Application of the Maximum Principle to a Continuous Path Determination Problem

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1. Introduction

Recently some technical methods based on the maximum principle have been developed. Most of them, however, seem to require a considerably extensive computer system to determine the initial conditions of the auxiliary vectors in comparison with the conventional PID control method, and seem to lack in investigations on the relationship between the cost required for such a computer system and the benefit derived from this type of control system.

To clarify these problematic points, it is necessary to investigate the problems: i. e. the operational time of the computer system, identification of the plant, non-linearity of control elements, disturbances to the control process, correction of errors resulting from component apparatus of the system, etc.

The authors have developed a unique method by which the optimal control technique in the linear system by Pearson and Chaudhuri et al.[3,4] is applied to the general solution of the system derived by the Pontryagin's existence theorem and thereby the time (T) required for the condition of the plant to be changed optimally to the target value is obtained directly. That is, a method has been developed by which T is increased or decreased in accordance with the determination of error function† as to its sign which varies with varying control process and thereby the optimal value is obtained, in an attempt to simplify the computer system involved. As a result, the optimal condition can be computed in a comparatively simple manner on a digital computer. Furthermore, it was found that an optimal control is possible, in principle, also with the analog techinque without necessarily resorting to the digital technique and with no loss of generality.

Then, as an example, the application of this control system to the submarine depth control problem was shown, together with investigations made to verify the validity. The following is a detailed report of this subject.

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† $\varepsilon(T)$ in the report.

2. System description and analysis

The maximum principle was established basically by Pontryagin in 1956. Since then many theories concerning this principle have been developed. In practice, however, most of them require an extensive computer system to determine the initial values of auxiliary variable vectors which determine optimal values of the system and can not replace the PID control system in the general process control.

The purpose of this paper is to present a method to obtain by the zero method, initial conditions of auxiliary variable vectors which are encountered in the application of the maximum principle to the general process control, to eliminate the complexity of the computer system and to derive a procedure for controlling with sequential correction of any deviation of the path which may occur during control operation.

The characteristic of the plant is assumed to be described by an n-th order linear process variable vector as shown in Eq. (1).

$$\left.\begin{aligned} &\dot{X} = AX + BU = f(X, U) \\ \text{or} \\ &\dot{x}_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}u_j = f_i(X, U), \qquad i = 1, 2, \cdots, n \end{aligned}\right\} \tag{1}$$

Now consider the case where the process changes between $t=0$ and $t=T$. Since the generality of the theory is not lost by taking $X(T)$ at $t=T$ at the origin of the coordinates, let $X(T)=0$, and the cost function is generally given as

$$J = \int_0^T F(X, U)\,dt \tag{2}$$

Then, control U that makes J minimum can be obtained by controlling H given by the following equation so that it is always maximum by the maximum principle.

$$H = \sum_{i=0}^{n} \psi_i f_i(X, U) - F(X, U) \tag{3}$$

where $\quad \dot{\psi}_i = -\dfrac{\partial H}{\partial x_i}, \quad i = 1, 2, 3, \cdots, n$

Eq. (3) is rewritten in vector form as

$$H = \Psi^{TR} f(X, U) - F(X, U) \tag{4}$$

The cost function J is given as

$$J=\int_0^T (X^{TR}CX+\lambda\|U\|^2)\,dt \tag{5}$$

where X^{TR} : transposed matrix of X
C : constant matrix
λ : constant
$\|U\|^2=U_1^2+U_2^2+\cdots+U_m^2$

Substituting Eq. (1) and Eq. (5) into Eq. (4) gives

$$H=\Psi^{TR}(AX+BU)-(X^{TR}CX+\lambda\|U\|^2) \tag{6}$$

The necessary condition for making H maximum with respect to U is obtained by differentiating Eq. (6) by U and finding such U that satisfies $\partial H/\partial U=0$. Therefore, the optimal value of U, U^0, should satisfy

$$U^0=\frac{1}{2\lambda}B^{TR}\Psi \tag{7}$$

On the other hand, from Eq. (4)

$$\dot{\Psi}=-A^{TR}\Psi+2CX \tag{8}$$

Denote initial conditions of X and Ψ by $X(0)$ and $\Psi(0)$ and let $\Psi(t)$ and $R(t)$ be the fundamental matrixes of $\dot{X}=AX$ and $\dot{\Psi}=-A^{TR}\Psi$, respectively, then the following relations hold true.

$$\dot{\Phi}(t)=A\Phi(t),\quad \Phi(0)=[1],\quad X(t)=\Phi(t)\,X(0) \tag{9}$$

$$\dot{R}(t)=-A^{TR}R(t),\quad R(0)=[1],\quad \Psi(t)=R(t)\,\Psi(0) \tag{10}$$

$$\Phi^{-1}(t)=R^{TR}(t) \tag{11}$$

Denoting the solution of Eq. (1) by the fundamental matrixes (1) gives

$$X(t)=\Phi(t)\Big(X(0)+\int_0^t \Phi^{-1}(t)\,BUdt\Big) \tag{12}$$

Under the optimal control, $X(T)=0$ at $t=T$, hence the following equation

$$-\Phi(T)\,X(0)=\Phi(T)\int_0^T \Phi^{-1}(t)\,BU^0(t)\,dt \tag{13}$$

The scalar product of Eq. (13) and $X(0)\Psi^{-1}(T)$ is given as

$$-X(0)\cdot X(0)=\int_0^T X(0)\cdot\Phi^{-1}(t)\,BU^0(t)\,dt \tag{14}$$

Substituting Eq. (7) and Eq. (11) into this gives

$$-X(0)\cdot X(0)=\frac{1}{2\lambda}\int_0^T X(0)\cdot R^{TR}(t)\,BB^{TR}\Psi(t)\,dt \tag{15}$$

Substitute Eq. (7) into Eq. (1) and solve it simultaneously with Eq. (8), then we have

$$\begin{bmatrix} \dot{X} \\ \dot{\Psi} \end{bmatrix} = \begin{bmatrix} A & \frac{1}{2\lambda} BB^{TR} \\ 2C & -A^{TR} \end{bmatrix} \begin{bmatrix} X \\ \Psi \end{bmatrix} \tag{16}$$

Denote the fundamental matrix of this equation by $V(t)$, and Eq. (17) holds true

$$\begin{bmatrix} X(t) \\ \Psi(t) \end{bmatrix} = V(t) \begin{bmatrix} X(0) \\ \Psi(0) \end{bmatrix} \tag{17}$$

where $V(t)$ is represented by a matrix of $2n \times 2n$. Denote each element of $V(t)$ by v_{ij} and put $V(t)$ as

$$\left.\begin{array}{lll} v_{ij}^{xx} = v_{ij}\,, & v_{ij}^{x\phi} = v_{ij+n} & \\ v_{ij}^{\phi x} = v_{i+nj}\,, & v_{ij}^{\phi\phi} = v_{i+nj+n}\,, & i, j = 1, 2, 3, \cdots, n \end{array}\right\} \tag{18}$$

and

$$V(t) = \begin{bmatrix} V_{xx}(t) & V_{x\phi}(t) \\ V_{\phi x}(t) & V_{\phi\phi}(t) \end{bmatrix} \tag{19}$$

then

$$X(t) = V_{xx}(t)\, X(0) + V_{x\phi}(t)\, \Psi(0) \tag{20}$$

$$\Psi(t) = V_{\phi x}(t)\, X(0) + V_{\phi\phi}(t)\, \Psi(0) \tag{21}$$

Putting $t = T$ in Eq. (20)

$$\begin{aligned} V_{x\phi}(T)\, \Psi(0) &= -V_{xx}(T)\, X(0) \\ \Psi(0) &= -V_{x\phi}^{-1}(T)\, V_{xx}(T)\, X(0) \end{aligned} \tag{22}$$

as $X(T) = 0$, which shows the relation between $\Psi(0)$ and $X(0)$.

Substituting Eq. (21) into Eq. (15) gives Eq. (23).

$$-X(0) \cdot X(0) = \frac{1}{2\lambda} \int_0^T X(0) \cdot R^{TR}(t)\, BB^{TR} \left(V_{\phi t}(t)\, X(0) + V_{\phi\phi}(t)\, \Psi(0) \right) dt \tag{23}$$

Now put $R(t) = [r_{ij}]$ and Eq. (24) holds true.

$$\begin{aligned} -2\lambda \sum_{j=1}^{n} \left(x_j(0) \right)^2 = \int_0^T \sum_{i=1}^{n} x_i(0) \Big(& \sum_{j=1}^{n} x_j(0) \sum_{m=1}^{n} v_{mj}^{\phi x} \sum_{l=1}^{n} r_{li} \sum_{k=1}^{n} b_{lk} \cdot b_{mk} \\ & + \sum_{j=1}^{n} \phi_j(0) \sum_{m=1}^{n} v_{mj}^{\phi\phi} \sum_{l=1}^{n} r_{li} \sum_{k=1}^{n} b_{lk} \cdot b_{mk} \Big) dt \end{aligned} \tag{24}$$

Therefore, putting

$$w_{ij}^{\phi x}(T)=\int_0^T \sum_{m=1}^{n} v_{mj}^{\phi x} \sum_{l=1}^{n} r_{li} \sum_{k=1}^{n} b_{lk} b_{mk} dt \tag{25}$$

$$w_{ij}^{\phi\phi}(T)=\int_0^T \sum_{m=1}^{n} v_{mj}^{\phi\phi} \sum_{l=1}^{n} r_{li} \sum_{k=1}^{n} b_{lk} b_{mk} dt \tag{26}$$

$$\left[w_{ij}(T)\right]=V_{x\phi}^{-1}(T)\, V_{xx}(T) \tag{27}$$

gives Eq. (28) and Eq. (29).

$$-2\lambda \sum_{j=1}^{n}\left(x_j(0)\right)^2=\sum_{i=1}^{n} x_i(0)\left(\sum_{j=1}^{n} x_j(0)\, w_{ij}^{\phi x}(T)+\sum_{j=1}^{n} \phi_i(0)\, w_{ij}^{\phi\phi}(T)\right) \tag{28}$$

$$\phi_j(0)=-\sum_{p=1}^{n} x_p(0)\, w_{jp}(T) \tag{29}$$

Since Eqs. (25) through (27) are determined, as the characteristic of the plant is determined, substituting Eqs. (25) through (27) for the characteristic of the plant is permitted. Thus, the initial conditions based on the maximum principle can be obtained easily by obtaining T or $\Psi_j(0)$ at the time when the solution $\Psi_j(0)$ of Eq. (29) satisfies Eq. (28) as T is increased.

3. Application to the actual control system

For the purpose of this article, a control system may be divided into blocks as shown in Fig. 1. The problem considered here is that the coefficient of Eq. (1) should be obtained from the signal given by the input of the plant within the block marked Identification, and the system be controlled to eliminate the difference between the output $X(t)$ and the set value $X_I(0)$ when the process of computing $w_{ij}^{\phi x}(T)$, $w_{ij}^{\phi\phi}(T)$, $w_{ij}(T)$ (where $i,j=1, 2, 3, \cdots, n$,

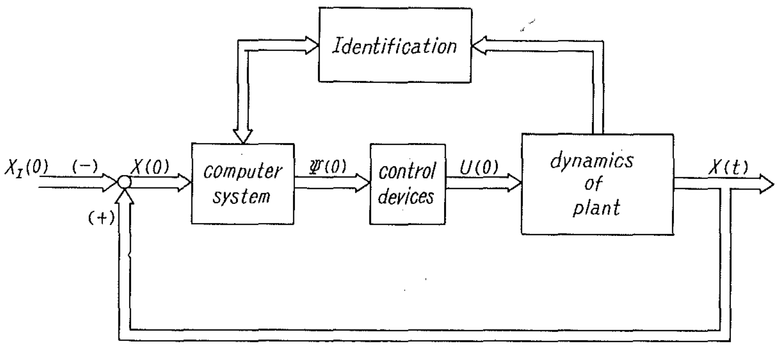

Fig. 1. Block diagram of general control system

Note: $X(0)=X(t)-X_I(0)$

$X(t)$: out put vector

$X_I(t)$: in put vector

$T=0, \Delta t, 2\Delta t, \cdots, p\Delta t, \cdots, n\Delta t)$ is in effect. Since the generality of error $X(0)$ which occurs when the set value $X_I(0)$ is given is not lost if the end value of $X(t)$, $X(T)$, is taken at the origin of the coordinates, it follows that

$$X(0) = -\left(X_I(0) - X(t)\right) \tag{30}$$

If $X(0)$ is given as the input to the computer system, $\Psi_j(0)$ is obtained by simultaneous solution of Eqs. (28) and (29), and by calculating it as the initial condition for Eq. (16), the system is controlled to satisfy Eq. (7), then it is the optimal control. However, the actual control is affected by indentification of the plant, non-linearity of control elements, disturbances, etc. and it is very difficult in many cases to carry out the control as computed. At this point, the problem of path determination arises which carries out the correction of initial condition $X(0)$ sequentially. The block diagram of Fig. 1 illustrates this method. The computer speed is increased to determine control $U(0)$ by the initial condition $\Psi(0)$ at the present time, and then similarly, by a new initial condition at the next time. This will be described in the following.

3.1. Determination of optimal time by error function

If the optimal time T^0 is given in Eq. (29), it is immediately possible to obtain $\Psi(0)$ from $X(0)$. In general, however, the value of T^0 which must satisfy Eq. (28) can not be obtained easily. Now, put arbitrary T in Eq. (23) and denote the resulting difference between the right and left side (hereinafter referred to as error function) by $\varepsilon(T)$. Let $\varepsilon(T)$ be

$$\varepsilon(T) = X(0)\,X(0) + \frac{1}{2\lambda}\int_0^T X(0)\cdot R^{TR}(t)\,BB^{TR}\left(V_{\psi x}(t)\,X(0) + V_{\psi\psi}(t)\,\Psi(0)\right)dt \tag{31}$$

then, $\varepsilon(T)$ is zero $T=T^0$ and $\varepsilon(T)$ must not be equal to zero for any T meeting the relation

$$0 < T < T^0 \tag{32}$$

Because, if $T=T_1$ which satisfies Eq. (32) exists and $\varepsilon(T_1)=0$, it means that a control with a control interval shorter than T^0 is possible and this fact is inconsistent with the optimal time T^0. If, therefore, T is taken for the abscissa and $\varepsilon(T)$ for the ordinate as shown in Fig. 2, $\varepsilon(T)$ intersects the abscissa only at $T=T^0$ for any T which satisfies Eq. (32). Therefore, the sign of $\varepsilon(T)$ is inverted between $T=T^0+\Delta t$, that is, when T is changed slightly in the positive direction from optimal time $T=T^0$ and $T=T^0-\Delta t$, that is, when the change is in the negative direction. The sign of $\varepsilon(T)$ is not inverted for $0<T<T^0$ as mentioned above. Thus, the following equation holds true.

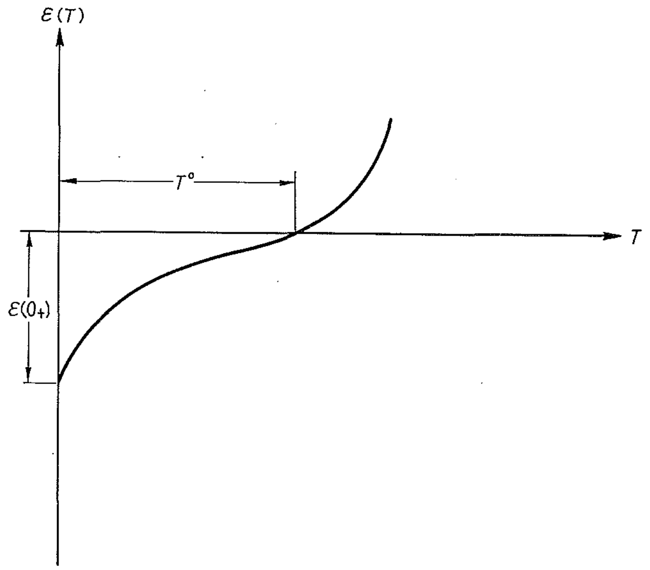

Fig. 2. Characteristics of error index $\varepsilon(T)$

$$\text{sign}\left\{\varepsilon(T)\right\} = -\text{sign}\left\{\varepsilon(T^0+\Delta t)\right\} \tag{33}$$

for $\quad 0<T<T^0$.

Since Eq. (33) must hold true even if T approaches zero infinitely, it can be put that

$$\text{sign}\left\{\lim_{T\to 0+}\varepsilon(T)\right\} = -\text{sign}\left\{\varepsilon(T^0+\Delta t)\right\} \tag{34}$$

On the other hand, the value of $\lim_{T\to 0+}\varepsilon(T)$ when T of $\varepsilon(T)$ approaches zero infinitely is obtained as follows. First, the procedure for obtaining of T^0 from Eq. (31) will be shown. Give $T=T_i$ and find $\Psi(0)$ from Eq. (22).

$$\Psi(0) = -V_{x\psi}^{-1}(T_i)\,V_{xx}(T_i)\,X(0) \tag{35}$$

Substitute this into Eq. (31).

$$\varepsilon(T_i) = X(0)\cdot X(0) + \frac{1}{2\lambda}\int_0^{T_i} X(0)\cdot R^{TR}(t)\,BB^{TR}\left(V_{\psi x}(t)\,X(0) + V_{\psi\psi}(t)\,\Psi(0)\right)dt \tag{36}$$

Let T be increased by the relation $T_i=T_i+\Delta t$ until $\varepsilon(T_i)=0$, then the following relation holds true for $\varepsilon(\Delta t)$.

$$\varepsilon(\Delta t) \fallingdotseq \varepsilon(0) + \Delta t\varepsilon(0)$$

increase or decrease T in accordance with the determination of $\varepsilon(T)$ as to

$$\doteqdot X(0)\cdot X(0)+\lim_{\delta\to 0}\frac{1}{2\lambda}\delta X(0)\cdot R^{TR}(\delta)\,BB^{TR}\Big(V_{\phi x}(\delta)\,X(0)$$
$$-V_{\phi\phi}(\delta)\,V_{x\phi}^{-1}(\delta)\,V_{xx}(\delta)\,X(0)\Big)+\frac{1}{2\lambda}\varDelta tX(0)\cdot R^{TR}(0)$$
$$\times BB^{TR}\Big(V_{\phi x}(0)\,X(0)-V_{\phi\phi}(0)\,V_{x\phi}^{-1}(\varDelta t)\,V_{xx}(\varDelta t)\,X(0)\Big) \qquad (37)$$

In this equation, from the definition of fundamental matrix,

$$R^{TR}(0)=[1]\,,\qquad V_{\phi x}(0)=[0]\,,\qquad V_{\phi\phi}(0)=[1] \qquad (38)$$

Since the value of $V_{xx}(\varDelta t)$ is the value at $t=\varDelta t$ given as the solution of Eq. (16),

$$V_{xx}(\varDelta t)\doteqdot[1]+A\varDelta t \qquad (39)$$

The same applies to $V_{xx}(\delta)$.
Since $V_{x\phi}^{-1}(\varDelta t)$ is the inverse matrix of $V_{x\phi}(\varDelta t)$ and $V_{x\phi}(\varDelta t)$ is the solution of Eq. (16),

$$V_{x\phi}(\varDelta t)\doteqdot\varDelta t\frac{1}{2\lambda}BB^{TR}$$

Therefore, it can be put that

$$V_{x\phi}^{-1}(\varDelta t)\doteqdot\frac{2\lambda}{\varDelta t}[BB^{TR}]^{-1} \qquad (40)$$

Substituting Eqs. (38), (39) and (40) into Eq. (37), we have

$$\varepsilon(\varDelta t)=-X(0)\cdot X(0)-\varDelta tX(0)\cdot AX(0) \qquad (41)$$

If $\varDelta t$ is selected sufficiently small in Eq. (41),

$$\varepsilon(\varDelta t)=-X(0)\cdot X(0)\leqq 0 \qquad (42)$$

The equality holds true only when $X(0)\cdot X(0)=0$. However, $X(t)$ is initially zero, and therefore, there is no need of control. Then,

$$-\text{sign}\left\{\varepsilon(T^0+\varDelta t)\right\}=\text{sign}\left\{\lim_{T\to 0+}\varepsilon(T)\right\}=\text{sign}\left\{\lim_{\varDelta t\to 0}\varepsilon(\varDelta t)\right\}<0 \qquad (43)$$

From the result so far obtained, a method is conceived to seek the optimal time T^0 of T, by which T is increased or decreased toward T^0 based on a judgement

$$\left.\begin{aligned}\varepsilon(T)<0:&\quad T=T+\varDelta t\\ \varepsilon(T)>0:&\quad T=T-\varDelta t\\ \varepsilon(T)=0:&\quad T=T^0\end{aligned}\right\} \qquad (44)$$

or an important relation is derived by which a servo mechanism which can positive or negative sign, if constructed, will enable a continuous obtaining of T^0. Throughout the description given above, $\varepsilon(T)$ was assumed to intersect abscissa T at $T=T^0$ as shown in Fig. 2. A special case is conceivable where $\varepsilon(T)$ is tangential to abscissa T at $T=T^0$. Under this condition,

$$\left.\begin{array}{lll} \varepsilon(T^0+\Delta t)<0 & \text{and} & \varepsilon'(T^0+\Delta t)<0 \\ \text{or} & & \\ \varepsilon(T^0)=0 & \text{and} & \varepsilon'(T^0)=0 \end{array}\right\} \tag{45}$$

That is, since the condition of $\varepsilon'(T)$ that satisfies Eq. (44) is reciprocal to Eq. (45), the following relation must hold true when T' is taken in the vicinity of T^0.

$$\varepsilon'(T')>0 \tag{46}$$

That is, the conditions of Eq. (47) must be satisfied.

$$X(0)\,R^{TR}(T')\,BB^{TR}\left(V_{\phi x}(T')+V_{\phi\phi}(T')\,V_{x\phi}^{-1}(T')\,V_{xx}(T')\right)X(0)>0 \tag{47}$$

The authors checked Eq. (47) for mere confirmation in marking the judgement on Eq. (44), but have no experience with a case reciprocal to Eq. (47), that is, the problem of tangential (T), in controlling many initial conditions of the submarine mentioned later. This special case is of considerable interest to us. In practice, however, it presents no significant trouble in the control because scanning is made from $T=0$ and such $T=T^0$ that makes $\varepsilon(T)=0$ is obtained.

3.2. Computing procedure

A linear plant has the characteristic given in the form of Eq. (1) in general. When non-linear elements are involved, the plant is controllable by the present method if Eq. (1) is given approximately as an equation of small perturbation. When Eq. (1) is given, the values of A and B are determined. Thus, by solving.

$$\frac{d}{dt}\begin{bmatrix} r_{11}(t)\,r_{12}(t) & \cdots & r_{1n}(t) \\ & r_{ij}(t) & \\ r_{n1}(t) & \cdots & r_{nn}(t) \end{bmatrix} = -\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & a_{ij} & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} r_{11}(t)\,r_{12}(t) & \cdots & r_{1n}(t) \\ & r_{ij}(t) & \\ r_{n1}(t) & \cdots & r_{nn}(t) \end{bmatrix} \tag{48}$$

for the initial conditions

$$\begin{bmatrix} r_{11}(0) & & r_{1n}(0) \\ & r_{ij}(0) & \\ r_{n1}(0) & \cdots & r_{nn}(0) \end{bmatrix} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

we can obtain $r_{ij}(t)$, $(t=0, \Delta t, 2\Delta t, n\Delta t)$.

Next, rewrite Eq. (16) in the form of Eq. (49) and obtain v_{ij} by solving Eq. (49)

$$\frac{d}{dt}\begin{bmatrix} v_{11}(t) & v_{12}(t) & \cdots & v_{1n}(t) \\ & & v_{ij}(t) & \\ v_{n1}(t) & \cdots & & v_{nn}(t) \end{bmatrix} = \begin{bmatrix} a_{11}a_{12}\cdots a_{1n} & d_{11}d_{12}\cdots d_{1n} \\ a_{ij} & d_{ij} \\ a_{n1}\cdots a_{nn} & d_{n1}\cdots d_{nn} \\ 2C_{11}2C_{12}\cdots 2C_{1n} & -a_{11}\cdots -a_{n1} \\ 2C_{ij} & -a_{12} \quad -a_{ij} \\ 2C_{n1}\cdots 2C_{nn} & -a_{1n}\cdots -a_{nn} \end{bmatrix} \begin{bmatrix} v_{11}(t)v_{12}(t)\cdots v_{1n}(t) \\ v_{i1}(t)\cdots v_{ij}(t)\cdots v_{in}(t) \\ v_{n1}(t)\cdots v_{nn}(t) \end{bmatrix} \tag{49}$$

for the initial conditions

$$[v_{ij}(0)] = [1]$$

$$[d_{ij}] = \frac{1}{2\lambda}[b_{ij}]\,[b_{ji}]$$

Classify $v_{ij}(t)$ $(t=0, \Delta t, 2\Delta t, 3\Delta t, \cdots, n\Delta t)$ obtained from Eq. (49) as shown by Eq. (18) and obtain $v_{ij}^{xx}(t)$, $v_{ij}^{x\phi}(t)$, $v_{ij}^{\phi x}(t)$ and $v_{ij}^{\phi\phi}(t)$. Using the result, calculate $w_{ij}^{\phi x}(T)$, $w_{ij}^{\phi\phi}(T)$ and $w_{ij}(T)$ from Eqs. (25), (26) and (27). The authors have

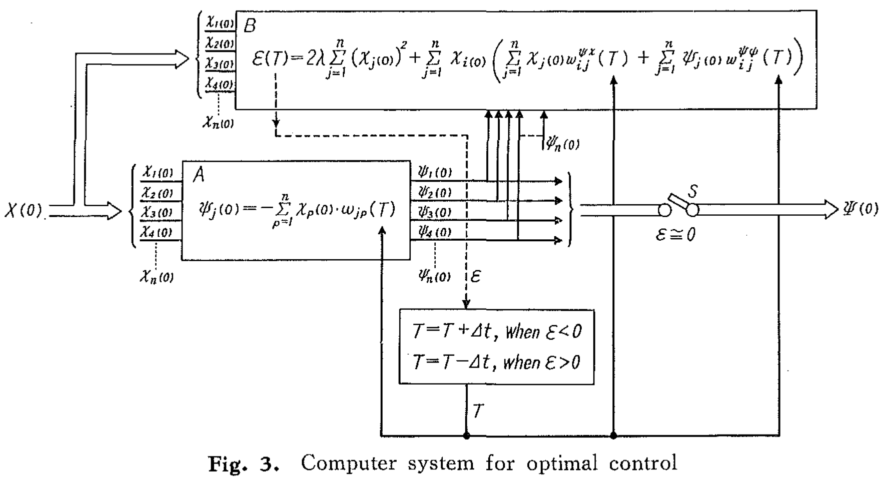

Fig. 3. Computer system for optimal control

applied the Runge–Kutta–Gill method and made the calculation of Eqs. (25) through (27) by the trapezoidal rule, that is, $\int_a^b f(x)\,dx=\frac{1}{2}(f_0+f_1)$. The block Identification shown in Fig. 1 represents the process up to the obtaining of w. In Fig. 1, if a difference $X(0)$ between the set value $X_I(0)$ and output $X(t)$ occurs, it is sent to the computer where the optimal control $U(0)$ corresponding to $X(0)$ is calculated. This block is shown in detail in Fig. 3. Referring to the figure, when $X(0)$ is given, a certain value of T is set and in block A, $\phi_i(0)$ is calculated by Eq. (29) and the result is used in block B to calculate $\varepsilon(T)$. The calculation is repeated by the relation $T=T+\Delta t$ or $T=T-\Delta t$ depending on whether $\varepsilon(T)<0$ or $\varepsilon(T)>0$ until $\varepsilon(T)\approx 0$, that is, $T=T^0$ is obtained. Now, close the switching circuit S and send $\Psi(0)$ to the next stage, then $\Psi(0)$ becomes the initial value of the initial variable vector for $X(0)\cdot\Psi(0)$ given in Fig. 3 is once held in Fig. 4.

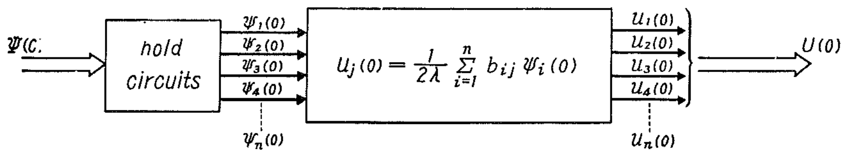

Fig. 4. Network of control devices

If control u_j is given by

$$u_j(0)=\frac{1}{2\lambda}\sum_{i=1}^{n} b_{ij}\phi_i(0) \tag{50}$$

the optimal value of u_j is obtained. The action of u_j is fed back to $X(0)$ through the characteristic of the plant and is given as input in Fig. 3, thus completing a closed control loop. This means that a chance is given for repeated correction of errors of the entire system including the computer errors. On the other hand, a certain optimal value T_1^0 is given by the judgement on Eq. (44) and thereby an optimal control $U_{ij}(T_1^0)$ is given. Thereafter, give $T_2=T_1^0-\Delta t$ as the estimated value of the initial value of T_2 for the second time onward, and the operation time becomes shortest. Generally, it is possible to limit the number of repetitions of calculation of Eqs. (28) and (29) to zero or 1 so that practically no repeated calculation is needed, by using

$$T_p=T_{p-1}^0-\Delta t\,, \qquad (p=1, 2, \cdots, n) \tag{51}$$

for the initial value of T_p.

4. Application to the 4th order linear model for submarine pitching motion control

It is already well known that the pitching motion of the submarine can be represented by the 4th order linear equations[9]. To be noted is that the equations of motions of the submarine have coefficients which are a function of ship speed and so vary frequently. When a digital computer is introduced in the control system, Eqs. (25) through (27) can be calculated easily and no problem arises in the calculation of these equations necessitated by any change of data given by the Identification. In the case of analog control, however, it is difficult to change the values of Eqs. (25) through (27) as described later and this may make the control impossible when the characteristic of the plant changes largely. In the case of the submarine, it may be assumed fixed as a result of normalization of the motion[9], and generally the following expression may be used.

$$\left.\begin{aligned} A_1\ddot{h}+A_2\dot{h}+A_3\ddot{\theta}+A_4\dot{\theta}+A_5\theta &= A_6\beta_{ea}-A_7\beta_{ef} \\ B_1\ddot{h}-B_2\dot{h}+B_3\ddot{\theta}+B_4\dot{\theta}-B_5\theta &= B_6\beta_{ea}+B_7\beta_{ef} \end{aligned}\right\} \tag{52}$$

where θ : pitching angle in degree
h : depth in meter
β_{ea} : angle of stern plane in degree
β_{ef} : angle of bow plane in degree

Now put

$$\left.\begin{aligned} &x=\dot{h}\,, \quad x_2=h, \quad x_3=\dot{\theta}\,, \quad x_4=\theta \\ &u_1=\beta_{ea}\,, \quad u_2=\beta_{ef} \end{aligned}\right\} \tag{53}$$

then the following relations hold true.

$$\left.\begin{aligned} a_{11} &= -\frac{1}{\Delta}\left(\frac{A_2}{A_1}+\frac{A_3B_2}{A_1B_3}\right), & a_{31} &= \frac{1}{\Delta}\left(\frac{B_2}{B_3}+\frac{B_1A_2}{B_3A_1}\right) \\ a_{13} &= -\frac{1}{\Delta}\left(\frac{A_4}{A_1}-\frac{A_3B_4}{A_1B_3}\right), & a_{33} &= -\frac{1}{\Delta}\left(\frac{B_4}{B_3}-\frac{B_1A_4}{B_3A_1}\right) \\ a_{14} &= -\frac{1}{\Delta}\left(\frac{A_5}{A_1}+\frac{A_3B_5}{A_1B_3}\right), & a_{34} &= \frac{1}{\Delta}\left(\frac{B_5}{B_3}+\frac{B_1A_5}{B_3A_1}\right) \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} b_{11} &= \frac{1}{\Delta}\left(\frac{A_6}{A_1}-\frac{A_3B_6}{A_1B_3}\right), & b_{31} &= \frac{1}{\Delta}\left(\frac{B_6}{B_3}-\frac{B_1A_6}{B_3A_1}\right) \\ b_{12} &= -\frac{1}{\Delta}\left(\frac{A_7}{A_1}+\frac{A_3B_7}{A_1B_3}\right), & b_{32} &= \frac{1}{\Delta}\left(\frac{B_7}{B_3}+\frac{B_1A_7}{B_3A_1}\right) \end{aligned}\right\} \tag{55}$$

where $\Delta=\left(1-\frac{B_1A_3}{A_1B_3}\right)$

Take for example a 1,000 ton class submarine to show values of A and B in the following.

$$A=\begin{pmatrix} -0.03739 & 0 & -0.02431 & -0.001671 \\ 1.0 & 0 & 0 & 0 \\ 0.1361 & 0 & -0.1113 & -0.002615 \\ 0 & 0 & 0 & 1.0 \end{pmatrix} \tag{56}$$

$$B=\begin{pmatrix} 0.0003664 & -0.0003043 \\ 0 & 0 \\ 0.002267 & 0.001067 \\ 0 & 0 \end{pmatrix} \tag{57}$$

$R(t)$ shown in Eq. (10) is the fundamental matrix of the equation expressed as

$$\dot{\Psi}(t)=-A^T\Psi(t)\,, \tag{58}$$

where T: transposed matrix of A

Now let $\Psi_1(0)$, $\Psi_2(0)$, $\Psi_3(0)$ and $\Psi_4(0)$ be

$$\Psi_1(0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad \Psi_2(0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \Psi_3(0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \Psi_4(0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{59}$$

then, $[r_{i1}]$ $(i=1, 2, 3, 4)$ is the solution of Eq. (58) for the initial condition taken as $\Psi_1(0)$. Generally, $[r_{ij}]$ $(i=1, 2, 3, 4)$ is the solution of Eq. (41) for the initial condition as $\Psi_j(0)$ ($\Psi_j(0)=1$ if $j=i$, $\Psi_j(0)=0$ if $j\neq i$). For $[v_{ij}]$, the solution can be obtained similarly from the eight initial conditions and Eq. (16). Fig. 6 shows the difference between the conventional automatic steering and the steering on the maximum principle presented when the ship is to change its depth 5 m in a horizontal travelling conditions. The control conditions are as follows.

(1) Automatic steering

In Eq. (52), let the relation between β_{ea} and β_{ef} be

$$\beta_{ef}=2\beta_{ea} \tag{59}$$

Assume a feedback to be represented by

$$\beta_{ea}=-k_\theta\theta+k_h h-k_{\dot\theta}\dot\theta \tag{60}$$

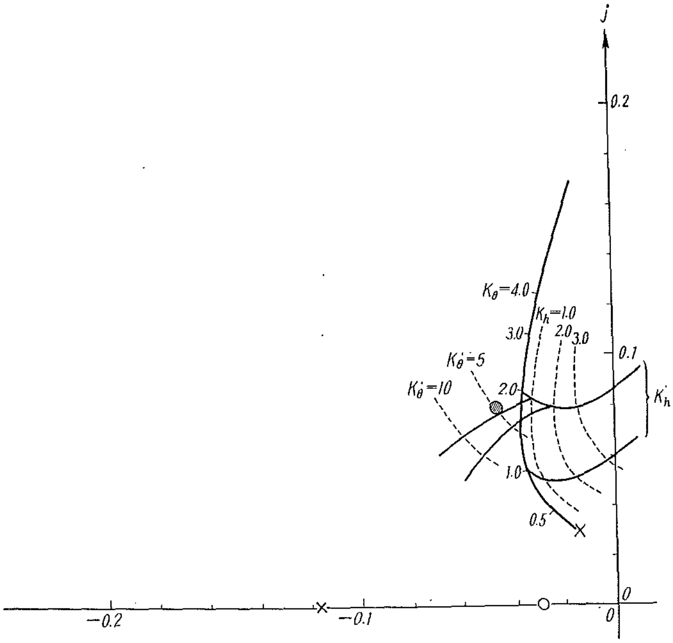

Fig. 5. Root locus for submarine control system

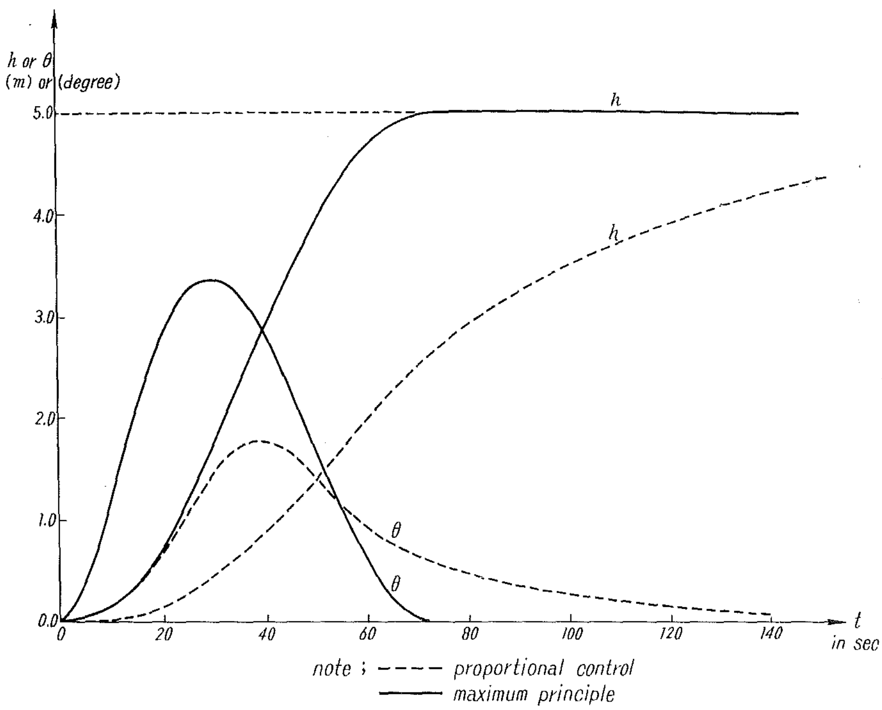

Fig. 6. Comparison of proportional control and optimal control for submarine in longitudinal motion

Then, for various values of elements k_θ, k_h and $k_{\dot{\theta}}$ of this feedback, the root locus for the control system is as shown in Fig. 5. If the values of k_θ, k_h and $k_{\dot{\theta}}$ are selected to correspond to the dot, that is, $k_\theta=2$, $k_h=1$ and $k_{\dot{\theta}}=5$, then the result of control of this system is as shown by the dotted line in Fig. 6.

(2) Steering on the maximum principle

The solid line in Fig. 6 shows the result of control obtained by the calculating method shown in Fig. 3 and Fig. 4 for such data as $[C]=[1]$, $\lambda=1$ and $\Delta t=2.0$ s in Fig. 6. Since the value of $\Psi(0)$ calculated for $X(0)$ in Fig. 3 is once held before being supplied as $U(0)$ in Fig. 4, the steering shown in Fig. 7 takes a stepped form, and the path shown in Fig. 6 is an approximate solution of the optimal path.

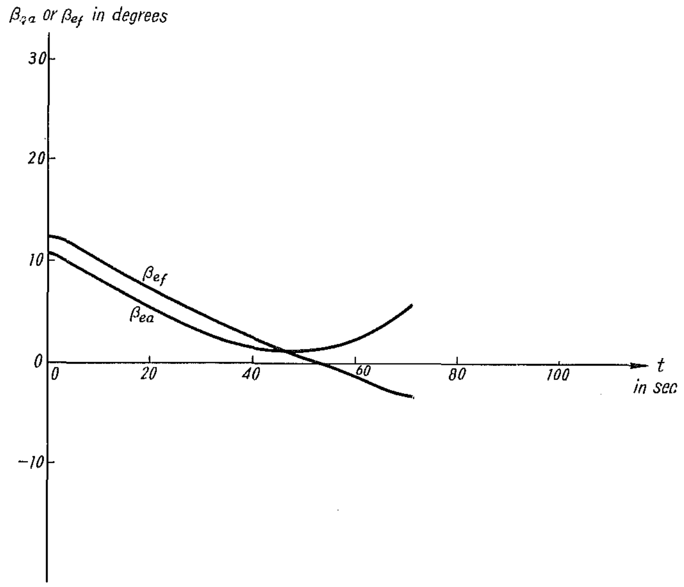

Fig. 7. Values of stern plane and bow plane

5. Extension of the scope, and control mode discussions

In the application of the maximum principle to the general process control, Eq. (5) may be construed as follows.

(1) Normal control $\quad J_1=\int_0^T\left(\sum_{j=1}^4 x_j^2(t)+\sum_{i=1}^2 u_i^2(t)\right)dt$

(2) Minimum deviation control $\quad J_2=\int_0^T \sum_{j=1}^4 x_j^2(t)\,dt$

(3) Minimum u control $\quad J_3=\int_0^T \sum_{i=1}^{2} u_i^2(t)\,dt$

(4) Shortest time control $\quad J_4=\int_0^T dt$

The normal control and the minimum u control can be determined uniquely by the method described in 3, whereas the minimum deviation control and the shortest time control can not be determined simply because of the restriction imposed in the derivation of Eq. (7). If such a control in which λ in Eq. (5) approaches zero as near as possible, is called the minimum deviation control, a considerably precise approximation of it can be obtained by the path correction method described in Fig. 3. The minimum deviation control in the strict sense shows up as a fluttering control, while the minimum deviation control modified as defined above to have λ which is selected very small allows the restriction on u to be alleviated so that calculated value of u is

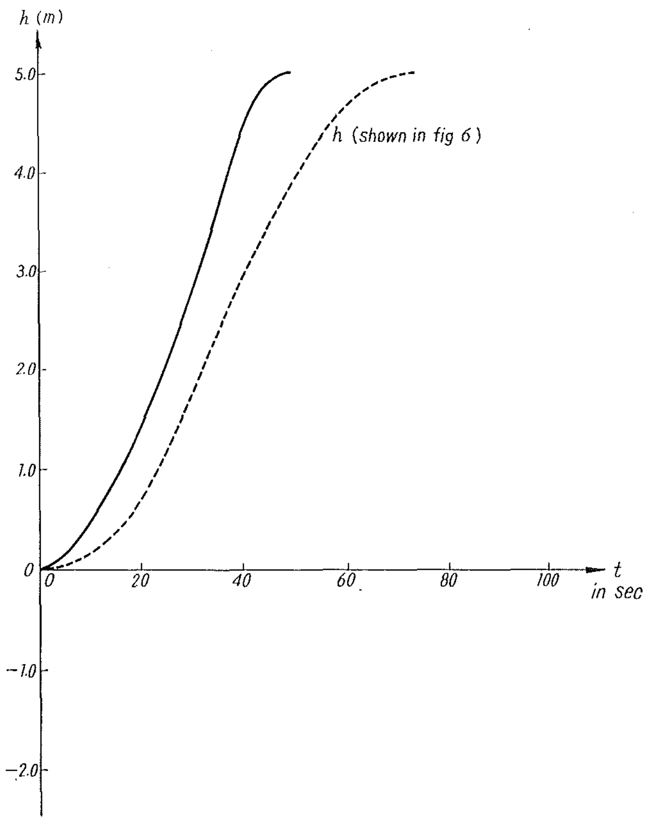

Fig. 8. Sub-optimal minimum error control

accordingly large and a saturation may be considered for it. Thus, the difference between the calculated value of u and the saturated value u_{max} can be corrected by the path correction to a considerably presice approximation. As an application of this concept to the submarine. Fig. 8 shows the result of control for the data

$$\left.\begin{aligned} &C=[1.0], \quad \lambda=0.001 \\ &u_{1max}=u_{2max}=25.0 \text{ degrees} \end{aligned}\right\} \tag{61}$$

when the submarine changes its depth 5 m.

This figure indicates that the control speed is considerably faster than the path of Fig. 6 shown by the dotted line and that each control becomes satuated. The minimum u control corresponds to $C=[0]$ in Eq. (5) and the control method equivalent to the normal steering may be used for this control. Fig. 9 shows the result of this control for the data $C=[0.0]$ and $\lambda=0.001$. In the case of the shortest time control, it is impossible to apply the path correction in this limited condition where λ approaches zero and other methods should be resorted to. Generally for the actual processes, however, the shortest time control should hardly be needed if the minimum deviation control

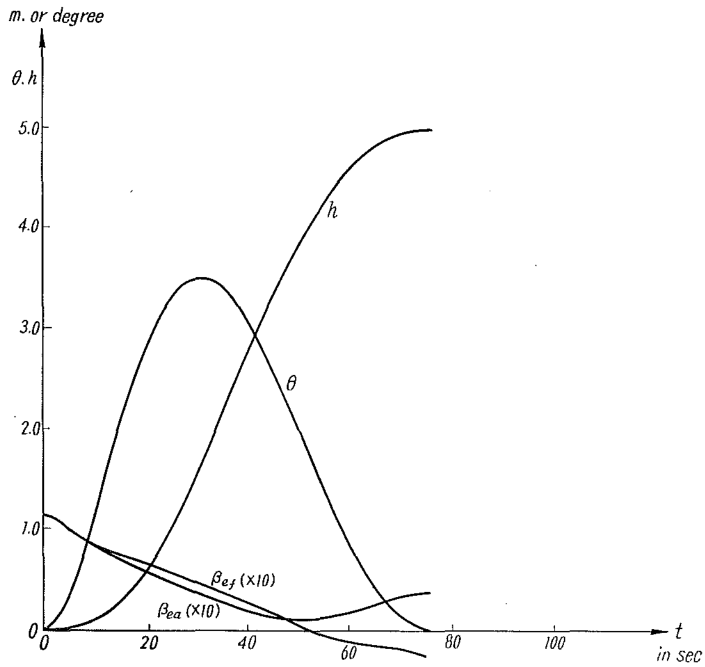

Fig. 9. *U* minimum control

and the minimum u control are possible. Especially in the case of a submarine, there is absolutely no need of this control, because depth hunting must be avoided for safety purpose, thus the meaning of the optimal condition becomes ambiguous in this limited case where the characteristics change severely, and the fluttering control is always accompanied by hazards: it is only necessary to change the mode of normal steering between specific ranges of λ. This description, however, does not mean that the shortest time control is meaningless, but that for the purpose of simplifying the computer system and offering a generalized approach in place of the conventional PID system, as intended by the authors, the shortest time control may be discounted from the consideration. From the discussion given above, it is seen that the computer system shown in Fig. 3 and Fig. 4 can be applied in the same form to normal control, minimum deviation control and minimum u control, and that the generality of the control exists including the cases where saturable control elements are involved and measurement errors of the plant are present. Especially, in the latter case of the measurement errors ε, very precise control is obtained if the characteristic of the plant is simulated precisely by the Runge-Kutta-Gill method and w is calculated by the first approximation of the Newton–Cotes' integration formula, that is, the trapezoidal rule. The last problem left in the application of the control method to general process control lies in the mode of simplification of the computer system shown in Fig. 3 and Fig. 4. At this point, the authors considered an analog computer system and have proposed a system for continuous control.

6. Control by analog computation

Taking for example the control of a submarine, it means that the control on the maximum principle can be carried out entirely by analog computation. Fig. 10 illustrates the technique for a fourth linear control system. A total of 48 potentiometers to generate function w_{ij}, $w_{ij}^{\phi x}$ and $w_{ij}^{\phi\phi}$ $(i, j=1, 2, 3, 4)$ are prepared (the number 48 will be used here for the clear explanation of the relationship between Fig. 3 and Fig. 4 although only 32 are actually needed) and mounted in such a manner as to interlock with each other on the same shaft which is driven by a motor supplied from the comparator which detects the polarity (+ or −) of $\varepsilon(T)$. Since the angular displacement of the potentiometer shaft is proportional to T, potentiometers provide outputs $w_{ij}(T)$, $w_{ij}^{\phi x}(T)$ and $w_{ij}^{\phi\phi}(T)$. As can be seen in the figure, composite output of potentiometers w_{11}, w_{12}, w_{13}, w_{14}, appears in the output of amplifier F_1 and corresponds to $\phi(0)$ which satisfies Eq. (29). Similarly, composite outputs of w_{21}, w_{22}, w_{23}, w_{24}, $\cdots$, and w_{41}, w_{42}, w_{43}, w_{44} appear in the outputs of amplifier

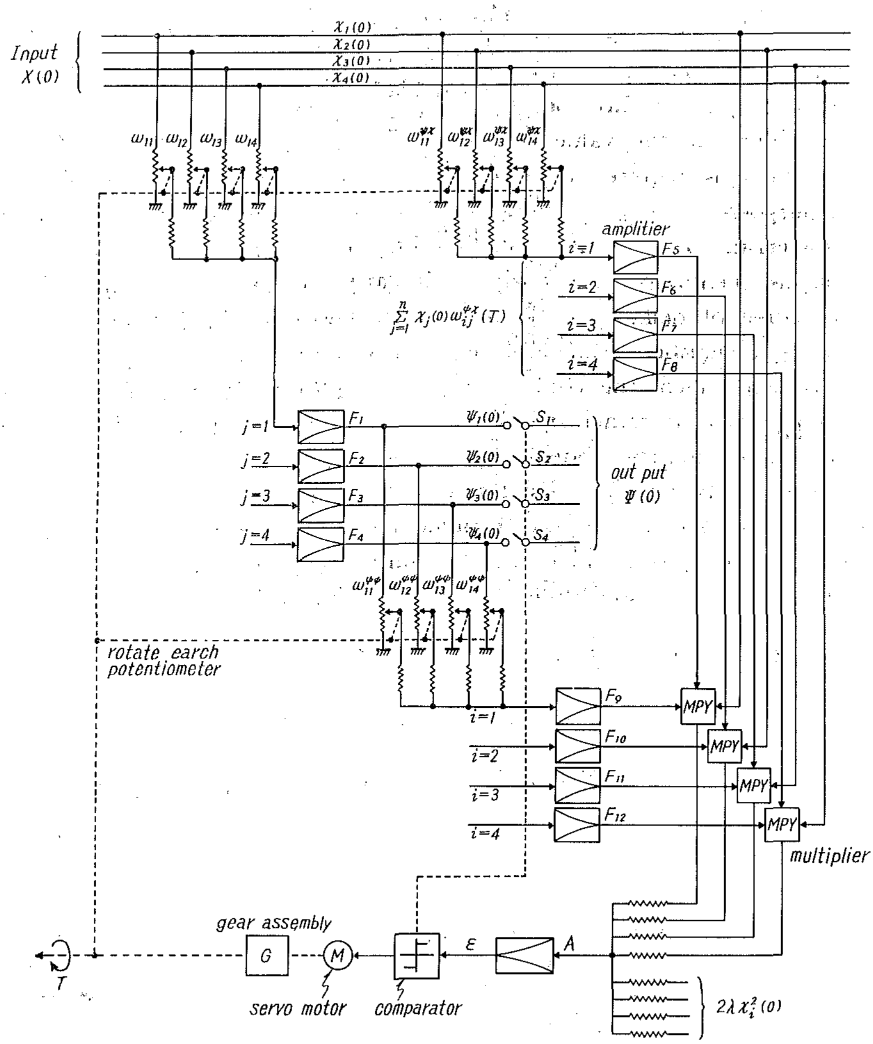

Fig 10. Analog computation technique

F_2, F_3 and F_4 and correspond to $\psi_2(0)$, $\psi_3(0)$ and $\psi_4(0)$, respectively. On the other hand, outputs of amplifier F_5, F_6, F_7 and F_8 obtained from $(w_{11}^{\psi x}, w_{12}^{\psi x}, w_{13}^{\psi x}, w_{14}^{\psi x}), \cdots, (w_{41}^{\psi x}, w_{42}^{\psi x}, w_{43}^{\psi x}, w_{44}^{\psi x})$ and outputs of amplifiers F_9, F_{10}, F_{11} and F_{12} obtained from $(w_{11}^{\psi\psi}, w_{12}^{\psi\psi}, w_{13}^{\psi\psi}, w_{14}^{\psi\psi}), \cdots, (w_{41}^{\psi\psi}, w_{42}^{\psi\psi}, w_{43}^{\psi\psi}, w_{44}^{\psi\psi})$ are summed appropriately in the output of multiplier MPY where sums are multiplied by input $x_1(0)$, $x_2(0), \cdots, x_2(0)$. Multiplier outputs are summed together in amplifier A to which another input $2\lambda(x_1^2(0)+x_3^2(0)+x_4^2(0))$ is also applied. Then, amplifier A provides output $\varepsilon(T)$ which satisfies Eq. (28) and is sent to the determinant mechanism where it is used to shift the potentiometer shaft so that T for $\varepsilon \cong 0$, that is, the optimal control time T^0 is obtained. When switch groups S_1, S_2, S_3 and S_4 are turned on, the calculated result $\Psi(0)$ is provided at the output to determine the value of control properly. The potentiometer used for this control is required only to have an ordinary class of precision for practical purposes: a slight difference between the potentiometer setting and the characteristic of the plant, if present, may be corrected by the path correction as mentioned previously. Similarly, a servo system having an ordinary degree of delay may be used with almost no inconvenience for the control of a submarine. Furthermore, as mentioned in the preceding section, once T is determined, the control may be continued in quite the same way as a general servo, or rather smoothly as compared against the digital system.

The analog control system, as compared with the digital system, is characterized by its very low cost for new installation: in this regard, it may be considered as one of the conventional mechanical servo systems of single function type, being convenient for the control of a submarine. To simplify the system of Fig. 10, substitute Eq. (29) into Eq. (28) and we obtain

$$\begin{aligned} -2\lambda \sum_{j=1}^{n} \left(x_j(0)\right)^2 &= \sum_{i=1}^{n} x_i(0) \Big(\sum_{j=1}^{n} x_j(0)\, w_{ij}^{\phi x}(T) \\ &\qquad - \sum_{j=1}^{n} w_{ij}^{\phi\phi}(T) \sum_{p=1}^{n} x_j(0)\, w_{jp}(T) \Big) \\ &= \sum_{i=1}^{n} x_i(0) \sum_{j=1}^{n} x_j(0) \Big(w_{ij}^{\phi x}(T) - \sum_{p=1}^{n} w_{ip}^{\phi\phi}(T)\, w_{pj}(T) \Big) \end{aligned} \tag{62}$$

Noting that

$$w_{ij}^{0}(T) = w_{ij}^{\phi x}(T) - \sum_{p=1}^{n} w_{ip}^{\phi\phi}(T)\, w_{pj}(T) \tag{63}$$

it is possible and yet practical to simulate by means of two sets of potentiometers of $w_{ij}(T)$ and $w_{ij}^{0}(T)$, $w_{ij}^{0}(T)$ being used for the generation of functions by potentiometers.

7. Conclusion

In summary, on-line application of a control device based on the maximum principle in place of the general PID control device is not so difficult, and especially upon the development of analog control systems, it has been found that the principle of conventional servo systems seeking continuously for the optimal value by zero-method can be applied also in this case. Whether the PID control device or the control device based on the maximum principle is superior can not be said generally, but, most PID control systems require various tests which are repeated after the device is manufactured to determine the optimal value and moreover, it is impossibly to determine at present whether the result of control is truly optimal or not. This difficulty increares in intensity in a complicated multi-variable control system. In contrast, the application of the maximum principle is advantageous in that most procedures are eliminated and that optimal control is insured even for a complicated system.

Notation

J: Cost function
X: Process vector, $X=[x_j]$, $j=1, 2, \cdots, n$
X^T: Transposed matrix of X
C: Constant vector
λ: Constant
U: Control vector, $U=[u_j]$, $j=1, 2, \cdots, n$
A: Constant of the plant, $A=[a_{ij}]$, $i, j=1, 2, \cdots, n$
B: Constant of the plant, $B=[b_{ij}]$, $i, j=1, 2, \cdots, n$
T: Optimal time required for the proccess change
t: Time
Ψ: Auxiliary vector, $\Psi=[\psi_j]$, $j=1, 2, \cdots, n$
H: Hamiltonian
A^T: Transposed matrix of A
B^T: Transposed matrix of B
Φ: Fundamental matrix of $\dot{X}=AX$
R: Fundamental matrix of $\dot{\Psi}=-A^T\Psi$
V: Fundamental matrix which satisfies Eq. (16)
$w_{ij}^{\phi x}$: Characteristic equation represented by Eq. (25)
$w_{ij}^{\phi\psi}$: Characteristic equation represented by Eq. (26)
w_{ij}: Characteristic equation represented by Eq. (27)
ε: Error of the determinant
A_i: Coefficient of the equation of motion of the submarine, $i=1, 2, \cdots, 7$
B_i: Coefficient of the equation of motion of the submarine, $i=1, 2, \cdots, 7$
θ: Pitching angle in degrees
$\dot{\theta}$: $d\theta/dt$ (deg/sec)
h: Depth in meters
$\dot{h}$: dh/dt (m/sec)
β_{ea}: Angle of stern plane in degress
β_{ef}: Angle of bow plane in degrees

k_θ : Pitching angle gain, β_{ea}/θ
k_h : Depth gain, β_{ea}/h
$k_{\dot\theta}$: Pitching angle velocity gain, $\beta_{ea}/\dot\theta$

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Appendix

(1) Flow chart for computation

The flow chart is shown in Fig. a-1. The coefficients a_{ij} and b_{ij}, and initial conditions $x_j(0)$ must be stored automatically from external sources.

(2) Symbols

Symbols employed in the computer are shown in Table 1.

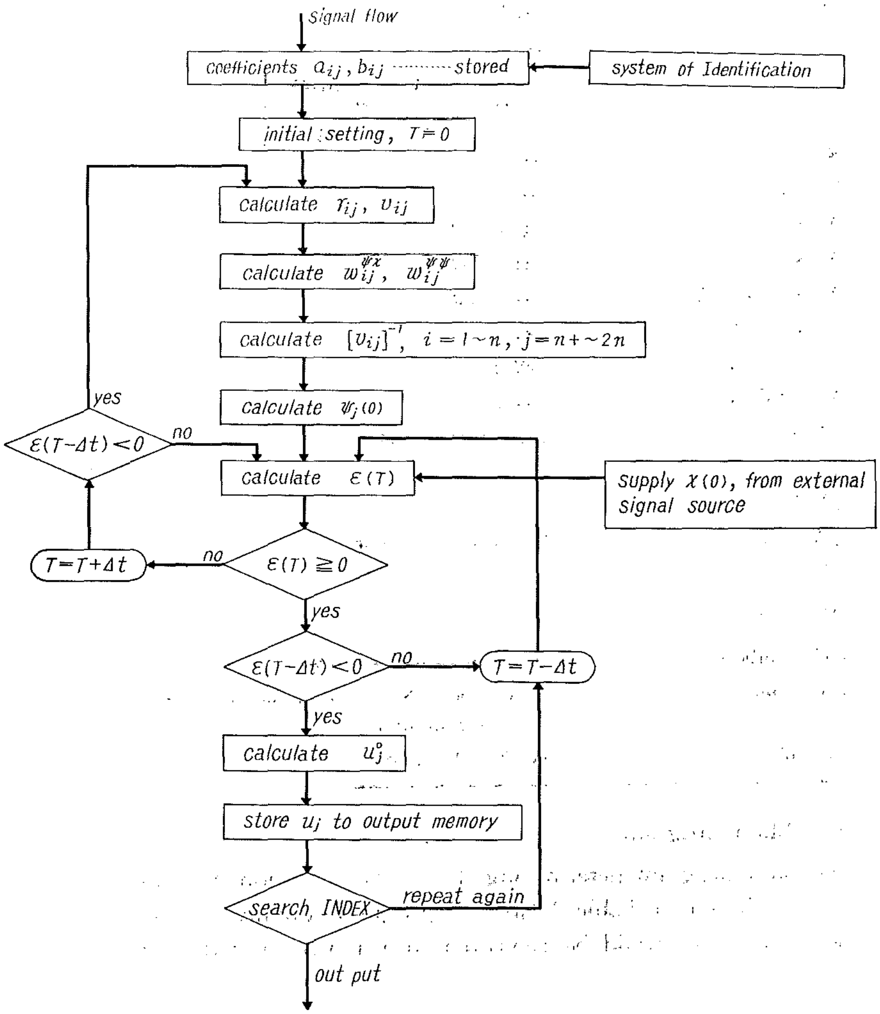

Fig. a-1. Flow chart

TABLE 1. Table of symbols employed in the computer

T	t
DELTAT	Δt
A (I, J)	a_{ij}
B (I, J)	b_{ij}
X (J)	x_j
U (J)	u_j
XŌ (J)	$x_j(0)$
FŌ (J)	$\psi_j(0)$
D (I, J)	d_{ij} $[d_{ij}] = \begin{bmatrix} A & \frac{1}{2\lambda} BB^{TR} \\ 2C & -A \end{bmatrix}$
R (I, J)	r_{ij}
V (I, J)	v_{ij}
AUXW (1, I, J)	$w_{ij}^{\psi x}(t+\Delta t)$
AUXW (2, I, J)	$w_{ij}^{\psi\psi}(t+\Delta t)$
AUXW (3, I, J)	$w_{ij}(t+\Delta t)$
AUXWŌ (1, I, J)	$w_{ij}^{\psi x}(t)$
AUXWŌ (2, I, J)	$w_{ij}^{xx}(t)$
AUXWŌ (3, I, J)	$w_{ij}(t)$
UMAX (J)	upper limit of u_j
QR (I, J) QV (I, J) QA (I, J) QX (I, J) AUXV (I, J)	auxiliary variables
DIF symbols	employed in subroutines
Computer	NEAC 2800 (NEC's scientific and engineering purpose computer)
Compiler	AUTOMATH 3800

(3) Main program

The main program determining the initial condition $\Psi(0)$ and the optimal control U is shown in Table 2. (as this program contains repeats itself causing a waste of time, it should be rewritten into a well-formed chart.)

```
                    * * * AUTOMATH 1800 SOURCE PROGRAM LISTING* * *
 IFN       EFN            PROGRAM: SUB5                 JOB:  CENTER                    PAGE: 01

 0001                DIMENSION FQ(4),A(4,4),B(4,4),XO(4),D(8,8)
                    1,R(4,4),QR(4,4),V(8,8),QV(8,8),AUXWO(3,4,4)
                    2,AUXW(3,4,4),AUXV(4,4),DIFK(8),DIFR(8),DIFU(8)
                    3,DIFA(8,8),DIFB(8,8),DIFY(8),DIFQ(8),DIF
                    4QA(8,8),DIFX(8),DIFC(8,8),DIFD(4,4)
                    5,UO(4),UMAX(4),QX(4)
 0002                DELTAT=2.0 (= Δt)
 0003                TO=2.0
 0004                ALFA=0.1
 0005                RAMDA=1.0 (= λ)
 0006                A(1,1)=-0.03739
 0007                A(1,2)=0.0
 0010                A(1,3)=-0.02431
 0011                A(1,4)=-0.001671
 0012                A(2,1)=1.0
 0013                A(2,2)=0.0
 0014                A(2,3)=0.0
 0015                A(2,4)=0.0
 0016                A(3,1)=0.1361
 0017                A(3,2)=0.0
 0020                A(3,3)=-0.1113
 0021                A(3,4)=-0.002615
 0022                A(4,1)=0.0
 0023                A(4,2)=0.0
 0024                A(4,3)=1.0
 0025                A(4,4)=0.0
 0026                B(1,1)=0.0003664
 0027                B(1,2)=-0.0003043
 0030                B(1,3)=0.0
 0031                B(1,4)=0.0
 0032                B(2,1)=0.0
 0033                B(2,2)=0.0
 0034                B(2,3)=0.0
 0035                B(2,4)=0.0
 0036                B(3,1)=0.002267
 0037                B(3,2)=0.001067
 0040                B(3,3)=0.0
 0041                B(3,4)=0.0
 0042                B(4,1)=0.0
 0043                B(4,2)=0.0
 0044                B(4,3)=0.0
 0045                B(4,4)=0.0
 0046                UMAX(1)=1000000.0
 0047                UMAX(2)=1000000.0
 0050                UMAX(3)=1000000.0
 0051                UMAX(4)=1000000.0
 0052                TSPD=0.001
 0053          1001  DEPTH=2.5
 0054          1002  DO 1003 I=1,4
 0055                XO(I)=0.0
 0056                UO(I)=0.0
```

```
* * * AUTOMATH 1800 SOURCE PROGRAM LISTING* * *
IFN     EFN     PROGRAM: SUB5          JOB:  CENTER          PAGE: 02

              QX(I)=0.0
        1003 CONTINUE
              DEPTH=2.0*DEPTH
              XO(2)=DEPTH
              IF(DEPTH-50.0)1,1,1004
        1004 STOP
           1 T=0.0
              DO 11 J=1,4
              DO 8 I=1,4
              R(I,J)=0.0
              QR(I,J)=0.0
           5 D(I,J)=A(I,J)
              D(I+4,J)=0.0
              D(I+4,J+4)=-A(J,I)
           8 CONTINUE
              D(J+4,J)=2.0
          10 R(J,J)=1.0
          11 CONTINUE
              DO 20 J=1,4
              DO 19 I=1,4
              AW=0.0
          15 DO 17 K=1,4
              AW=AW+B(I,K)*B(J,K)
          17 CONTINUE
              D(I,J+4)=AW/(2.0*RAMDA)
          19 CONTINUE
          20 CONTINUE
              DO 27 J=1,8
              DO 25 I=1,8
              V(I,J)=0.0
              QV(I,J)=0.0
          25 CONTINUE
              V(J,J)=1.0
          27 CONTINUE
         115 DO 122 J=1,4
              DO 121 I=1,4
              DO 120 K=1,3
              AUXWO(K,I,J)=0.0
              AUXW(K,I,J)=0.0
         120 CONTINUE
         121 CONTINUE
         122 CONTINUE
         200 DO 205 J=1,4              }
              DO 205 I=1,4             }
              DIFA(I,J)=-A(J,I)        }
              DIFB(I,J)=R(I,J)         } SUBROUTINE INITIAL SETTING
              DIFQA(I,J)=QR(I,J)       }
         205 CONTINUE                  }
              DIFXA=T                  }
              DIFXOA=DELTAT            }
              NDIFA=4                  }
```

```
                    * * * AUTOMATH 1800 SOURCE PROGRAM LISTING* * *
IFN        EFN            PROGRAM: SUB5                    JOB1   CENTER                    PAGE: 03

                KDIFA=1                   SUBROUTINE INITIAL SETTING
           210 GO TO 9840         -->    SUBROUTINE FOR CALCULATION OF HOMOGENOUS EQUATION
          9051 DO 215 J=1,4       <--
                DO 215 I=1,4
                QR(I,J)=DIFQA(I,J)
                R(I,J)=DIFB(I,J)   (SOLUTION OF r_ij)
           215 CONTINUE
                DO 221 J=1,8          }
                DO 221 I=1,8          }
                DIFA(I,J)=B(I,J)      }
                DIFB(I,J)=V(I,J)      }
           220 UIFQA(I,J)=QV(I,J)     } INITIAL SETTINGS FOR SUBROUTINE
           221 CONTINUE               }
                DIFXA=T               }
                DIFXOA=DELTAT         }
                NDIFA=8               }
           225 KDIFA=2                }
                GO TO 9840        -->    SUBROUTINE FOR CALCULATION OF HOMOGENOUS EQUATION
          9052 DO 231 J=1,8       <--
                DO 231 I=1,8
                QV(I,J)=UIFQA(I,J)
           230 V(I,J)=DIFB(I,J)   (SOLUTION OF V_ij)
           231 CONTINUE
           300 DO 321 KV=1,2
                DO 304 J=1,4
                JJJJ=J+4*(KV-1)
                DO 304 I=1,4
                IIII=I+4
                AUXV(I,J)=V(IIII,JJJJ)
           304 CONTINUE
           305 DO 321 J=1,4
                DO 321 I=1,4
                AW1=0.0
                DO 318 M=1,4
                AW2=0.0
           310 DO 316 L=1,4
                AW3=0.0
                DO 314 K=1,4
                AW3=AW3+B(L,K)*B(M,K)
           314 CONTINUE
           315 AW2=AW2+AW3*R(L,I)
           316 CONTINUE
                AW1=AW1+AW2*AUXV(M,J)
           318 CONTINUE
                AUXW(KV,I,J)=AUXW(KV,I,J)+(AUXWO(KV,I,J)+AW1)*DELTAT/2.0
           320 AUXWO(KV,I,J)=AW1
           321 CONTINUE
                T=DIFXA
                IF(T-TO+DELTAT/2.0) 200,200,328
           328 A=A
           400 DO 403 J=1,4          INITIAL SETTINGS FOR SUBROUTINE
```

```
* * * AUTOMATH 1800 SOURCE PROGRAM LISTING* * *
IFN      EFN        PROGRAM: SUBS                JOB: CENTER                PAGE: 04

              DO 403 I=1,4               }
              DIFC(I,J)=V(I,J+4)         } INITIAL SETTINGS FOR SUBROOTINE
         403  CONTINUE                   }
              NDIFC=4                    }
         405  KDIFC=1                    }
              GO TO 9830  ------->  SUBROUTINE FOR CALCULATION OF INVERSE MATRIX
        9041  DO 410 J=1,4  <------
              DO 410 I=1,4
              AUXV(I,J)=DIFC(I,J)    SOLUTION OF [V_ij]^-1
         410  CONTINUE
              DO 418 J=1,4
              DO 418 I=1,4
              AW=0.0
              DO 416 K=1,4
         415  AW=AW+AUXV(I,K)*V(K,J)
         416  CONTINUE
              AUXW(3,I,J)=AW
         418  CONTINUE
              DO 425 I=1,4
         420  AW=0.0
              DO 423 J=1,4
              AW=AW-AUXW(3,I,J)*XO(J)
         423  CONTINUE
              FO(I)=AW
         425  CONTINUE
         500  AW1=0.0
              DO 507 I=1,4
              AW2=0.0
              DO 505 J=1,4
              AW2=AW2+AUXW(1,I,J)*XO(J)+AUXW(2,I,J)*FO(J)
         505  CONTINUE
              AW1=AW1+AW2*XO(I)+2.0*RAMDA*XO(I)**2
         507  CONTINUE
              GO TO 515
         509  IF(ERROR1)200,514,514
         514  GO TO 600
         515  ERROR1=AW1     (= ε(τ))
              AW=0.0
              DO 519 J=1,4
              AW=AW+AUXW(3,J,2)*AUXW(2,2,J)
         519  CONTINUE
              ERROR2=2.0*RAMDA+AUXW(1,2,2)-AW
              GO TO 509
         600  GO TO 601
         601  GO TO 602
         602  TFINE=T*TSPD
              PRINT604,T,TFINE,ERROR1
         604  FORMAT(1H4,5X,3E12.4)
              PRINT606,(FO(J),J=1,4)
         606  FORMAT(1H2,5X,4E12.4)
              PRINT608,(XO(J),J=1,4)
```

```
* * * AUTOMATH 1800 SOURCE PROGRAM LISTING* * *
IFN        EFN        PROGRAM: SUB5              JOB: CENTER                 PAGE: 05

            608 FORMAT(1H2,5X,4E12.4)
                PRINT610,(UO(J),J=1,4)
            610 FORMAT(1H2,5X,4E12.4)
            611 GO TO 700
            700 T=DELTAT
            701 DO 708 J=1,4
                DO 705 I=1,4
                DIFA(I,J)=A(I,J)
                DIFB(I,J)=B(I,J)
            705 CONTINUE                      } INITIAL SETTINGS FOR SUBROUTINE
                DIFY(J)=XO(J)
                DIFQ(J)=QX(J)
            708 CONTINUE
                A=A
            710 DIFX=T
                DIFXO=DELTAT
                NDIF=4
                MDIF=1
                KDIF=1
                GO TO 9000 ------>  SUBROUTINE TO SOLVE DIFFERENTIAL EQUATION
           9015 T=DIFX.    <------
                IF(T+DELTAT/2.0-TFINE)710,710,718
            718 DO 725 I=1,4
                AW=0.0
                DO 722 J=1,4
                AW=AW+B(J,I)*FO(J)
            722 CONTINUE
                UO(I)=AW/(2.0*RAMDA)
                XO(I)=DIFY(I)
                QX(I)=DIFQ(I)
            725 CONTINUE
            726 T=TFINE/TSPD
                IF(T-2.0-DELTAT/2.0)728,728,727
            727 GO TO 1
            728 GO TO 1002
           9011 DO 735 J=1,4
                IF(ABS(UO(J))-ABS(UMAX(J)))732,732,734
            732 DIFU(J)=UO(J)                 } CALLED FROM SUBROUTINE 9000
                GO TO 737
            734 DIFU(J)=SIGN(UMAX(J),UO(J))
            737 A=A
            735 CONTINUE
            736 GO TO 9001
           9021 PAUSE 9021
           9022 PAUSE 9022
           9023 PAUSE 9023
           9024 PAUSE 9024
           9025 PAUSE 9025
           9026 PAUSE 9026
           9027 PAUSE 9027
           9031 PAUSE 9031
```

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                                                                                                   PAGE: 06
                    SUBPROGRAMS REFERENCED

              INTRINSIC FUNCTION    ABS
              INTRINSIC FUNCTION    SIGN
              INTRINSIC FUNCTION    SQRT

                                    OVERLAY SIZE          1790
                                    IR SPAN SIZE           178
```

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                    JOB    SUMMARY                       CENTER
                                                                                  PAGE: 07
          PROGRAM NAME        OVERLAY SIZE  IR-SPAN  NC-ARRAY SIZE

          SUB5       FIRST  MAIN       3376      262     1304



                    COMMON  BLOCK NAME  SIZE

                            UNLABELED      0



                    LOGICAL UNIT S/D BUFFER SIZE

                          3  DOUBLE       23
                    TOTAL AMOUNT OF MEMORY REQUIRED FOR JOB   005457
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