Stability of a Reactor with an External Control System

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Abstract

When a reactor is externally controlled, the parameters governing the control must be chosen in close connection with the nuclear and thermal properties and the out-put of the reactor, in order to attain a stability in the vicinity of the operating power.

To obtain this relation, non-linear integral equations containing the effects of fast and slow responding internal feedbacks and delayed neutrons are adopted and connected with the external control system. The method of introducing the special Liapunov's function which was developed by LUR'YE et al. is applied to the entire system and the required relation was obtained. The results of numerical caluculation are presented in the form of diagrams. These relations and diagrams will be usefull for the designing and operation of external control systems for reactors.

This paper also deals briefly with some modifications on the reactor dynamic equations to obtain a more concrete representation of the dynamical behaviour of heterogeneous reactors. It is desirable that the equations directly contain the effect of the moderator temperature, which consists of three thermal time constants and is the solution of heat equations between the heterogeneous elements of the reactor. The new equations are solved by expansion, and the region of stability was determined.

I. Introduction

When external control is applied to the nuclear reactor, the values given to the parameters which govern the external control must be properly chosen in close connection with the nuclear and thermal properties and the out-put of the reactor, in order to maintain the entire system, which now includes the external control system, in a stable condition.

To actually obtain the relations, the reactor kinetic equations of the non-linear integral equation type is first adopted. These contain the effects of two kinds of internal feedbacks and that of the delayed neutrons.

One side of the feedback responds to the variation of the reactor out-put relatively rapidly mainly due to the variation of the fuel temperature and the coolant density, and the other side responds relatively slowly, chiefly due to the variation of the moderator temperature.

Next, in order to obtain equations which describe the dynamical behaviour of the entire system, the above mentioned reactor kinetic equations are converted into non-linear simultaneous differential equations from which two specific equilibrium points can be obtained, one of which corresponds to the "zero power" and the other corresponds to the "non-zero power" of the reactor out-put. Then, these are joined to the equations which describe the dynamics of the variables of the external control system.

In order to examine the stability, especially near the operating power of the reactor, the system of equations is expanded in the vicinity of the equilibrium point which corresponds to the "non-zero power" and are converted to approximate linear simultaneous differential equations, by means of neglection of the terms of higher orders.

Then the methods of introducing the special Liapunov's function, which was developed by LUR'YE et al is applied to the linearlized system of equations and, as a result, the relations which are required to satisfy the stability of the entire system near the operating power of the reactor were obtained.

These relations were numerically caluculated by varying the values of the nuclear and thermal constants and the out-put of the reactor over a wide range, and the results are presented in the forms of diagrams. These relations and the diagrams are considered to be quite useful for the practical designing and the operation of the external control system of the reactor.

Other problems connected with the stability of the heterogeneous reactor are also discussed briefly in this paper. The dynamic equations of reactors with only internal feedbacks was dealt with by ANDREIEV et al, but in order to obtain the region of stability in a more concerete form, it would be desirable that the equations of the heat exchange between the heterogenous constructing elements are directly connected to the reactor kinetic equations. From this point of view, it was clarified in the dynamic equations that the internal feedback of a relatively slow response is governed chiefly by the temperature of the moderator which is given as the solution of the thermal equations and is expressed by a form which includes three thermal constants. These new equations are solved by a more expanded method and more concrete regions of stability are gained and shown. The case where the external control system is added to the above expanded reactor kinetic equations is also discussed briefly.

II. General Principles

(1) Dynamic equations of a reactor with internal feedbacks only

Taking into account the effects of the two kinds of internal feedbacks and the delayed neutrons represented by the equivalent one group, which were stated in the introduction, and under a condition where the out-put of the reactor does not change extremly rapidly, one can adopt the following reactor kinetic equations

$$w(t) = \frac{\beta}{\tau_0\{\beta-\rho(t)\}}\int_{-\infty}^{t} w(t')\exp\left(-\frac{t-t'}{\tau_0}\right)dt' \tag{1}$$

$$\rho(t) = \rho_0 + \mathrm{E}w(t) + F\int_{-\infty}^{t} w(t')\exp\left(-\frac{t-t'}{\tau}\right)dt' \ , \tag{2}$$

where,

$w(t)$ = out-put of the reactor,
$\rho(t)$ = reactivity at time t,
ρ_0 = reactivity at the time of the initial start up (cold reactor),
τ = thermal time constant of the reactor,
τ_0 = mean generating time of the delayed neutrons,
E = coefficient of the internal feedback which corresponds to relatively quick response,
F = coefficient of the internal feedback which corresponds to relatively slow response.

Now, by means of the substitutions, $w(t)=x_2^*$ and $\tau\dot{w}(t)=x_1$, Equations (1) and (2) are reduced to the following non-linear simultaneous differential equations

$$\left.\begin{aligned} \frac{dx_1}{dt} &= X(x_1,\ x_2^*) \\ \frac{dx_2^*}{dt} &= \frac{1}{\tau}x_1 \end{aligned}\right\} , \tag{3}$$

from which one can obtain the following two equilibrium points in the phase plane, one of which corresponds to the "zero power operation" and the other to the "non-zero power" operation of the reactor

Equilibrium point A: $(x_{10}=0,\quad x_{20}=0)$

Equilibrium point B: $(x_{1p}=0,\quad x_{2p}=-\rho_0/(E+F\tau))$.

Then by means of expansion of Equation (3) in the vicinity of the equilibrium point B, and the neglection of terms of higher orders, one obtains the following linear simultaneous differential equations

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 \\ \frac{dx_2}{dt} &= a_{21}x_1 \end{aligned}\right\}, \tag{4}$$

wherein x_2 is the variation of the reactor out-put from the equilibrium point denoted by

$$x_2 = x_2^* - x_{2p}, \tag{5}$$

and the coefficients are given by the following expressions

$$\left.\begin{aligned} a_{11} &= -\frac{\frac{\rho_0}{\beta}\left(ms+1\right)+1}{\left(\frac{\rho_0}{\beta}s+1\right)\tau} \\ a_{12} &= -\frac{\frac{\rho_0}{\beta}m}{\left(\frac{\rho_0}{\beta}s+1\right)\tau} \\ a_{21} &= \frac{1}{\tau} \\ a_{22} &= 0 \end{aligned}\right\}. \tag{6}$$

In equation (6), m and s are the abbreviations for

$$m = \tau/\tau_0, \qquad s = E/(E+F\tau). \tag{7}$$

If one views x_1, x_2 as the components of the 2-vector $\boldsymbol{x}$ and a_{11}, a_{12}, a_{21}, $a_{22}(=0)$ as the elements of the 2×2 matrix $\boldsymbol{A}$, Equation (5) can assume the following simpler aspect

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}. \tag{8}$$

The characteristic roots of Equation (4) and hence of Equation (8) are gained from the relation

$$|\boldsymbol{A} - \lambda\boldsymbol{I}| = 0, \tag{9}$$

and are represented by the following expression

$$\lambda=\frac{-\left[\frac{\rho_0}{\beta}(ms+1)+1\right]\pm\sqrt{\left[\frac{\rho_0}{\beta}(ms+1)+1\right]^2-4\frac{\rho_0}{\beta}m\left(\frac{\rho_0}{\beta}s+1\right)}}{2\tau\left(\frac{\rho_0}{\beta}s+1\right)} . \tag{10}$$

A reactor with internal feedbacks only can be ascertained to be stable near the equilibrium point which corresponds to the "non-zero power" operation by knowing the fact that both of the characteristic roots (10) have negative real parts.

(2) Dynamic equations of a reactor with an external control system

When an external control system is applied to the nuclear reactor, the equations describing the dynamical behaviour of the entire system near the equilibrium point, which corresponds to the "non-zero power" operation, are given by the following expressions

$$\left.\begin{aligned}&\frac{dx}{dt}=a_{11}x_1+a_{12}x_2+h_1\xi\\&\frac{dx_2}{dt}=a_{21}x_1\\&\frac{d\xi}{dt}=f(\sigma)\\&\sigma=g_1x_1+g_2x_2-r\xi\end{aligned}\right\} , \tag{11}$$

where ξ is the amount of control given to the reactor, σ is the signal by which the control mechanism is actuated and r, g_1, g_2 are the parameters of the control system whose values should be decided properly in the course of the designing and operation of the external control system. The function $f(\sigma)$ represents the characteristic of the servomoter and is assumed to possess the following properties

$$\sigma f(\sigma)>0 \qquad (\sigma \neq 0), \qquad f(0)=0 . \tag{12}$$

If one transforms x_1 and x_2 to the new variables y_1 and y_2 by the relations

$$y_1=\frac{dx_1}{dt}, \qquad y_2=\frac{dx_2}{dt}, \tag{13}$$

Equation (11) is converted into the the following equations

$$\left.\begin{aligned}\frac{dy_1}{dt}&=a_{11}y_1+a_{12}y_2+h_1f(\sigma)\\ \frac{dy_2}{dt}&=a_{21}y_1\\ \frac{d\sigma}{dt}&=g_1y_1+g_2y_2-rf(\sigma)\end{aligned}\right\}. \tag{14}$$

Equation (14) is also shown by a simpler aspect

$$\left.\begin{aligned}\frac{d\boldsymbol{y}}{dt}&=\boldsymbol{A}\boldsymbol{y}+\boldsymbol{h}f(\sigma)\\ \frac{d\sigma}{dt}&=\boldsymbol{g}'\boldsymbol{y}-rf(\sigma)\end{aligned}\right\}. \tag{15}$$

wherein $\boldsymbol{y}$ is the 2-vector whose components are y_1 and y_2 and both $\boldsymbol{h}$ and $\boldsymbol{g}$ are also 2-vectors which are denoted by

$$\boldsymbol{h}=\begin{bmatrix}h_1\\0\end{bmatrix},\qquad \boldsymbol{g}'=[g_1,\quad g_2]\,. \tag{16}$$

In this transformation of the variables, the stability situation in the new variables is the same as in the initial variables under the satisfication of the following condition

$$r+\boldsymbol{g}'\boldsymbol{A}^{-1}\boldsymbol{h}\neq 0\,. \tag{17}$$

Now, the special Liapunov's function which is denoted by the following expression

$$V(\boldsymbol{y},\sigma)=\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y}+\int_0^\sigma f(\sigma)d\sigma \tag{18}$$

is applied to Equation (14) or (15). If this function satisfies the following conditions

$$\left.\begin{aligned}&V(0,\,0)=0 &&\text{(a)}\\ &V(\boldsymbol{y},\,\sigma)>0 &&\text{(b)}\\ &\dot{V}(\boldsymbol{y},\,\sigma)\leqq 0 &&\text{(c)}\end{aligned}\right\} \tag{19}$$

near the equilibrium point corresponding to the "non-zero power" operation, the stability of the entire system near the equilibrium point is established under the satisfication of Condition (17). In Equation (18), the matrix $\boldsymbol{B}$ is an arbitrary positive symmetric matrix.

By virture of the fact that $f(\sigma)$ is restricted by Condition (12) and also the fact that the matrix $\boldsymbol{B}$ is the positive symmetric matrix, the Condition (a) and (b) in Equation (19) are obviously satisfied, and only the Condition (c) remains to be examined. Hence after differentiating Equation (18) and introducing a symmetric matrix $\boldsymbol{C}$ which is defined by the following relation

$$\boldsymbol{C} = -(\boldsymbol{A}'\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A}) = \boldsymbol{C}' \;, \tag{20}$$

one obtains

$$\dot{V} = -\boldsymbol{y}'\boldsymbol{C}\boldsymbol{y} - \gamma f(\sigma) + 2f(\sigma)\left(\boldsymbol{B}\boldsymbol{h} + \frac{1}{2}\boldsymbol{g}\right)'\boldsymbol{y} \leqq 0 \tag{21}$$

and hence,

$$r \geqq \left(\boldsymbol{B}\boldsymbol{h} + \frac{1}{2}\boldsymbol{g}\right)'\boldsymbol{C}^{-1}\left(\boldsymbol{B}\boldsymbol{h} + \frac{1}{2}\boldsymbol{g}\right) \tag{22}$$

is the necessary and sufficient condition to obtain the stability of the entire system.

(3) Derivation of the conditions required for the stability of the entire system

In order to maintain the reactor with an external control system in stable condition in the vicinity of the operationg power, the numerical parameters h_1, r, g_1 and g_2 must have suitable values which should be chosen in such a way as to be closely related with the nuclear and thermal properties and out-put of the reactor.

These relations can be derived in the following two noteworthly special cases; one is the case where all the characteristic roots of the reactor system are real and negative, and the other is the case were the characteristic roots form conjugate complex roots whose real parts are negative.

(a) The characteristic roots are all real and negative

We set the characteristic roots (10) as

$$\left.\begin{aligned} \lambda_1 &= -\alpha + \delta \\ \lambda_2 &= -\alpha - \delta \end{aligned}\right\} , \tag{23}$$

where α and δ are given by

$$\left.\begin{aligned} \alpha &= \frac{\dfrac{\rho_0}{\beta}(ms+1)+1}{2\tau\left(\dfrac{\rho_0}{\beta}s+1\right)} \\ \delta &= \frac{\sqrt{\left\{\dfrac{\rho_0}{\beta}(ms+1)+1\right\}^2 - 4\dfrac{\rho_0}{\beta}m\left(\dfrac{\rho_0}{\beta}s+1\right)}}{2\tau\left(\dfrac{\rho_0}{\beta}s+1\right)} \end{aligned}\right\} . \tag{24}$$

In Equation (24), the quantity under the square root sign is positive in this case.

If one uses the transformation of the coordinate $\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{x}$, choosing matrix $\boldsymbol{P}$ as

$$\boldsymbol{P} = \begin{bmatrix} -\lambda_1 & -\lambda_2 \\ -a_{21} & -a_{21} \end{bmatrix}, \tag{25}$$

the matrix $\boldsymbol{A}$ is reduced to the following diagonal matrix

$$\boldsymbol{A}^* = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \tag{26}$$

and the vectors $\boldsymbol{h}$ and $\boldsymbol{g}'$ are also transformed to the following vectors

$$\boldsymbol{h}^* = \boldsymbol{P}^{-1}\boldsymbol{h} = \begin{pmatrix} -\dfrac{h_1}{2\delta} \\ \dfrac{h_1}{2\delta} \end{pmatrix} \tag{27}$$

$$\boldsymbol{g}^* = \boldsymbol{g}'\boldsymbol{P} = \left[-g_1\lambda_1 - g_2 a_{21}, \quad -g_1\lambda_2 - g_2 a_{21} \right]. \tag{28}$$

Then, if one chooses an arbitrary diagonal matrix for $\boldsymbol{C}$, the matrix $\boldsymbol{B}$ is obtained likewise as the diagonal matrix from the Relation (20).

Now, in order to obtain the minimum value of r which satisfies the previously gained Relation (22), the values given in Equations (26), (27) and (28) and also the above mentioned matrix $\boldsymbol{B}^*$ and $\boldsymbol{C}^*$ are substituted into Equation (22). In the course of inquiry for the minimum value of r, all elements of the matrix $\boldsymbol{C}^*$ and $\boldsymbol{B}^*$ are eliminated and the required relation which satisfies the stability of the entire system is obtained as

$$\frac{r_{\min}}{h_1 g_2} \geqq \tau_0 \left[\frac{\beta}{\rho_0} + s \right]. \tag{29}$$

It is noticeable that Equation (29) does not contain parameter g_2. This fact means that the control signal depends on the amount of the variation but does not depend on the variational speed of the reactor out-put in this case.

(b) The characteristic roots are the conjugate complex whose real parts are negative

In this case we set the characteristic roots (10) as

$$\left.\begin{aligned} \lambda &= -\alpha + i\delta' \\ \bar{\lambda} &= -\alpha - i\delta' \end{aligned}\right\}, \tag{30}$$

where δ' is equivalent to $i\delta$.

Now, if one uses the first transformation of the coordinate

$$\boldsymbol{y} = \boldsymbol{P}_1\boldsymbol{z} \tag{31}$$

choosing the matrix $\boldsymbol{P}_1$ as

$$\boldsymbol{P}_1 = \begin{bmatrix} -\lambda & -\bar{\lambda} \\ -a_{21} & -a_{21} \end{bmatrix}, \tag{32}$$

the reactor system is converted into the following diagonal expressions

$$\begin{pmatrix} \dfrac{dz}{dt} \\ \dfrac{d\bar{z}}{dt} \end{pmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix}, \tag{33}$$

where z and $\bar{z}$ are the pairs of the conjugate complex variables denoted by

$$\left.\begin{array}{l} z = u_1 + iu_2 \\ \bar{z} = u - iu_2 \end{array}\right\}. \tag{34}$$

Next, by the introduction of the another matrix $\boldsymbol{P}_2$ denoted by

$$\boldsymbol{P}_2 = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}, \tag{35}$$

Equation (34) is represented by the following matrix form

$$\boldsymbol{z} = \boldsymbol{P}_2\boldsymbol{u}\,. \tag{36}$$

If one combines Equation (31) with Equation (36), one obtains the relation

$$\boldsymbol{y} = \boldsymbol{P}_1\boldsymbol{P}_2\boldsymbol{u}\,. \tag{37}$$

Equation (37) indicates that the matrix $\boldsymbol{P}_1\boldsymbol{P}_2$ is again the matrix which transforms the real coordinate $\boldsymbol{y}$ into the real coordinate $\boldsymbol{u}$ and therefore $\boldsymbol{P}_1\boldsymbol{P}_2$ is the real matrix. In transforming Equation (14) by this matrix $\boldsymbol{P}_1\boldsymbol{P}_2$, one obtains

$$\boldsymbol{A}^* = \begin{bmatrix} -\alpha & -\delta' \\ \delta' & -\alpha \end{bmatrix} \tag{38}$$

$$\boldsymbol{h}^* = \boldsymbol{P}_2^{-1}\boldsymbol{P}_1^{-1}\boldsymbol{h} = \begin{bmatrix} 0 \\ h_1/2\delta' \end{bmatrix} \tag{39}$$

$$\boldsymbol{g}^* = \boldsymbol{g}'\boldsymbol{P}_1\boldsymbol{P}_2 = \left[2(g_1\alpha - g_2/\tau),\ 2g_1\delta'\right]. \tag{40}$$

By virture of the special form of matrix $\boldsymbol{A}^*$, if one chooses an arbitrary diagonal matrix for $\boldsymbol{C}$, the matrix $\boldsymbol{B}$ becomes also the diagonal matrix.

Substituting Equations (38), (39) and (40) and also the above mentioned $\boldsymbol{B}^*$ and $\boldsymbol{C}^*$ into Equation (22), an inquiry for the minimum value of r which satisfies Condition (22) follows. In the course of inquiry for the minimum value, all elements of the matrix $\boldsymbol{B}^*$ and $\boldsymbol{C}^*$ are eliminated as before, and one can obtain the required relation to satisfy the stability of the entire system, namely

$$\frac{r_{\min}}{h_1 g_2} \geqq \tau_0 \frac{\left(\frac{\rho_0}{\beta}s+1\right)m}{\frac{\rho_0}{\beta}(ms+1)+1}\left(\sqrt{\frac{\left\{\frac{g_1}{g_2}(ms+1)-2s\right\}\frac{\rho_0}{\beta}+\frac{g_1}{g_2}-2}{4\frac{\rho_0}{\beta}\left(\frac{\rho_0}{\beta}s+1\right)m-\left[\frac{\rho_0}{\beta}(ms+1)+1\right]^2}+\left(\frac{g_1}{g_2}\right)^2}+\frac{g_1}{g_2}\right). \tag{41}$$

It is noticeable in this case that the relation includes g_1/g_2 as a parameter. This fact does mean that the control signal depends not only on the quantity of the variation but also on the variational speed of the reactor out-put. The value of g_1/g_2 is considered to play the roll of the "method of the external control".

(4) Improvement of the reactor kinetic equations

Although Equations (1) and (2) are fairly concrete equations which take into consideration the effects of the heterogenuity of the reactor, to make the argument more practical, it would be more desirable to take into account the phenomena of the heat exchange between the heterogeneous constructing element ostensibly into the kinetic equations of the reactor. In this respect, it is made clear in this section that the internal feedback of the relatively slow response is mainly governed by the temperature of the moedrator which is given by the solution of the thermal equations and hence includes three thermal time constants therein.

The equations of the heat exchange between the heterogeneous elements are given for the unit length of the representative cell by the following expressions

$$\left.\begin{aligned} C_m\frac{dT_m}{dt} &= \mu_m w(t)+H_3(T_C-T_m) \\ C_f\frac{dT_f}{dt} &= \mu_f w(t)+H_2(T_C-T_f) \\ C_C\frac{dT_C}{dt} &= H_2(T_f-T_C)+H_3(T_m-T_C)-\frac{u}{L}C_C(T_{C_{\rm out}}-T_{C_{\rm in}}) \\ T_C &\approx \frac{1}{2}(T_{C_{\rm out}}+T_{C_{\rm in}}) \end{aligned}\right\} \tag{42}$$
,

where,

T = statistically weighted mean temperature,
C = heat capacity per unit length along the channel,
$w(t)$ = power production per unit length along the channel,
μ = proportion of power generated,
H_2 = heat transfer coefficient between the fuel and the coolant,
H_3 = heat transfer coefficient between the moderator and the coolant,
u = coolant velocity,
L = channel length,

and the suffixes f, m and c are related to fuel, moderator and coolant respectively.

Solving Equation (42), one obtains the temperature of moderator as

$$T_m = \sum_{i=1}^{3} B_i \int_0^t w(t')\exp\left(-\frac{t-t'}{\tau_i}\right)dt' , \tag{43}$$

where, B_i is the constant which is given as the function of only the thermal properties of the reactor.

Assuming that this temperature contributes mainly to the internal feedback of the relatively slow response, one obtains new reactor kinetic equations which contain the effects of the heat exchanges between the heterogeneous elements

$$w(t) = \frac{\beta}{\tau_0[\beta-\rho(t)]}\int_0^t w(t')\exp\left(-\frac{t-t'}{\tau_0}\right)dt' \tag{44}$$

$$\rho(t) = \rho_0 + Ew(t) + F'\int_0^t \sum_{i=1}^{3} B_i w(t')\exp\left(-\frac{t-t'}{\tau_i}\right)dt' . \tag{45}$$

By means of the following substitutions

$$w(t) = x_4^* , \quad \dot{w}(t) = x_3 , \quad \ddot{w}(t) = x_2 , \quad \dddot{w}(t) = x_1 , \tag{46}$$

one can reduce Equation (44) and (45) into the following non-linear simultaneous differential equations

$$\left.\begin{aligned} \frac{dx_1}{dt} &= \frac{f(x_1, x_2, x_3, x_4^*)}{g(x_1, x_2, x_3, x_4^*)} \\ \frac{dx_2}{dt} &= x_1 \\ \frac{dx_3}{dt} &= x_2 \\ \frac{dx_4^*}{dt} &= x_3 \end{aligned}\right\} , \tag{47}$$

where, $f(x_1, x_2, x_3, x_4^*)$ and $g(x_1, x_2, x_3, x_4^*)$ are the polynomials regarding the four variables x_1, x_2, x_3 and x_4^*, and the maximum orders of them are six and five respectively. The coefficients of these polinomials are all obtained as the function of the nuclear and thermal properties and the out-put of the reactor.

From Equation (47), one obtains the following two equilibrium points in the (x_1, x_2, x_3, x_4^*) space, one of which corresponds to the "zero power" and the other corresponds to the "non-zero power" of the reactor out-put, namely

$$\text{Equilibrium point A}': \quad (x_{10}=0, \quad x_{20}=0, \quad x_{30}=0, \quad x_{40}=0)$$

$$\text{Equilibrium point B}': \quad (x_{1p}=0, \quad x_{2p}=0, \quad x_{3p}=0, \quad x_{4p}=-\alpha_5/\alpha_1).$$

In the above equilibrium point, α_1 and α_5 are the coefficients of the terms of x_4^{*6} and x_4^{*5} in the polynomial f respectively.

Now, expanding Equation (47) near the equilibrium point which corresponds to the "non-zero power" operation, and neglecting the terms of the higher orders, one obtains the following simultaneous linear differential equations

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \\ \frac{dx_4}{dt} \end{pmatrix} = \begin{pmatrix} -\gamma_1 & -\gamma_2 & -\gamma_3 & -\gamma_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{48}$$

where, x_4 is the amount of variation of the reactor out-put from the equilibrium point B′ denoted by

$$x_4 = x_4^* - x_{4p}, \tag{49}$$

and the coefficients γ_1, γ_2, γ_3 and γ_4 are given by the following expressions

$$\left.\begin{aligned} -\gamma_1 &= \frac{\alpha_4 x_{4p} + \alpha_{14}}{\beta_1 x_{4p} + \beta_4} \\ -\gamma_2 &= \frac{\alpha_3 x_{4p} + \alpha_{12}}{\beta_1 x_{4p} + \beta_4} \\ -\gamma_3 &= \frac{\alpha_2 x_{4p} + \alpha_9}{\beta_1 x_{4p} + \beta_4} \\ -\gamma_4 &= \frac{-\alpha_5}{\beta_1 x_{4p} + \beta_4} \end{aligned}\right\}. \tag{50}$$

In Equation (50), α_2, α_3, α_4, α_9, α_{12} and α_{14} are the coefficients of the terms of $x_4^{*^5}x_3$, $x_4^{*^5}x_2$, $x_4^{*^5}x_3$, $x_4^{*^4}x_3$, $x_4^{*^4}x_2$ and $x_4^{*^4}x_1$ in the polynomial f, and β_1 and β_4 are the coefficients of the terms of $x_4^{*^5}$ and $x_4^{*^4}$ in the polynomial g respectively.

One can judge the stability of the system by ascertaining that all the characteristic roots of Equation (48) and hence the roots of the following algebraic equation

$$\lambda^4+\gamma_1\lambda^3+\gamma_2\lambda^2+\gamma_3\lambda+\gamma_4=0 \tag{51}$$

have negative real parts.

It is not always necessary to obtain the characteristic roots of Equation (48), if one applies the "Condition to be the Hurwitz's polinomial" to Equation (51). The necessary and sufficient condition by which Equation (51) becomes the Hurwitz's polinomial is given by the following expressions

$$\left.\begin{array}{ll} \gamma_1>0 & \text{(a)} \\ \gamma_2>0 & \text{(b)} \\ \gamma_3>0 & \text{(c)} \\ \gamma_4>0 & \text{(d)} \\ \gamma_2^2-4\gamma_4>0 & \text{(e)} \\ \gamma_1\gamma_2\gamma_3-\gamma_3^2-\gamma_1^2\gamma_4>0 & \text{(f)} \end{array}\right\} . \tag{52}$$

By the investigation of Equation (52), one can judge the statility of the heterogeneous reactor relatively easily.

Because the constants γ_1, γ_2, γ_3 and γ_4 are given as slightly complicated functions of the reactor properties, it is desirable to use some approximation by which the regions of stability can be observed directly from the nuclear and thermal constants and the out-put of the reactor.

Now, assuming that the following relation exists among the heat capacities of the heterogeneous constructing elements, which is a relation generally satisfied in most reactors (especially in the gas cooled reactor), namely

$$C_m \gg C_f \gg C_C\,, \tag{53}$$

the γ in Equation (52) can be expressed in extremly simple forms. If the signs of inequality in Equation (52) are replaced by the signs of equality, the curves which show the boundary of the stable region are obtained. It is obvious that the region which is located on the stable sides of all of the curves is the true region of stability. After making some simplications, the conditions (c) and (d) in Equation (52) become quite identical conditions, and the curves corresponding to (a), (b), (c), (d) and (e) are reduced to the following simple hyperbolas

$$\left.\begin{array}{llll}\dfrac{\rho_0}{\beta}=-\dfrac{1}{s^*} & \text{(a)} & \dfrac{\rho_0}{\beta}=\dfrac{-1}{Gs^*+H} & \text{(c, e)} \\ \dfrac{\rho_0}{\beta}=\dfrac{-1}{Is^*+J} & \text{(b)} & \dfrac{\rho_0}{\beta}=\dfrac{-1}{Ls^*+M} & \text{(d)}\end{array}\right\} , \tag{54}$$

where s^* is the constant which is given by the following expanded form of s in Equation (7)

$$s^*=E\Big/\Big(E+\sum_{i=1}^{3}B_i\tau_iF'\Big), \tag{55}$$

and H, G, J, I, L and M are given by the following expressions

$$\left.\begin{aligned}
H&=\frac{\tau_2}{\sum_{i=1}^{3}\tau_iB_i}\left\{\frac{2\tau_2}{\tau_2-\tau_0}\cdot\frac{\tau_0}{\tau_1}B_1+\frac{3\tau_2-\tau_0}{\tau_2-\tau_0}B_2+\frac{-2\tau_0^2+\tau_0\tau_2-\tau_2^2}{\tau_2(\tau_2-\tau_0)}B_3\right\}\\
G&=1+\frac{\tau_2}{\tau_0}H\\
J&=\frac{\tau_3}{\sum_{i=1}^{3}\tau_iB_i}\left\{\frac{8\tau_2^2}{\tau_0(\tau_2-\tau_0)}B_1+\frac{\tau_2}{\tau_0}B_2+\frac{2\tau_0^2+\tau_2^2}{\tau_0(\tau_2-\tau_0)}B_3\right\}\\
I&=\frac{\tau_3}{\tau_0}-J\\
M&=\frac{1}{\sum_{i=1}^{3}\tau_iB_i}\left\{\tau_1B_1+\left(\tau_1+\frac{2\tau_2}{\tau_2-\tau_0}\cdot\frac{\tau_0^2}{\tau_3}\right)B_2+\frac{2\tau_0^2}{\tau_2-\tau_0}B_3\right\}\\
L&=1+M
\end{aligned}\right\} . \tag{56}$$

In this simplification the reactor out-put which corresponds to the equilibrium point B' is obtained as

$$x_{4p}=-\alpha_5/\alpha_1\approx-\rho_0\Big/\Big(E+\sum_{i=1}^{3}B_i\tau_iF'\Big), \tag{57}$$

and is recognized to have the expanded form of x_{2p} which have been defined in II. (1).

The curve which corresponds to (f) in Equation (54) is obtained after the simplification as follows

$$a_1\left[\frac{\beta}{\rho_0}\cdot\frac{1}{s^*}\right]^3+\left(a_2+\frac{b_2}{s^*}\right)\left[\frac{\beta}{\rho_0}\cdot\frac{1}{s^*}\right]^2$$
$$+\left(a_3+\frac{b_3}{s^*}+\frac{c_3}{s^{*2}}\right)\left[\frac{\beta}{\rho_0}\cdot\frac{1}{s^*}\right]+\left(a_4+\frac{b_4}{s^*}+\frac{c_4}{s^{*2}}+\frac{d_4}{s^{*3}}\right)=0, \quad (58)$$

wherein a_i, b_i, c_i, and d_i are the constants whose values are obtained as the relatively simple functions of the thermal time constants and the mean generating time of the delayed neutrons.

It can be recognized that the curve given by Equation (58) is approximized by the hyperbola in the region where s^* has a relatively large value. In addition, under the assumption that the Condition (53) holds, the coefficient a_1 becomes negligibly small and one can neglect the term of the third order in the region of the practically conceivable values of s^* and ρ_0/β. Thus one can reduce Equation (59) into a more simple form

$$\frac{\rho_0}{\beta}\approx\frac{1}{s^*}$$
$$\cdot\frac{2\left(a_2+\frac{b_2}{s^*}\right)}{-\left(a_3+\frac{b_3}{s^*}+\frac{c_3}{s^{*2}}\right)\pm\sqrt{\left(a_3+\frac{b_3}{s^*}+\frac{c_3}{s^{*2}}\right)^2-4\left(a_2+\frac{b_2}{s^*}\right)\left(a_4+\frac{b_4}{s^*}+\frac{c_4}{s^{*2}}+\frac{d_4}{s^{*3}}\right)}}, \quad (59)$$

from which one can obtain the curve directly.

Now, in order to deal with the cases where the external control system is applied to the system denoted by Equation (48), one must obtain the four characteristic roots solving Equation (51) directly. If one deals with only the cases where the real parts of all the roots are negative, one can use the methods applied in II. (2), and obtain the relations which are required to attain the stability of the entire system.

III. Numerical Caluculation and Discussion

Equation (29) and (41) are numerically caluculated by varying the values of the nuclear and thermal constants and the out put of the reactor over a wide range, and the results are shown in Fig. 1~Fig. 5.

Fig. 1 corresponds to the case in which both of the characteristic roots are real and negative. This diagram shows the minimum value of r/h_1g_1 which is required to satisfy the stability of the entire system as the function of ρ_0/β. Since the operating power of the reactor is given by $-\rho_0/(E+F\tau)$, ρ_0/β is the value which is proportional to the reactor out-put. Fig. 1 shows that the

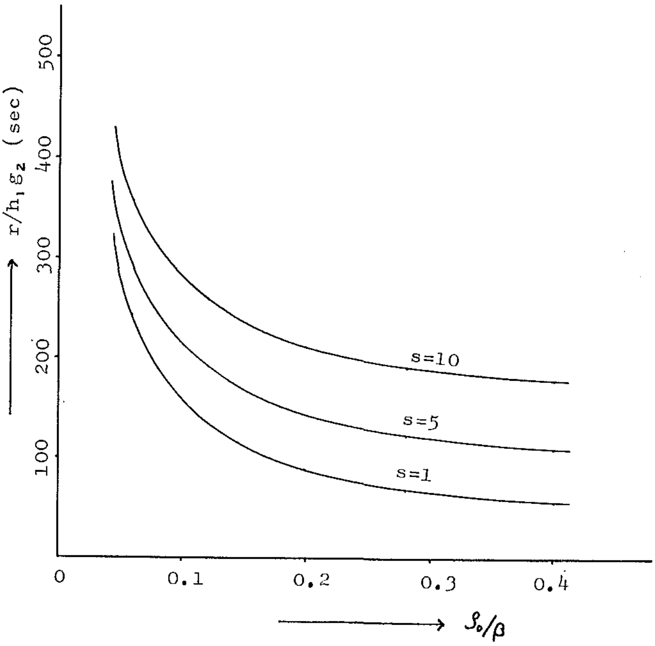

Fig. 1. Minimum Value of r/h_1g_2 vs. ρ_0/β in the Case of Non-oscillatory Stable.

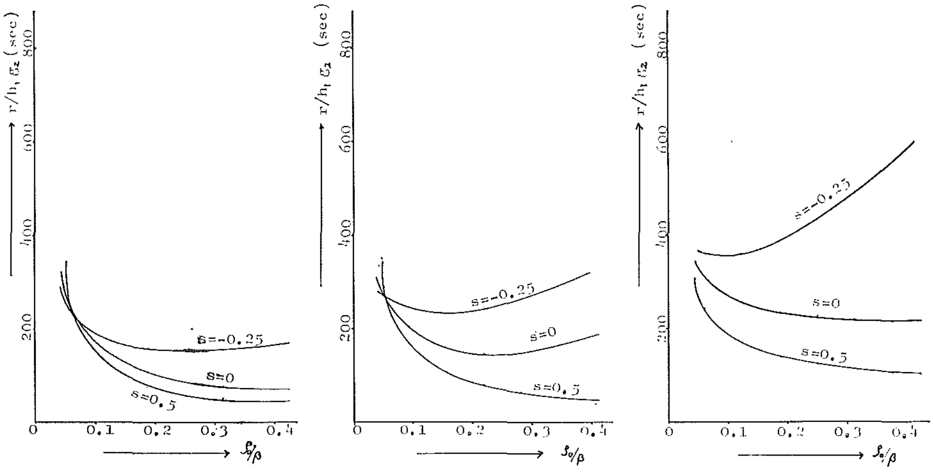

Fig. 2. Minimum Value of r/hg vs. ρ_0/β in the Case of Oscillatory Stable.

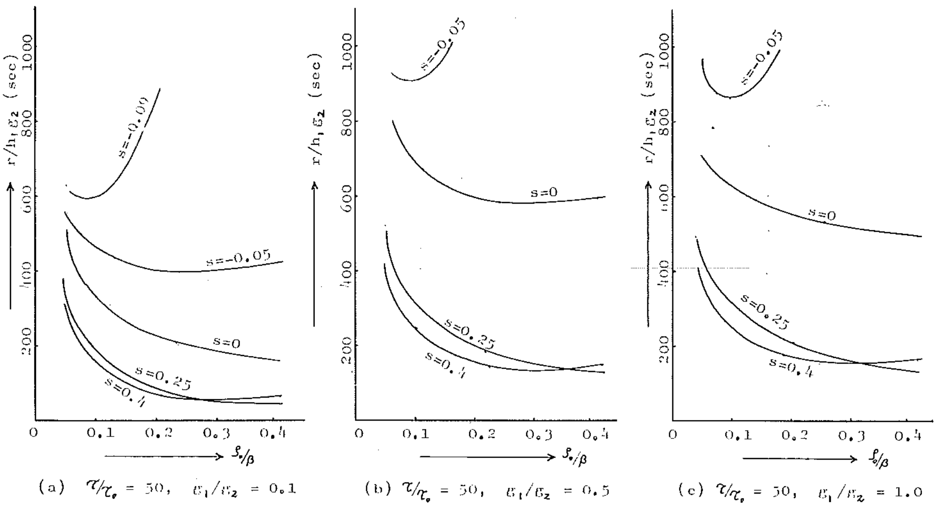

Fig. 3. Minimum Value of r/h_1g_2 vs. ρ_0/β in the Case of Oscillatory Stable.

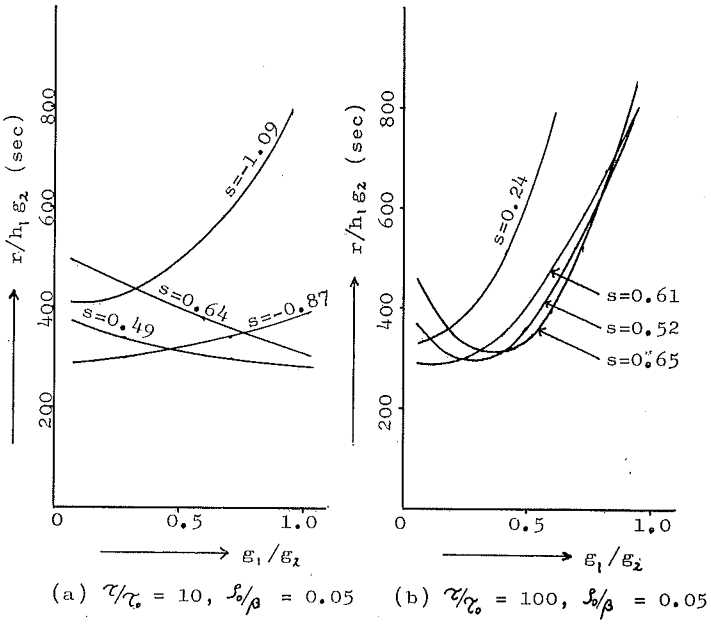

Fig. 4. Minimum Value of r/h_1g_2 vs. g_1/g_2 in the Case of Oscillatory Stable.

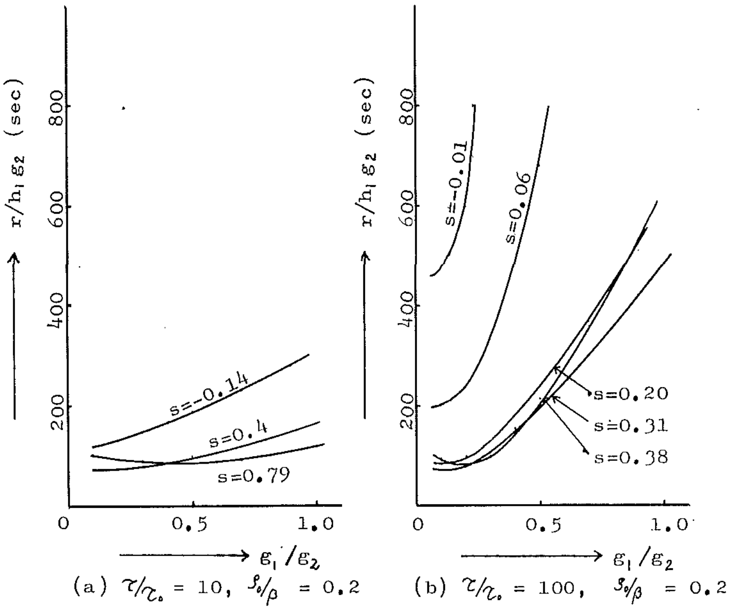

Fig. 5. Minimum Value of r/h_1g_2 vs. g_1/g_2 in the Case of Oscillatory Stable.

increase of the operating power gives a deeper depth of the region of stability.

The diagrams shown in Fig. 2~Fig. 5 correspond to the cases in which the characteristic roots are the complex conjugate whose real parts are negative, and hence the cases in which the reactor is oscillatory stable near the equilibrium point corresponding to the power operation. Fig. 2 and Fig. 3 show the minimum value of r/h_1g_1 versus ρ_0/β, taking τ/τ_0 and g_1/g_2 as the numerical parameters. In these cases $s(=E/(E+F\tau))$ has a value which is restricted to a relatively narrow range, and the smaller the value of s becomes, the narrower the region of stability becomes. This reduction of the region of stability in the smaller value of s is extreamly remarkable when the thermal time constant of the reactor becomes large. Fig. 4 and Fig. 5 show the region of stability, taking g_1/g_2 as the transverse axis and ρ_0/β and τ/τ_0 as the numerical parameters. Fig. 4 and Fig. 5 correspond to the cases in which the values of ρ_0/β are 0.05 and 0.2 respectively. These curves indicate the fact that there exists a properly chosen value of g_1/g_2 which minimizes the value of r/h_1g_2 existing on the boundary line, and this fact is considered to be quite important in the designing and operation of the external control of the reactor. In these diagrams one also notices the fact that, in the case of the larger thermal time constant of the reactor, the inclination of the curves become extreamly sharp and a small variation of the g_1/g_2 produces a large variation of the depth of the region of the stability. This is also considered to be an important problem which might affect the stability of the reactor system.

Equations (54) and (59) are numerically caluculated for the actual natural uranium gas cooled type reactor, and the results are shown in Fig. 6. The stable region is given as the region which is situated on the stable sides of all of the five lines (a), (b), (c, e), (d) and (f) which are mutually crossed. This practical method allows also the caluculation of the effect of the coolant speed upon the depth of the stable region and the result is shown in Fig. 7.

In summing up, when one applies an external control system to a reactor with internal feedbacks, one must choose parameters governing the control in

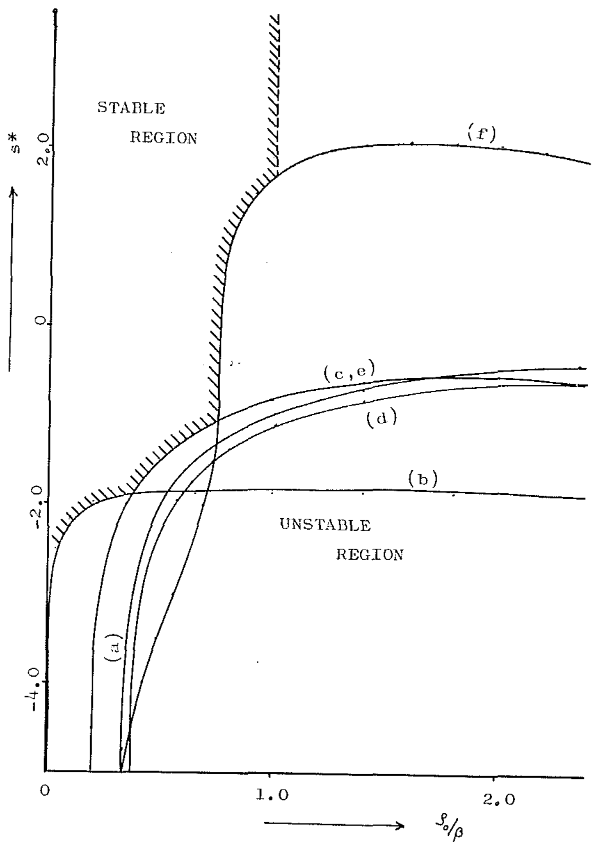

Fig. 6. Stable Region of the Reactor with Internal Feedbacks only.

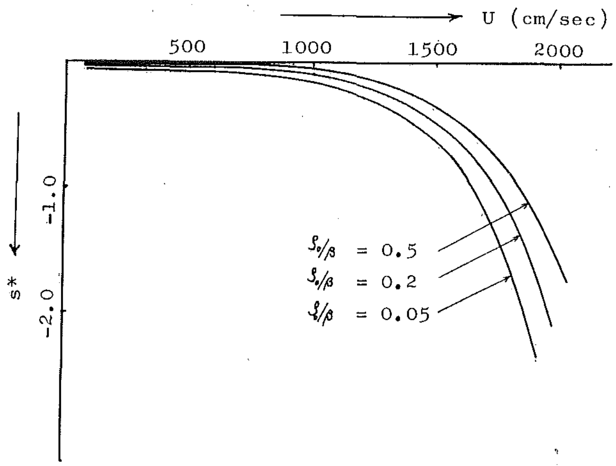

Fig. 7. Effect of the Coolant Speed upon the Reactor Stabillity.

close relation with the nuclear and thermal properties and the out-put of the reactor, in order to attain stability in the vicinity of the operating power. This fact should be considered specially in the course of the designing and operation of the external control of the reactor, and this paper gives an estimation of the problems.

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