1	1	0	0	0	0	0	1	1	1	1
0-	1	1	1	1	0	0	0	0	0	1	1	1	1
k_1	-5		-3		-1	0	1		3		5		7

TABLE 7. The sesults of stability analysis for the system expressed in Fig. 6, which have been derived by long range digital computation and the author's prediction.

k_2													
10-	1	1	1	1	0	0	1	1	1	1	1	1	1
	1	1	1	1	0	0	1	1	1	1	1	1	1
8-	1	0	0	0	0	0	1	1	1	1	1	1	1
	1	1	0	0	0	0	0	1	1	1	1	1	1
6-	1	1	1	0	0	0	0	1	1	1	1	1	1
	1	1	1	0	0	0	0	1	1	1	1	1	1
4-	1	1	1	1	0	0	0	0	1	1	1	1	1
	1	1	1	1	0	0	0	0	0	0	1	1	1
2-	1	1	1	1	0	0	0	0	0	0	1	1	1
	1	1	1	1	0	0	0	0	0	0	1	1	1
0-	1	1	1	1	0	0	0	0	0	0	1	1	1
k_1	-5		-3		-1	0	1		3		5		7

Now, in the application of the method, the following remarks should be understood.

Remark 1. Almost all misjudgements introduced in the use of the method are attributed to computation errors of the system responses. Nevertheless, except for the special case as shown in Example 2, the use of a specially accurate computing method is not required.

Remark 2. In the special case shown in Example 2, the use of stabilizing factor a is desirable. Moreover, the use of a may be effective from a standpoint of rapid computation.

Remark 3. When the variation of coefficients is discontinuous, the system responses may also become discontinuous. In such a case, the time origin t_0 should be selected so as to avoid such a point from a standpoint of accuracy.

Conclusions

As a result of considerations for the theorem derived by R. E. Kalman and J. E. Bertram, a general method, although it is a numerical method, for the stability analysis of linear systems with periodic coefficients has been derived. The method is easily applicable to almost all linear systems, and is able to give automatically results of sufficiently high accuracy in accordance with the requirements.

In the use of the method, the selection of time origin t_0 and the use of stabilizing factor a should be especially heeded.

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