On the Equations of Motion of the Automobile and the Steady-State Circular Turn

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Abstract

General theory on the motion of automobiles is so difficult that many papers, adopting numerous assumptions and simplifications, have concentrated on the treatment of two or three degrees of freedom on the longitudinal, lateral and yaw motions, where any of motions of sprung mass and effects thereof on the automotive performance have been ignored.

In this paper, dynamical equations of motion of six degrees of freedom, taking into account the bounce, pitch and roll motions of the sprung mass, were established and geometrical and dynamical conditions on the automotive motion were presented in detail. Applying the theory to problems of steady-state circular turn, the quantities governing the performance were calculated. As these equations, however, constitute a complicated system of simultaneous equations difficult to treat mathematically, an iteration method for computing the accurate value was proposed. And it was shown to be effective within a convergence domain in which the forward velocity has an upper limit and radius of turn is larger than the critical value. It is seen from the results of calculation of steer angle of the front wheels obtained here, that an automobile having 'neutral-steer' characteristics presents a slight tendency of 'under-steer' and at the same time the 'over-steer' characteristics are weakened.

1. Introduction

Recently theoretical studies on mathematical models of numerous types have been developed for analyzing the performance of automobiles in such problems as dynamic stability and control. However, the general theory treating precise models is so difficult that numerous assumptions and simplifications have been used in the mathematical treatment. And many papers in this field have concentrated on the treatment of two or three degrees of freedom on the longitudinal, lateral and yaw motions of an automobile; while hardly any of

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the motions of sprung mass and the effect thereof on the automotive performance have been treated[1)~7)].

In this paper, dynamical equations of motion of six degrees of freedom, taking into account the bounce, pitch and roll motions of the sprung mass, are established and geometrical and dynamical conditions on the motion of automobiles are presented in detail. Applying the theory to the problem of a steady-state circular turn, the quantities governing the automotive performance—side-slip angle, load distribution on the tires, the motion of the sprung mass, the cornering forces and frictional forces of the tires etc.—are calculated. These equations, however, constitute a complicated system of simultaneous (differential) equations difficult to treat mathematically and, hence, an iteration method for computing the accurate value of the quantities, will be proposed.

2. The coordinate system

The coordinate systems for the analysis can be used in many different ways and particularly a system of coordinates embedded in the sprung mass has several advantages for the purpose. In problems studied here, it will be desirable to use the two independent coordinate systems[8)].

One of them designates the location of the center of gravity and the orientation of the automobile as an entire system, for describing the longitudinal, lateral and yaw motions. The other designates the deflection of the sprung mass such as bounce, pitch and roll motions, which are usually regarded as small in some sense so that linear analyses are applicable.

As seen in **Fig. 1**, $OXYZ$ is a Cartesian coordinate system, fixed in space with origin O at a height h above the road and with XY-plane in the horizontal

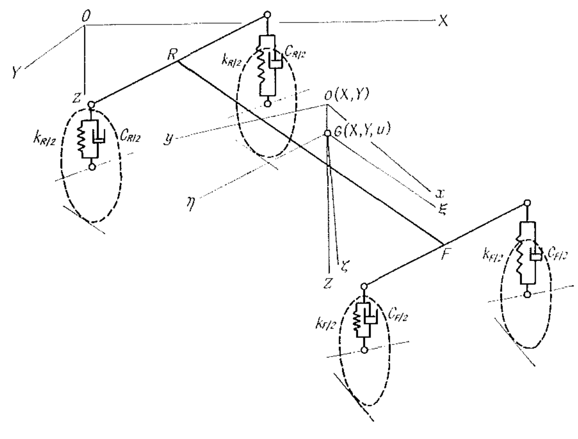

Fig. 1. A mathematical model of the automobile.

plane and with Z-axis vertical downward. The height h is that of the center of gravity of the sprung mass in the static equilibrium configuration. Another Cartesian coordinate system $oxyz$ is chosen in such a way that origin o and xy-plane moves only in XY-plane. If the x, y and z-axis are taken to be fixed at the longitudinal, lateral and vertical axes of the sprung mass through the center of gravity, (X, Y) coordinates of origin o will present the location of the center of gravity and the rotation of $oxyz$ coordinate system about z-axis will represent yaw motion.

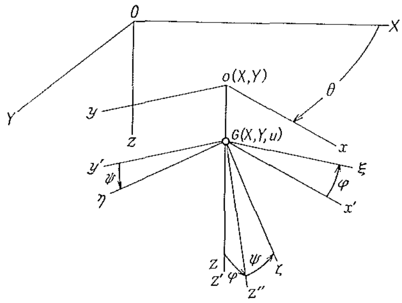

Fig. 2. Coordinate systems used for the analysis.

$oxyz$ coordinate system is again moved to $G\xi\eta\zeta$ system in the following way. As seen in **Fig. 2**, $oxyz$ coordinate system is translated to $Gx'y'z'$ system by a vertical displacement (bounce motion) u of the center of gravity and then $Gx'y'z'$ system is rotated to $G\xi y'z''$ system by pitch motion φ about Gy'-axis and finally $G\xi y'z''$ system is rotated to $G\xi\eta\zeta$ system by roll motion ϕ about $G\xi$-axis. Hence, the transformation between $Gx'y'z'$ and $G\xi\eta\zeta$ systems is written by

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ \sin\varphi\sin\phi & \cos\phi & \cos\varphi\sin\phi \\ \sin\varphi\cos\phi & -\sin\phi & \cos\varphi\cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{1}$$

Pitch and roll motions of the sprung mass are usually small and, hence, the transformation (1) will be simplified as

$$\begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\varphi \\ 0 & 1 & \phi \\ \varphi & -\phi & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \tag{2}$$

3. The equations of motion and geometrical and dynamical conditions

The usual assumption is also adopted in a model treated here for the convenience of mathematical treatment[5].

1) The road is level and for simplicity smooth.
2) The automobile has a vertical longitudinal plane of symmetry not only with regard to the mass distribution, but also with regard to any of the other system parameters.

3) The position of the center of gravity of the sprung mass relative to the unsprung mass is not changed, regardless of how the sprung mass may be deflected.
4) No front or rear axle streerings occur during the motion of automobile.
5) The steer angle of the front wheels is represented by the mean value of that of the left and right wheels.
6) The forward velocity and radius of turn are expressed by the velocity and radius of curvature at the center of gravity respectively.
7) Aerodynamic force is assumed to act on the sprung mass horizontally and its vertical component is ignored.
8) Gyromoments produced by the rotation of wheels are small enough to be ignored.
9) Camber angles produced on the wheels by centrifugal force are neglected for simplicity of the treatment, although they are not always small.
10) Engine torque reactions are not considered here.

Under these assumptions, two groups of the equations governing the motion of the automobile as an entire system and the motion of the sprung mass are obtained (See Fig. 3).

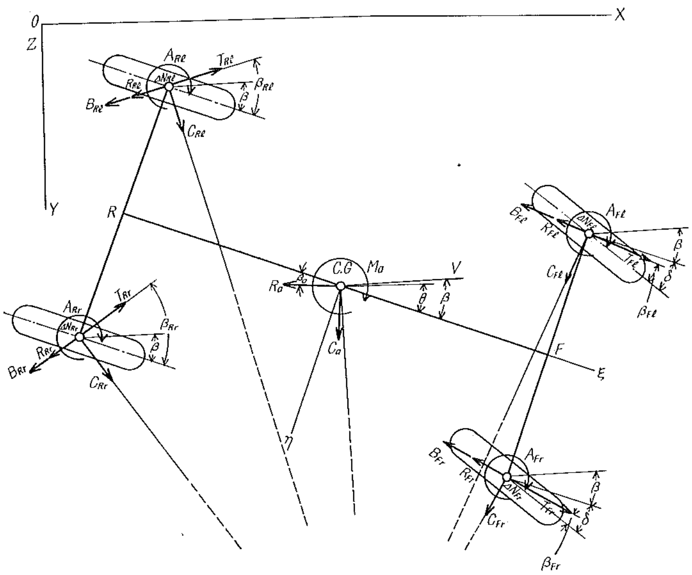

Fig. 3. Forces and moments acting on the automobile.

I) **The longitudinal, lateral and yaw motion.**

The motion in a forward direction is governed by the components of driving and braking forces and frictional forces of the tires and aerodynamic force acting on the automobile.

$$\begin{aligned}\frac{W}{g}\dot{V} = \; & (T-B-R)_{Fl}\cos(-\beta_{Fl}+\beta+\delta)-C_{Fl}\sin(-\beta_{Fl}+\beta+\delta)\\ &+(T-B-R)_{Fr}\cos(-\beta_{Fr}+\beta+\delta)-C_{Fr}\sin(-\beta_{Fr}+\beta+\delta)\\ &+(T-B-R)_{Rl}\cos(-\beta_{Rl}+\beta) \quad -C_{Rl}\sin(-\beta_{Rl}+\beta)\\ &+(T-B-R)_{Rr}\cos(-\beta_{Rr}+\beta) \quad -C_{Rr}\sin(-\beta_{Rr}+\beta)\\ &-R_a\cos(-\beta_a+\beta) \quad -C_a\sin(-\beta_a+\beta) \end{aligned} \tag{3}$$

The equations of motion in the lateral direction will be

$$\begin{aligned}\frac{W}{g}\frac{V^2}{r} = \; & (T-B-R)_{Fl}\sin(-\beta_{Fl}+\beta+\delta)+C_{Fl}\cos(-\beta_{Fl}+\beta+\delta)\\ &+(T-B-R)_{Fr}\sin(-\beta_{Fr}+\beta+\delta)+C_{Fr}\cos(-\beta_{Fr}+\beta+\delta)\\ &+(T-B-R)_{Rl}\sin(-\beta_{Rl}+\beta) \quad +C_{Rl}\cos(-\beta_{Rl}+\beta)\\ &+(T-B-R)_{Rr}\sin(-\beta_{Rr}+\beta) \quad +C_{Rr}\cos(-\beta_{Rr}+\beta)\\ &-R_a\sin(-\beta_a+\beta) \quad +C_a\cos(-\beta_a+\beta) \end{aligned} \tag{4}$$

The rotatory motion around the vertical axis through the center of gravity—yaw motion—will be governed by

$$\begin{aligned}&\frac{d}{dt}(J_\zeta\omega_\zeta-J_{\zeta\xi}\omega_\xi)+\omega_\xi(J_\eta\omega_\eta)-\omega_\eta(J_\xi\omega_\xi-J_{\zeta\xi}\omega_\zeta)\\ &= (T-B-R)_{Fl}\left\{ l_F\sin(-\beta_{Fl}+\delta)+\frac{1}{2}t_F\cos(-\beta_{Fl}+\delta)\right\}\\ &\quad +(T-B-R)_{Fr}\left\{ l_F\sin(-\beta_{Fr}+\delta)-\frac{1}{2}t_F\cos(-\beta_{Fr}+\delta)\right\}\\ &\quad +(T-B-R)_{Rl}\left\{-l_R\sin(-\beta_{Rl}) +\frac{1}{2}t_R\cos(-\beta_{Rl})\right\}\\ &\quad +(T-B-R)_{Rr}\left\{-l_R\sin(-\beta_{Rr}) -\frac{1}{2}t_R\cos(-\beta_{Rr})\right\}\\ &\quad +C_{Fl}\left\{ l_F\cos(-\beta_{Fl}+\delta)-\frac{1}{2}t_F\sin(-\beta_{Fl}+\delta)\right\}\\ &\quad +C_{Fr}\left\{ l_F\cos(-\beta_{Fr}+\delta)+\frac{1}{2}t_F\sin(-\beta_{Fr}+\delta)\right\}\\ &\quad +C_{Rl}\left\{-l_R\cos(-\beta_{Rl}) -\frac{1}{2}t_R\sin(-\beta_{Rl})\right\}\\ &\quad +C_{Rr}\left\{-l_R\cos(-\beta_{Rr}) +\frac{1}{2}t_R\sin(-\beta_{Rr})\right\}\\ &\quad +M_a+A_{Fl}+A_{Fr}+A_{Rl}+A_{Rr} \end{aligned} \tag{5}$$

(II) Bounce, pitch and roll motion of the sprung mass.

The motion of the center of gravity of the sprung mass in the vertical direction—bounce motion—is caused by the dynamic components of vertical reactions on the tires

$$\frac{W}{g}\ddot{u} = -\Delta N_{Fl} - \Delta N_{Fr} - \Delta N_{Rl} - \Delta N_{Rr} \tag{6}$$

The rotatory motion of the sprung mass around the lateral axis—pitch motion—will be governed by

$$\begin{aligned}
&\frac{d}{dt}(J_\eta \omega_\eta) + \omega_\zeta (J_\xi \omega_\xi - J_{\zeta\xi}\omega_\zeta) + \omega_\xi (J_\zeta \omega_\zeta - J_{\zeta\xi}\omega_\xi) \\
&= (\Delta N_{Fl} + \Delta N_{Fr}) l_F - (\Delta N_{Rl} + \Delta N_{Rr}) l_R \\
&\quad + \{(T-B-R)_{Fl} \cos(-\beta_{Fl} + \delta) - C_{Fl} \sin(-\beta_{Fl} + \delta) \\
&\quad + (T-B-R)_{Fr} \cos(-\beta_{Fr} + \delta) - C_{Fr} \sin(-\beta_{Fr} + \delta) \\
&\quad + (T-B-R)_{Rl} \cos(-\beta_{Rl}) - C_{Rl} \sin(-\beta_{Rl}) \\
&\quad + (T-B-R)_{Rr} \cos(-\beta_{Rr}) - C_{Rr} \sin(-\beta_{Rr})\} h \\
&\quad + \{-R_a \cos(-\beta_a) - C_a \sin(-\beta_a)\} h_a
\end{aligned} \tag{7}$$

Also the motion around the longitudinal axis—roll motion—will be

$$\begin{aligned}
&\frac{d}{dt}(J_\xi \omega_\xi - J_{\zeta\xi}\omega_\zeta) + \omega_\eta (J_\zeta \omega_\zeta - J_{\zeta\xi}\omega_\xi) - \omega_\zeta (J_\eta \omega_\eta) \\
&= (\Delta N_{Fl} - \Delta N_{Fr}) \frac{1}{2} t_F + (\Delta N_{Rl} - \Delta N_{Rr}) \frac{1}{2} t_R \\
&\quad - \{(T-B-R)_{Fl} \sin(-\beta_{Fl} + \delta) + C_{Fl} \cos(-\beta_{Fl} + \delta) \\
&\quad + (T-B-R)_{Fr} \sin(-\beta_{Fr} + \delta) + C_{Fr} \cos(-\beta_{Fr} + \delta) \\
&\quad + (T-B-R)_{Rl} \sin(-\beta_{Rl}) + C_{Rl} \cos(-\beta_{Rl}) \\
&\quad + (T-B-R)_{Rr} \sin(-\beta_{Rr}) + C_{Rr} \cos(-\beta_{Rr})\} h \\
&\quad - \{-R_a \sin(-\beta_a) + C_a \cos(-\beta_a)\} h_a
\end{aligned} \tag{8}$$

Geometrical and dynamical conditions on the motion of the automobile will be described in detail here. Referring to **Fig. 4**, side-slip angles of the tires are written by

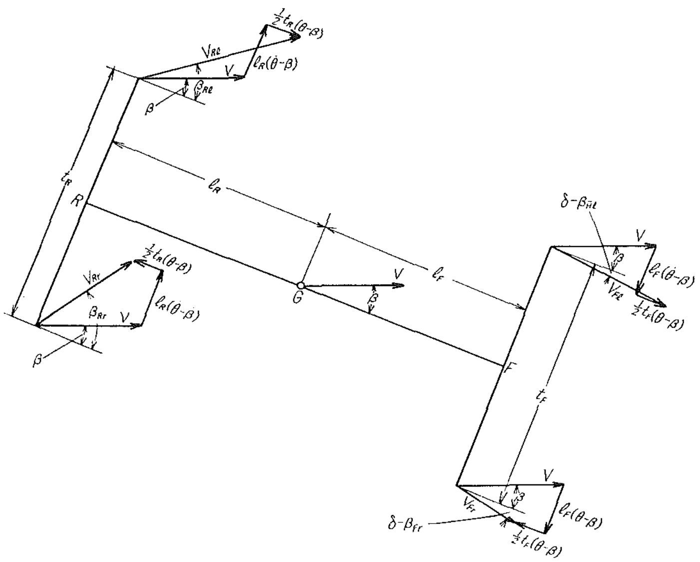

Fig. 4. Velocity at the position of the tires.

$$\left.\begin{aligned} \tan(-\beta_{Fl}+\delta) &= \frac{-r\sin\beta+l_F}{r\cos\beta+t_F/2}, \quad \tan(-\beta_{Fr}+\delta) = \frac{-r\sin\beta+l_F}{r\cos\beta-t_F/2} \\ \tan(-\beta_{Rl}) &= \frac{-r\sin\beta-l_R}{r\cos\beta+t_R/2}, \quad \tan(-\beta_{Rr}) = \frac{-r\sin\beta-l_R}{r\cos\beta-t_R/2} \end{aligned}\right\} \quad (9)$$

As pitch and roll angles are usually small, components of an angular velocity around the coordinate axes of the sprung mass will be approximately

$$\left.\begin{aligned} \omega_\xi &= \dot\phi-\dot\theta\sin\varphi \approx \dot\phi \\ \omega_\eta &= \dot\theta\cos\varphi\sin\phi+\dot\varphi\cos\phi \approx \dot\varphi \\ \omega_\zeta &= \dot\theta\cos\varphi\cos\phi-\dot\phi\sin\phi \approx \dot\theta \end{aligned}\right\} \quad (10)$$

Yawing velocity is represented by the forward velocity and radius of turn of the center of gravity.

$$\frac{V}{r} = \dot\theta-\dot\beta \quad (11)$$

Frictional forces of the tires may be written in the form

$$R_i = \mu_{r0}(1+m\beta_i^n)N_i \qquad (i=Fl,\ Fr;\ Rl,\ Rr) \quad (12)$$

The cornering forces on the tires may be expressed by modified Fiala's formula[9], whose original expression has been used by many authors[10)11)] for calculation of the automotive performance

$$C_i = K_i\beta_i\left\{1+\gamma_1\left(\frac{K_i\beta_i}{\mu_i N_i}\right)+\gamma_2\left(\frac{K_i\beta_i}{\mu_i N_i}\right)^2\right\} \tag{13}$$

where K_i are constants governed by contact length and contact pressure between the tires and the ground, form and dimension of the tread and elasticity of the tires, etc. and μ_i are friction coefficients between the tires and the road in the lateral direction. The expression (13) for the cornering forces is applicable to the motion of an automobile with small side-slip angles, where the quantities γ_1 and γ_2 are the correcting coefficients, for which $\gamma_1=-1/3$, $\gamma_2=1/27$ were obtained theoretically by Fiala[9] or $\gamma_1=-0.0668$, $\gamma_2=-0.1032$ as were proposed by other authors[11] from many experimental results. And K_i are the quantities corresponding to 'cornering powers' which are variables rather than constants and may be written in the form

$$K_i = K_0(1+\kappa_1 N_i+\kappa_2 N_i^2+\kappa_3 N_i^3) \tag{14}$$

where κ_1, κ_2 and κ_3 are the correcting coefficients. The values of friction coefficients of the tires in the lateral direction are also usually affected by the vertical reactions on the tires and may be

$$\mu_i = \mu_0(1+\lambda_1 N_i+\lambda_2 N_i^2) \tag{15}$$

where λ_1 and λ_2 are the correcting coefficients that should be determined experimentally from the characteristics of the tires and the road. And, hence the cornering forces on the tires may be written by

$$C_i = K_0\beta_i\left\{1+\gamma_i\left(\frac{K_0\beta_i}{\mu_0 N_i}\right)+\gamma_2\left(\frac{K_0\beta_i}{\mu_0 N_i}\right)^2+\lambda_1 N_i+\lambda_2 N_i^2+\cdots\right\} \tag{16}$$

The expressions for the cornering forces are not necessarily correct for all combinations of the tires and the road, which have been studied by many authors[9),12)~15)]. For aligning torques produced by the tires, Fiala's formula is adopted again

$$A_i = \alpha_i\beta_i l\left\{1+a_1\left(\frac{K_0\beta_i}{\mu_i N_i}\right)+a_2\left(\frac{K_0\beta_i}{\mu_i N_i}\right)^2+a_3\left(\frac{K_0\beta_i}{\mu_i N_i}\right)^3\right\} \tag{17}$$

where a_1, a_2, a_3, are the correcting coefficients, for which $a_1=-1$, $a_2=1/3$, $a_3=-1/27$ were also obtained theoretically by Fiala.

The vertical reactions on the tires are conveniently divided into two parts, one of which is equal to the weight of the sprung mass statically distributed to four tires and the other is a dynamic component changed by the motion of the sprung mass. The former is simply written as

$$(N_{Fl})_{st}=(N_{Fr})_{st}=\frac{1}{2}\frac{l_R}{l}W, \quad (N_{Rl})_{st}=(N_{Rr})_{st}=\frac{1}{2}\frac{l_F}{l}W \tag{18}$$

Dynamic components of the vertical reactions are obtained by introducing the deflection of the sprung mass and equivalent spring stiffness of suspension mechanism at the position of the tires. Deflections at the position of the tires are written by bounce, pitch and yaw angles

$$\left.\begin{array}{lll} u-l_F\varphi-\frac{1}{2}t_F\phi, & u-l_F\varphi+\frac{1}{2}t_F\phi & \text{(Front wheel)} \\ u+l_R\varphi-\frac{1}{2}t_R\phi, & u+l_R\varphi+\frac{1}{2}t_R\phi & \text{(Rear wheel)} \end{array}\right\} \quad (19)$$

Introducing equivalent spring stiffnesses $2\times k_F/2$, $2\times k_R/2$ and equivalent damping coefficients $2\times c_F/2$, $2\times c_R/2$ of the suspension mechanism of the tires, dynamic reacions are

$$\left.\begin{aligned} \Delta N_{Fl} &= \frac{1}{2}\left(k+c\frac{d}{dt}\right)_F\left(u-l_F\varphi-\frac{1}{2}t_F\phi\right) \\ \Delta N_{Fr} &= \frac{1}{2}\left(k+c\frac{d}{dt}\right)_F\left(u-l_F\varphi+\frac{1}{2}t_F\phi\right) \\ \Delta N_{Rl} &= \frac{1}{2}\left(k+c\frac{d}{dt}\right)_R\left(u+l_R\varphi-\frac{1}{2}t_R\phi\right) \\ \Delta N_{Rr} &= \frac{1}{2}\left(k+c\frac{d}{dt}\right)_R\left(u+l_R\varphi+\frac{1}{2}t_R\phi\right) \end{aligned}\right\} \quad (20)$$

Components of aerodynamic force acting on the automobile may be expressed in the usual way

$$R_a = C_r\frac{1}{2}\rho V^{*2}S, \quad C_a = \frac{dC_c}{d\beta}\beta\frac{1}{2}\rho V^{*2}S \quad (21)$$

and aerodynamic moment around the yaw axis is

$$M_a = \frac{dC_m}{d\beta}\beta\frac{1}{2}\rho V^{*2}Sl \quad (22)$$

where V^* is the resultant velocity of the wind on the automobile

$$V^* = \sqrt{(V\cos\theta - w\cos\chi)^2 + (V\sin\theta - w\sin\chi)^2} \quad (23)$$

Quantities regarded as variables with respect to time among many quantities appearing in this study, will be

$$V, r(\text{or } T, \delta);\ \beta, \beta_i;\ \theta, \varphi, \phi;\ \omega_\xi, \omega_\eta, \omega_\zeta;\ u;\ A_i, C_i, N_i, R_i$$
$$(i = Fl, Fr, Rl, Rr)$$

If they can be solved from the equations introduced here, the dynamic behavior of the automobile becomes clear, but the mathematical treatment for the equations will be generally difficult.

4. The steady-state circular turn

The theory is applied to problems of steady-state circular turn[5)16)]. In this case, the change rate of the quantities $V, u, \varphi, \phi, \beta$ are all zero and the mathematical treatment will become considerably simple. From the equations (6), (7), (8) and (20), bounce and pitch angle of the stationarily turning automobile are respectively

$$u = \frac{k_F l_F - k_R l_R}{k_F k_R l^2}\left[J_{\zeta\xi}\left(\frac{V}{r}\right)^2 + \frac{W}{g}\frac{V^2}{r}h\sin\beta + \left\{R_a\cos(-\beta_a) + C_a\sin(-\beta_a)\right\}(h-h_a)\right] \quad (24)$$

and

$$\varphi = \frac{k_F + k_R}{k_F k_R l^2}\left[J_{\zeta\xi}\left(\frac{V}{r}\right)^2 + \frac{W}{g}\frac{V^2}{r}h\sin\beta + \left\{R_a\cos(-\beta_a) + C_a\sin(-\beta_a)\right\}(h-h_a)\right] \quad (25)$$

which are found to be caused by dynamical moments due to inertia around the lateral axis of the sprung mass, centrifugal force and aerodynamic force. However, these are all practically small enough to be ignored except in a special case. And the roll angle is

$$\phi = -\frac{4}{k_F t_F^2 + k_R t_R^2}\left[\frac{W}{g}\frac{V^2}{r}h\cos\beta + \left\{R_a\sin(-\beta_a) - C_a\cos(-\beta_a)\right\}(h-h_a)\right] \quad (26)$$

Remembering that the contribution of the aerodynamic moment around the longitudinal axis is small, the roll angle of the sprung mass is proportional to the centrifugal force due to a circular turn and height of the center of gravity of the sprung mass above the road. The negative sign for the rool angle indicates that the sprung mass is rolled outwards from the turning circle. Neglecting small quantities, deflection of the sprung mass will be simply

$$u = \varphi = 0\,, \quad \phi = -\frac{4}{k_F t_F^2 + k_R t_R^2}\frac{W}{g}\frac{V^2}{r}h\cos\beta \quad (27)$$

Roll motion of the sprung mass will be followed by the non-uniform distribution of vertical reactions on the tires

$$\left.\begin{aligned}\Delta N_{F_r^l} &= \pm\frac{k_F t_F h}{k_F t_F^2 + k_R t_R^2}\frac{W}{g}\frac{V^2}{r}\cos\beta \\ \Delta N_{R_r^l} &= \pm\frac{k_R t_R h}{k_F t_F^2 + k_R t_R^2}\frac{W}{g}\frac{V^2}{r}\cos\beta\end{aligned}\right\} \tag{28}$$

which are proportional to the roll angle and are governed by dimension and stiffness of the suspension mechanism. The change of vertical reactions on the tires have some effects on frictional forces and cornering forces and, hence, on the general performance of the automobile.

The equation (9) will be written approximately in a simple form, when the radius of turn is not extremely small and the side-slip angle not so large

$$\left.\begin{aligned}\beta_{Fl} = \beta_{Fr} &= -\frac{l_F}{r} + \beta + \delta \quad (= \beta_F) \\ \beta_{Rl} = \beta_{Rr} &= \frac{l_R}{r} + \beta \qquad (= \beta_R)\end{aligned}\right\} \tag{29}$$

where small differences between the side-slip angles of the left and right wheels are considered to be ignored for simplicity.

The equations on the longitudinal, lateral and yaw motion are respectively

$$T = R_{Fl} + R_{Fr} + R_{Rl} + R_{Rr} + (C_{Fl} + C_{Fr})\frac{l_F}{r} - (C_{Rl} + C_{Rr})\frac{l_R}{r} + R_a \tag{30}$$

$$\begin{aligned}\frac{W}{g}\frac{V^2}{r} = &-(R_{Fl} + R_{Fr})\frac{l_F}{r} + (-T + R_{Rl} + R_{Rr})\frac{l_R}{r} \\ &+ C_{Fl} + C_{Fr} + C_{Rl} + C_{Rr} + C_a\end{aligned} \tag{31}$$

and

$$\begin{aligned}0 = &-R_{Fl}\left\{ l_F\left(\frac{l_F}{r} - \beta\right) + \frac{1}{2}t_F\right\} - R_{Fr}\left\{ l_F\left(\frac{l_F}{r} - \beta\right) - \frac{1}{2}t_F\right\} \\ &+ \left(\frac{T}{2} - R_{Rl}\right)\left\{-l_R\left(-\frac{l_R}{r} - \beta\right) + \frac{1}{2}t_R\right\} + \left(\frac{T}{2} - R_{Rr}\right)\left\{-l_R\left(-\frac{l_R}{r} - \beta\right) - \frac{1}{2}t_R\right\} \\ &+ C_{Fl}\left\{ l_F - \frac{1}{2}t_F\left(\frac{l_F}{r} - \beta\right)\right\} + C_{Fr}\left\{ l_F + \frac{1}{2}t_F\left(\frac{l_F}{r} - \beta\right)\right\} \\ &+ C_{Rl}\left\{-l_R - \frac{1}{2}t_R\left(-\frac{l_R}{r} - \beta\right)\right\} + C_{Rr}\left\{-l_R + \frac{1}{2}t_R\left(-\frac{l_R}{r} - \beta\right)\right\} \\ &+ M_a + A_{Fl} + A_{Fr} + A_{Rl} + A_{Rr}\end{aligned} \tag{32}$$

Substituting the expressions for frictional forces, cornering forces, aligning torques on the tires and aerodynamic force into the equations (30)~(32), the following equations are obtained

$$T=\left\{1+m\left(\frac{l_R}{l}\beta_F^n+\frac{l_F}{l}\beta_R^n\right)\right\}\mu_{r0}W+\Delta_1 \tag{33}$$

$$\Delta_1=\frac{2}{r}(K_F\beta_F l_F-K_R\beta_R l_R)+\frac{l_F}{r}(\Delta C_{Fl}+\Delta C_{Fr})$$
$$-\frac{l_R}{r}(\Delta C_{Rl}+\Delta C_{Rr})+C_r\frac{1}{2}\rho V^2S \tag{33$'$}$$

$$\frac{W}{g}\frac{V^2}{r}=2(K_F\beta_F+K_R\beta_R)+\Delta_2 \tag{34}$$

$$\Delta_2=-\left\{T+m(\beta_F^n-\beta_R^n)\mu_{r0}W\frac{l_F}{l}\right\}\frac{l_R}{r}+\Delta C_{Fl}+\Delta C_{Fr}$$
$$+\Delta C_{Rl}+\Delta C_{Rr}+\frac{dC_c}{d\beta}\beta\frac{1}{2}\rho V^2S \tag{34$'$}$$

and

$$0=2(K_F\beta_F l_F-K_R\beta_R l_R)+l\Delta_3 \tag{35}$$

$$l\Delta_3=-(1+m\beta_F^n)\mu_{r0}\frac{1}{2}\frac{l_R}{l}W\left\{1+\Delta N_{Fl}\Big/\left(\frac{1}{2}\frac{l_R}{l}W\right)\right\}\left\{l_F\left(\frac{l_F}{r}-\beta\right)+\frac{1}{2}t_F\right\}$$
$$-(1+m\beta_F^n)\mu_{r0}\frac{1}{2}\frac{l_R}{l}W\left\{1+\Delta N_{Fr}\Big/\left(\frac{1}{2}\frac{l_R}{l}W\right)\right\}\left\{l_F\left(\frac{l_F}{r}-\beta\right)-\frac{1}{2}t_F\right\}$$
$$+\left\{\frac{T}{2}-(1+m\beta_R^n)\right\}\mu_{r0}\frac{1}{2}\frac{l_F}{l}W\left\{1+\Delta N_{Rl}\Big/\left(\frac{1}{2}\frac{l_F}{l}W\right)\right\}$$
$$\times\left\{-l_R\left(-\frac{l_R}{r}-\beta\right)+\frac{1}{2}t_R\right\}$$
$$+\left\{\frac{T}{2}-(1+m\beta_R^n)\right\}\mu_{r0}\frac{1}{2}\frac{l_F}{l}W\left\{1+\Delta N_{Rr}\Big/\left(\frac{1}{2}\frac{l_F}{l}W\right)\right\}$$
$$\times\left\{-l_R\left(-\frac{l_R}{r}-\beta\right)-\frac{1}{2}t_R\right\}$$
$$+\Delta C_{Fl}\left\{l_F-\frac{1}{2}t_F\left(\frac{l_F}{r}-\beta\right)\right\}+\Delta C_{Fr}\left\{l_F+\frac{1}{2}t_F\left(\frac{l_F}{r}-\beta\right)\right\}$$
$$+\Delta C_{Rl}\left\{-l_R-\frac{1}{2}t_R\left(-\frac{l_R}{r}-\beta\right)\right\}+\Delta C_{Rr}\left\{-l_R+\frac{1}{2}t_R\left(-\frac{l_R}{r}-\beta\right)\right\}$$
$$-2(\alpha_F\beta_F+\alpha_R\beta_R)+\frac{dC_m}{d\beta}\beta\frac{1}{2}\rho V^2Sl \tag{35$'$}$$

where

$$\Delta C_i=K_0\beta_i\left\{\gamma_1\left(\frac{K_0\beta_i}{\mu_0N_i}\right)+\gamma_2\left(\frac{K_0\beta_i}{\mu_0N_i}\right)^2+\kappa_1N_i+\kappa_2N_i^2\right\} \tag{36}$$

Only the first terms on the right-hand sides of the equations (33)~(35) have been treated in the usual studies, however, by considering more accurate values of the cornering forces, the load distribution on the tires, the aligning torques and aerodynamic forces, the second terms including the correcting terms Δ_1, Δ_2, Δ_3 must be also treated.

Ignoring the terms Δ_1, Δ_2, Δ_3, the radius and forward velocity of the steady-state circular turn are

$$r=\frac{l}{\delta-\left(\frac{K_R}{K_F+K_R}-\frac{l_F}{l}\right)\left(\frac{1}{K_F}+\frac{1}{K_R}\right)\left[\frac{1}{m}\left(\frac{T}{\mu_{r0}W}-1\right)\Big/\left\{\left(\frac{l_R}{l}\right)^{n+1}\frac{1}{K_F^n}+\left(\frac{l_F}{l}\right)^{n+1}\frac{1}{K_R^n}\right\}\right]^{1/n}} \tag{37}$$

and

$$V=\left[\frac{1}{m}\left(\frac{T}{\mu_{r0}W}-1\right)\Big/\left\{\left(\frac{l_R}{l}\right)^{n+1}\frac{1}{K_F^n}+\left(\frac{l_F}{l}\right)^{n+1}\frac{1}{K_R^n}\right\}\right]^{\frac{1}{2n}}\sqrt{r\Big/\left(\frac{1}{2}\frac{W}{g}\right)} \tag{38}$$

Steer angle of front wheels necessary to maintain a circular turn will be

$$\delta=\frac{l}{r}+\frac{1}{2}\left(\frac{K_R}{K_F+K_R}-\frac{l_F}{l}\right)\left(\frac{1}{K_F}+\frac{1}{K_R}\right)\frac{W}{g}\frac{V^2}{r} \tag{39}$$

The first term on the right-hand side which is called 'Ackerman's angle' corresponds to the geometrical angle without side-slip motion and the quantity $K_R/(K_F+K_R)-l_F/l$ in the second term is the 'static margin' and if it is positive, the automobile has 'over-steer' characteristics and if inversely negative, 'under-steer' characteristics.

The driving force is

$$\frac{T}{\mu_{r0}W}=1+\frac{m}{2^n}\left\{\left(\frac{l_R}{l}\right)^{n+1}\frac{1}{K_F^n}+\left(\frac{l_F}{l}\right)^{n+1}\frac{1}{K_R^n}\right\}\left(\frac{W}{g}\frac{V^2}{r}\right)^n \tag{40}$$

which presents the ratio of driving force to frictional forces of all tires without side-slip. It will be found that always $T/\mu_{r0}W>1$, i.e. driving force, must be larger than the frictional forces of the tires without side-slip to maintain a steady-state circular turn. Side-slip angle at the center of gravity becomes

$$\beta=-\frac{l_R}{r}+\frac{1}{2}\frac{l_F}{l}\frac{1}{K_R}\frac{W}{g}\frac{V^2}{r} \tag{41}$$

The results presented in the equations (39)~(41), are the same with those treated in the usual theory and, to expect a more accurate calculation, the correcting terms presented in the equations (33)′~(35)′ must be brought into the calculation. That is,

$$\delta=\delta_{\text{Ack.}}+\frac{1}{2}\left(\frac{K_R}{K_F+K_R}-\frac{l_F}{l}\right)\left(\frac{1}{K_F}+\frac{1}{K_R}\right)\frac{W}{g}\frac{V}{r}$$
$$-\frac{1}{2}\frac{1}{l}\left(\frac{K_R}{K_F+K_R}-\frac{l_F}{l}\right)\left(\frac{1}{K_F}+\frac{1}{K_R}\right)\varDelta_2-\frac{1}{2}\left(\frac{1}{K_F}+\frac{1}{K_R}\right)\varDelta_3 \quad (42)$$

$$\frac{T}{\mu_{r0}W}=1+\left[m\left(\frac{l_R}{l}\right)^{n+1}\frac{1}{K_F^n}\left\{1-\varDelta_2\Big/\left(\frac{W}{g}\frac{V^2}{r}\right)-\varDelta_3\Big/\left(\frac{l_R}{l}\frac{W}{g}\frac{V^2}{r}\right)\right\}^n\right.$$
$$\left.+m\left(\frac{l_F}{l}\right)^{n+1}\frac{1}{K_R^n}\left\{1-\varDelta_2\Big/\left(\frac{W}{g}\frac{V^2}{r}\right)+\varDelta_3\Big/\left(\frac{l_F}{l}\frac{W}{g}\frac{V^2}{r}\right)\right\}^n\right]\left(\frac{W}{2g}\frac{V^2}{r}\right)^n$$
$$+\frac{\varDelta_1}{\mu_{r0}W} \quad (43)$$

and

$$\beta=-\frac{l_R}{r}+\frac{1}{2}\frac{l_F}{l}\frac{1}{K_R}\frac{W}{g}\frac{V^2}{r}\left\{1-\varDelta_2\Big/\left(\frac{W}{g}\frac{V^2}{r}\right)+\varDelta_3\Big/\left(\frac{l_F}{l}\frac{W}{g}\frac{V^2}{r}\right)\right\} \quad (44)$$

A process of the numerical calculation is explained in **Fig. 5.** At first, for given values of r and V, the (first) approximate values for β, δ, β_i, N_i, A_i, C_i, R_i are calculated by the equations written in the brackets of **Fig. 5**

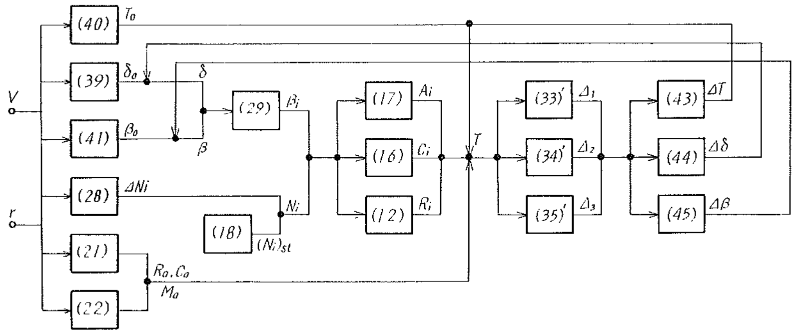

Fig. 5. A diagram of the numerical calculation.

respectively and the correcting terms $\varDelta_1$, $\varDelta_2$, $\varDelta_3$ are evaluated by putting those values into the equations (33)'~(35)', and then the correcting values for T, δ, β are calculated and fed back to the positions indicated in **Fig. 5.** By repeating the same process of calculation, the required accurate solutions will be expected to be obtained iteratively, only if they converge.

5. Some results of numerical calculaion

Applying the theory to the automobile of standard type, some results of the numerical calculation carried out on the problem of steady-state circular turn and some consideration will be presented.

Dimension of the automobile used here are

$$W = 1.235\,\text{kg};\quad 1 = 2.69\,\text{m};\quad l_F = 1.28\,\text{m},\quad l_R = 1.41\,\text{m};$$
$$t_F = 1.36\,\text{m},\quad t_R = 1.38\,\text{m};\quad h = 0.56\,\text{m};\quad S = 2.49\,\text{m}^2;$$
$$C_r = 0.3,\quad dC_c/d\beta = 0.03\,\text{deg}^{-1}.,\quad dC_m/d\beta = 0.03\,\text{deg}^{-1}.$$

The equivalent stiffnesses of the suspension mechanism at the wheels are conveniently assumed to be $2\times k_F/2 = 2\times 2.92\times 10^3$ kg/m, $2\times k_R/2 = 2\times 2.65\times 10^3$ kg/m and, therfore, the natural frequency of the sprung mass will be 1.5cps.

The numerical values concerning frictional forces of the tires $\mu_{r0}=0.012$; $m=0.12$, $n=2$ are used here and the coefficients of aligning torques produced by the tires are assumed to be $\alpha_F=\alpha_R=0.457$ kg/deg.. In **Fig. 6**, the broken

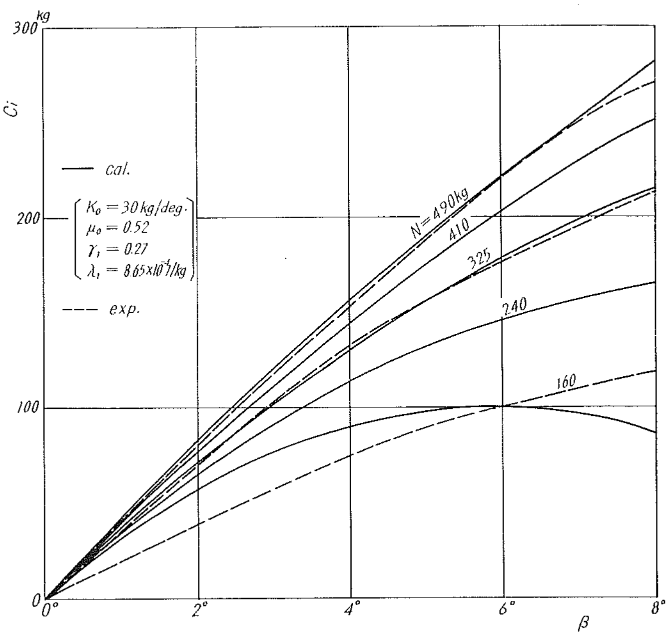

Fig. 6. Cornering forces on the tires.

lines are the cornering forces obtained experimentally for $N=160$ kg, 325 kg, 490 kg; while the full lines are the results calculated from the equation (16), applying $K_0=30$ kg/deg, $\mu_0=0.52$, $\gamma_1=0.27$, $\lambda_1=8.65\times10^{-4}\,\text{kg}^{-1}$, which will be found to agree considerably with the experimental results on the small side-slip motion, except for the case in which the cornering forces are extremely small, but which have a property to saturate or inversely decrease owing to the side-

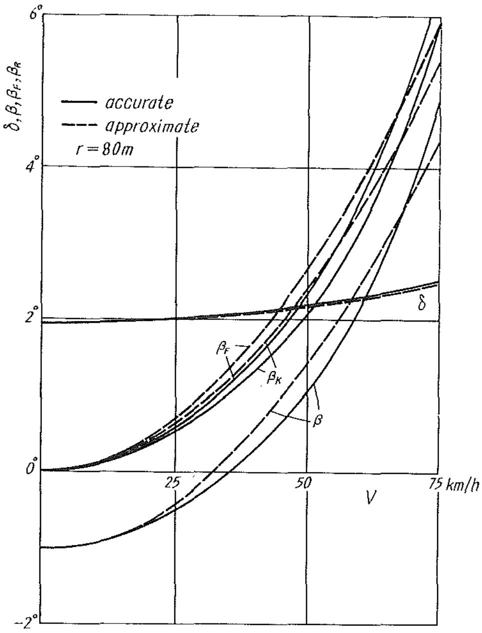

Fig. 7-1. Steer angle of front wheels and side-slip angles.

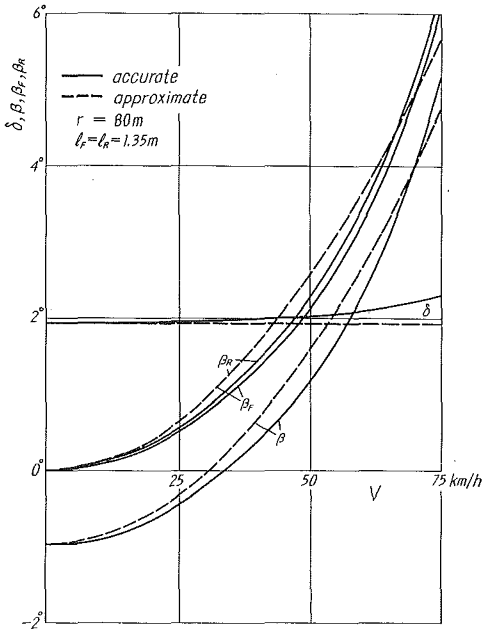

Fig. 7-2. Steer angle of front wheels and side-slip angles.

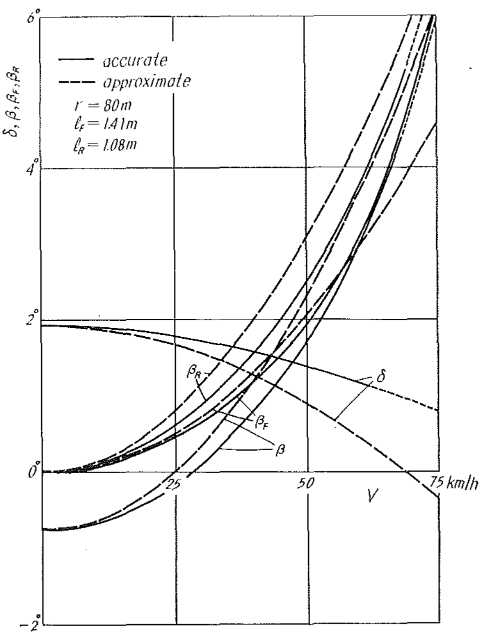

Fig. 7-3. Steer angle of front wheels and side-slip angles.

slip angles being large. The expressions and their coefficients for the cornering forces which give more accurate agreement with many experimental results must be studied in the future.

Fig. 7 is the result of a numerical calculation of steer angle of front wheels and side-slip angles of the center of gravity and the tires during a steady-state circular turn with radius of 80 m, where the broken lines present approximate values and the full lines are the results obtained by the iteration method presented here. And each figure of **Fig. 7–1～7–3** presents a numerical example on the automobile of a standard type having 'undeer-steer', 'neutral-steer' and 'over-steer' characteristics respectively. It is found from these curves that the accurate values of steer angle of the front wheels to maintain a circular turn need be larger than the approximate one and, hence, the automobile having 'neutral-steer' characteristics presents a slight tendency of 'under-steer' characteristics and the automobile having 'over-steer' characteristics weakens this tendency. Also the accurate side-slip angles are found to be all smaller than the approximate one for ordinary speed. **Fig. 8** shows the steer angle of the front wheels and side-slip angles of the automobile whose weight increases to 1.735 kg, from which the effects on them are hardly found. **Fig. 9** gives the steer angle of the front wheels and side-slip angle of the center of gravity

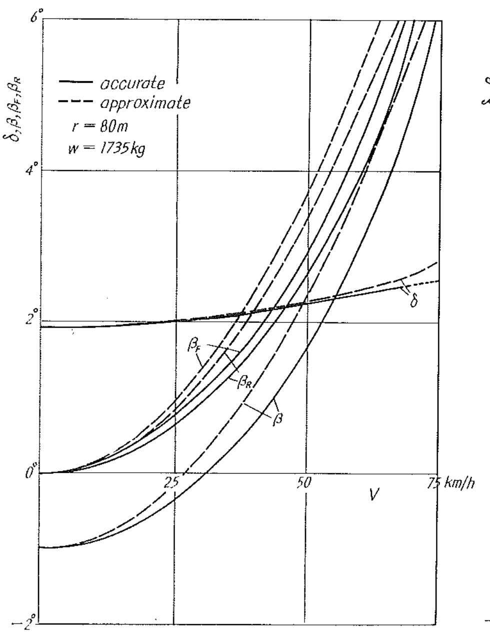

Fig. 8. Steer angle of front wheels and side-slip angles.

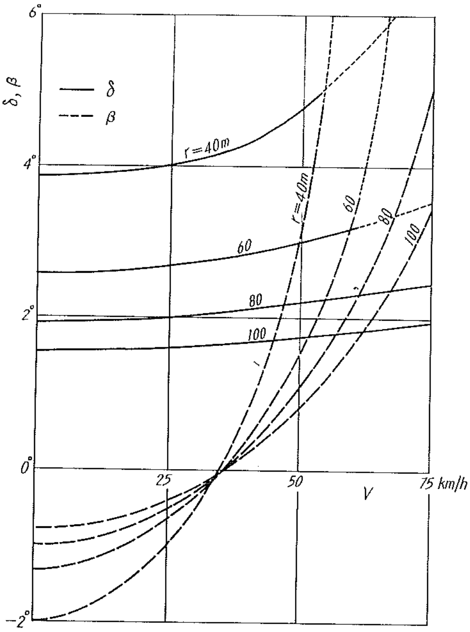

Fig. 9. Steer angle of front wheels and side-slip angles.

whose radii are varied in many ways, frcm which it will be concluded that the steer angle and absolute values of side-slip angles become larger when forward velocity increases and radius of turn inversely decreases. The automobile has a property in which 'under-steer' characteristics become remarkable when the radius of turn is small, as seen in the equation (39).

Fig. 10 shows the roll angle of the sprung mass calculated by the equation (27), which increases when the forward velccity is large and the radius of turn small. The roll motion is caused naturally by the change of load

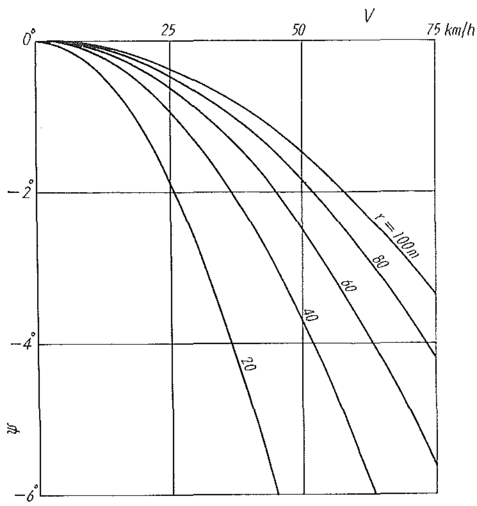

Fig. 10. Roll angle of sprung mass.

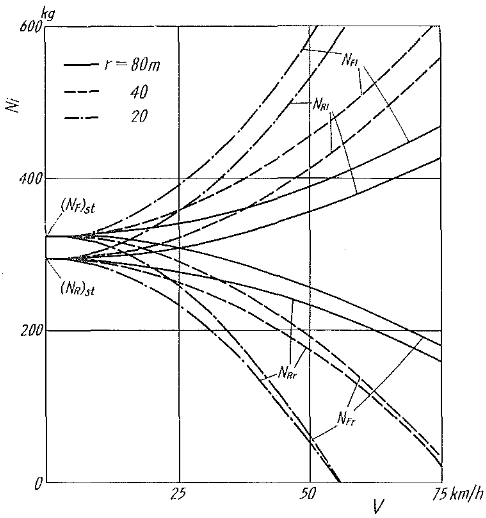

Fig. 11. Vertical reactions on the tires.

distribution on the tires, as seen in **Fig. 11**, and the vertical reactions on the right wheels are smaller, while the reactions on the left wheels are larger than the reactions of the automobile stopping or going straight on, when it is turned in the right direction. When the forward velocity increases or the radius decreases to a critical value where the vertical reactions on the tires on either side become zero, the automobile will be in a dangerous state in as much as the wheels on that side lift from the ground and, therefore, the forces acting on the tires become zero. The exact driving force to maintain a circular turn is larger than the approximate one and becomes large, as seen in **Fig. 12**. The broken lines here present the cornering forces of the front and rear wheels obtained by approximate calculation where the values on both sides are not distinguished from each other, but practically the difference of the cornering forces between right and left wheels is produced by the varying load distribution on the tires.

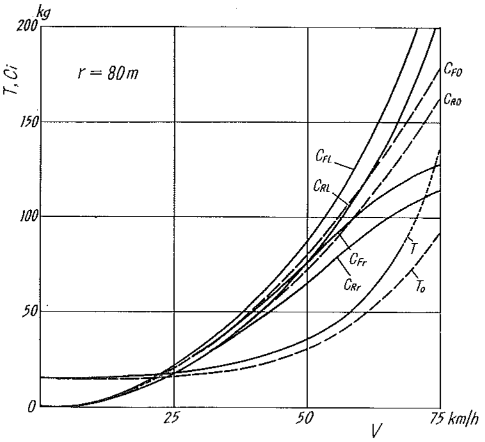

Fig. 12. Driving force and cornering forces of the tires.

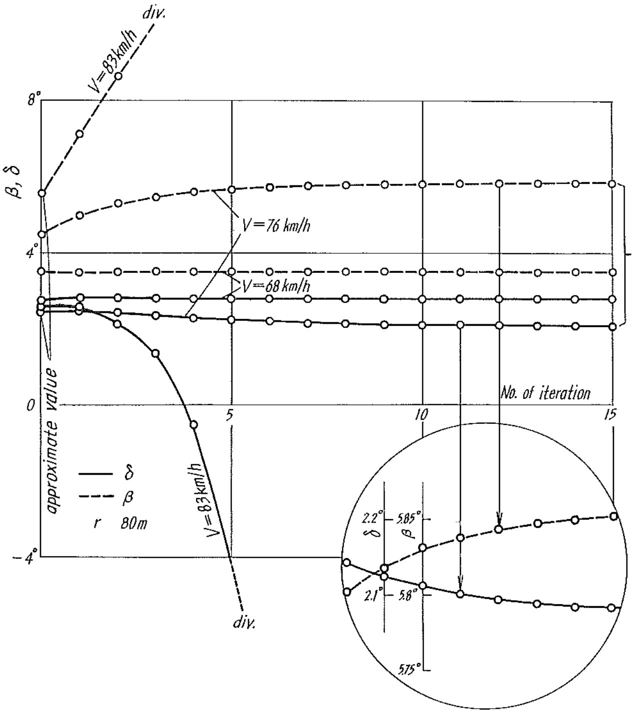

Fig. 13. Steer angle and side-slip angle changed by number of iteration.

The results obtained by the iteration method will be effective only if the numerical value tends to a definite one and all curves presented in this paper are drawn with only convergent values. **Fig. 13** presents an example of the numerical values of the steer angle of the front wheels and side-slip angle changed by the number of iterative calculations, both of which rapidly converge when the velocity is small and the radius large and eventually diverges as the velocity becomes large and the radius small. And **Fig. 14** presents a

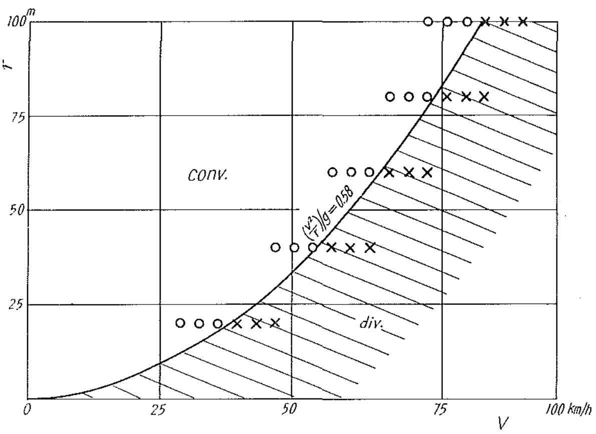

Fig. 14. Convergence domain of numerical calculation.

convergence domain for these numerical values, from which it will be seen that the iteration method is effective for $(V^2/r)/g<0.58$ and another method must be studied for the case of larger velocity and smaller radius.

6. Conclusion

In this paper, for the purpose of studying the performance of an automobile theoretically, the following treatment was carried out and conclusions were obtained.

1. Two Cartesian coordinate systems were introduced, one of which presents the automotive motion as an entire system such as longitudinal, lateral and yaw motions, and the other designates the motion of the sprung mass such as bounce, pitch and roll motions.
2. Dynamical equations of motion of six degrees of freedom taking into account the motion of the sprung mass were established and geometrical and dynamical conditions on the motion were presented in detail.
3. Applying the theory to the problem of a steady-state circular turn, the

quantities governing the automotive performance were calculated.

4. The equations on those quantities are so difficult to treat mathematically that an iteration method was proposed and the process for computing their accurate value was presented in a diagram. It was shown to be effective for practical use, from the results of numerical calculation on an automobile of standard type and at the same time a convergence domain where this method is effective, was determined.
5. It was found, from the results of accurate calculations of steer angle of the front wheels obtained by the method, that the automobile having 'neutral-steer' characteristics presents a slight tendency of 'undeer-steer' and also the 'over-steer' characteristics is weakened.

The authors wish to express their thanks to Prof. K. Hata for his kind suggestions.

Notation

A_i : Aligning torques produced by the tires. ($i = Fl, Fr; Rl, Rr$)

a_1, a_2, a_3 : Correcting coefficients of aligning torques.

B_i : Braking forces.

C_a : Cornering force produced by aerodynamic force.

C_i : Cornering forces on the tires.

$C_r, dC_c/d\beta$: Coefficients of aerodynamic forces.

$dC_m/d\beta$: Coefficient of aerodynamic moment.

h : Height of the center of gravity of the sprung mass in the equilibrium configuration above the road.

h : Vertical distance between the aerodynamic center and the center of gravity.

J_ξ, J_η, J_ζ : Moments of intertia of the sprung mass around the axes through the center of gravity.

$J_{\zeta\xi}$: Product of inertia of the sprung mass around the lateral axis.

K_i : Cornering stiffnesses of the tires.

$k_F, k_R; c_F, c_R$: Equivalent spring stiffnesses and damping coefficients of the front and rear wheels.

l : Horizontal distance between the front and rear wheel axes.

l_F, l_R : Horizontal distances between the front and rear wheel axes from the center of gravity of the sprung mass.

M_a : Aerodynamic moment.

m, n : Constants concerning frictional forces of the tires.

N_i : Vertical reactions on the tires.

R_a : Aerodynamic force.
R_i : Frictional forces of the tires.
r : Radius of turn.
S : Projected area of an automobile to the vertical plane in the longitudinal direction.
T_i : Driving forces of the tires.
t_F, t_R : Front and rear tracks.
u : Bounce of the sprung mass.
V : Forward velocity of the automobile.
V^* : Resultant velocity of the wind to the automobile.
W : Weight of the sprung mass.
w : Wind velocity to the ground.
XYZ : Space coordinate system.
xyz : Coordinate system fixed in the sprung mass in a static equilibrium configuration.
α_i : Coefficients of aligning torques on the tires.
β : Side-slip angle of the center of gravity.
β_i : Side-slip angles of the tires.
γ_1, γ_2 : Correcting coefficients of cornering forces.
δ : Steer angle of the front wheels.
$\xi\eta\zeta$: Coordinate system fixed in the deflected sprung mass.
θ : Yaw angle.
κ_2, κ_1 : Correcting coefficients of 'cornering powers'.
λ_2, λ_1 : Correcting coefficients of friction in the lateral diretion.
μ_i : Friction coefficients between the tires and the road in the lateral direction.
μ_{r0} : Friction coefficients of the tires without side-slip.
ρ : Air density on the ground.
φ : Pitch angle of the sprung mass.
χ : Orientation of the wind.
ψ : Roll angle of the sprung mass.
$\omega_\xi, \omega_\eta, \omega_\zeta$: Components of angular velocity of the sprung mass around the $\xi\eta\zeta$ axes.

Subscripts:

F : Front wheel position. R : Rear wheel position.
l : Left wheel. r : Right wheel.

Note : A dot over a symbol means time rate of change.

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