Rippling of CEF-Type Electron Beams

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Contents

Abstract

The Waters' condition of zero rippling of thin electron ribbons is modified to a CEF-type electron beam.

There is no rippling of the electron beam when $Q=-2$, where Q is the Nunn-Rowe space-charge parameter in CEF-type traveling-wave devices. This means that the radial resonant frequency of the edge-of-the-beam electron $\sqrt{2}\,\Omega_0$ is equal to the beam plasma frequency ω_q for a beam possessing $Q=-2$, where Ω_0 is the unperturbed spatial angular velocity of an optical-axis electron. The radial resonant angular frequency of electrons within the beam is given by $\sqrt{2+Q}\,\Omega_0$ and the beam stiffness S is also shown by $\sqrt{2+Q}\,\Omega_0$ It should be noted that the condition of no rippling does not occur for zero space-charge, but rather for a particular value of Q.

1. Introduction

Recently it has been shown that certain types of photodemodulators may

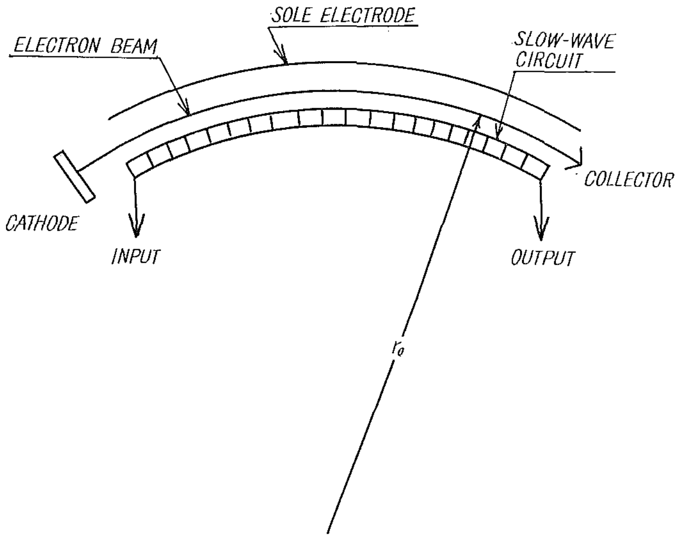

Fig. 1. The basic geometry of the CEF-type traveling-wave device.

make effective use of CEF-type electron beams[1),2)], in which the centrifugal force of the electrons in the interaction region is balanced by an equal and opposite radial electric field, as shown in Fig. 1. The space-charge forces giving rise to a tendency of the beam to increase in thickness are compensated by purely radial electrostatic fields between concentric cylinders. One of important considerations in the use of such beams is the variation of the beam thickness.

This paper deals with the variations in the thickness of thin CEF-type beams. The beams treated will be considered to be very thin and the charge densities due to the electrons sufficiently small, so that first order approximations to the forces which cause variations in the thickness may be used. The derivation will be based on Waters' paraxial-ray equation for two-dimensional systems having a curved optical axis[3)]. The conditions of no rippling will be expressed in terms of various parameters, namely, the Nunn-Rowe space-charge parameter[4)], the radial resonant frequency of the edge-of-the-beam electron, the beam plasma frequency and the beam stiffness.

2. The Paraxial-Ray Equation in a CEF-Type Focusing System

Figure 2 is a schematic cross section of a typical electron ribbon-beam[3)]. The beam may be pictured to have been formed by a suitable electron gun, the final electrode of which is the anode, at potential V_a relative to the cathode

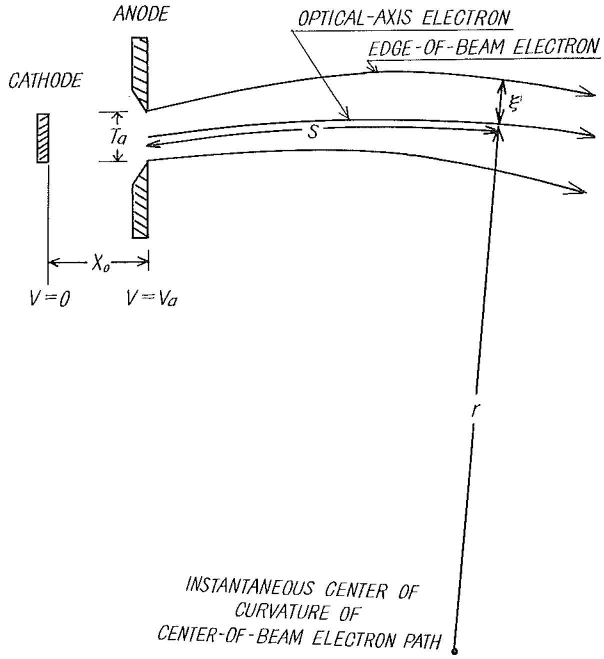

Fig. 2. Schematic cross section of a thin electron ribbon.

of the gun. Having passed through the anode, the beam must further be pictured to travel along a curved path under the influence of electrostatic forces; these forces will be the result of the field produced between suitable electrodes not shown in Fig. 2.

For each point along the optical axis it is convenient to describe the motion in a system of circular polar coordinates, centered at the instantaneous center of curvature. In such a system the equation of motion of the optical-axis electron in the radial direction is

$$m\ddot{r} = m\frac{v^2}{r} - eE_r\,, \tag{1}$$

where m is the electron mass, e is the electron charge ($e>0$), r is the radius of curvature, v is tangential velocity and E_r is the electric field along the radius vector, produced by the focusing system.

For the edge-of-the-beam electron the equation corresponding to Eq. (1) is

$$m\left[\ddot{r} + \left(\frac{d^2\xi}{dt^2}\right)\right] = m\frac{v^2}{r} + \delta\left(m\frac{v^2}{r}\right) - eE_r + \delta(-eE_r) + F_{SC}\,, \tag{2}$$

where ξ is the separation between the optical-axis electron and the edge-of-the-beam electron, measured along the radius vector, δ is a first-order difference and F_{SC} is the space-charge force.

From Eqs. (1) and (2), the basic electron beam equation becomes

$$m\left(\frac{d^2\xi}{dt^2}\right)=\delta\left(\frac{mv^2}{r}\right)+\delta(-eE_r)+F_{SC}. \tag{3}$$

In order to eliminate time from Eq. (3), we will need

$$v^2=\frac{2eV}{m},$$

$$\delta(r)=\xi,$$

$$E_r=\frac{2V}{r},$$

and

$$\frac{d^2\xi}{dt^2}=v^2\frac{d^2\xi}{ds^2}+v\frac{dv}{ds}\frac{d\xi}{ds},$$

where s is the position measured along the optical axis. The polar-coordinate form of Laplace's equation can be written variously as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}=0,$$

and

$$\frac{1}{r}\frac{\partial V}{\partial r}+\left(\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}\right)=-\frac{\partial^2 V}{\partial s^2}+\frac{1}{r}\frac{\partial V}{\partial r}=-\frac{\partial^2 V}{\partial s^2}-\frac{2V}{r^2},$$

since $rd\theta=ds$. Finally, it follows that

$$2V\xi''+V'\xi'+[4\kappa^2V+V'']\xi=\frac{F_{SC}}{e}, \tag{4}$$

where $\xi'=d\xi/ds$, $V'=dV/ds$ and $\kappa=\frac{1}{r}$.

Figure 3 shows a short section of a thin electron ribbon. At the edge of the beam a first approximation to the outward-directed electric field can be obtained through the use of Gauss' law:

$$\int_a \mathbf{E}_\xi\cdot d\mathbf{a}=\frac{q}{\varepsilon_0}=2WE_\xi ds=2\rho\xi Wds/\varepsilon_0$$

where ρ is the charge density and ε_0 is the permittivity of free space. The current density J_0 is

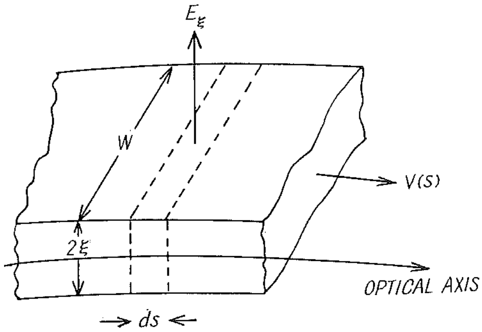

Fig. 3. Short length of an electron ribbon.

$$-J_0 = \rho v = -I_0/2\xi W, \tag{5}$$

where I_0 is the total current, so that the space-charge force is given by

$$F_{SC} = \pm \frac{eI_0}{2\varepsilon_0 v W}. \tag{6}$$

The space-charge term on the right side of Eq. (6) is to be taken with the + sign when $\xi > 0$ and with the − sign when $\xi < 0$. This means that the direction of the space-charge force will always be away from the optical axis. From Eqs. (4) and (6), it follows that

$$2V\xi'' + V'\xi' + [4\kappa^2 V + V'']\xi = \frac{\pm I_0}{2\varepsilon_0 \left(\frac{2e}{m}\right)^{1/2} W V^{1/2}}. \tag{7}$$

Equation (7) is the desired beam equation. If the actual current has been created by an equivalent planar diode having cathode-anode spacing X_0, anode potential V_a, and a slit with a length W and width T_a, through which the current is ejected, as shown in Fig. 2, a normalized form of the equation may be obtained. Using $\phi = V/V_a$ and $\mu = 2\xi/T_a$, Eq. (6) can be written

$$2\phi\mu'' + \phi'\mu' + [4\kappa^2\phi + \phi'']\mu = \frac{\pm 4}{9X_0^2\phi^{1/2}}. \tag{8}$$

This is a paraxial-ray equation for two-dimensional systems having a curved optical axis, which include a first-order space-charge term, as given by Waters. A solution of Eq. (8) yields the beam thickness as a function of path position.

The basic model of the CEF-type traveling-wave device is considered as shown in Fig. 1. The interaction region consists of a coaxial-cylindrical structure with a slow-wave circuit situated along the inner concentric cylinder and a sole electrode along the outer cylinder. A strip-shaped electron beam is injected along a tangent to the optical axis in the cross-section plane of the system, and a static electric field is applied between the sole and circuit electrodes. The d-c potential of the circuit measured relative to the cathode electrode is chosen sufficiently larger than that of the sole so that the centrifugal force of the electrons in the interaction region is balanced by an equal and opposite radial electric field force. In the CEF-type focusing system the curvature is constant, $\kappa=1/r_0$, where r_0 is the radius of the optical-axis. So Eq. (8) becomes

$$2\phi\mu''+\phi'\mu'+\left[\frac{4}{r_0^2}\phi+\phi''\right]=\frac{\pm 4}{9X_0^2\phi^{1/2}}. \tag{9}$$

Equation (9) is the paraxial-ray equation for CEF-type systems. If r_0 is infinity, i. e., the optical axis is a constant line, Eq. (9) is reduced to the paraxial-ray equation with space-charge in an O-type electron beam.

3. Conditions of Zero Rippling

3.1 Analytic Solutions of Electron-Ribbon Properties

Now let us consider the case for which $\phi=V/V_a=1$ in a thin CEF-type electron ribbon. This condition means that the electrons move at constant speed along their trajectories and the optical-axis electron moves along a circle. If a slow wave is also progressing along this same optical axis, at constant velocity, the condition $\phi=1$ means that the electron ribbon will move in synchronism with the wave. Then Eq. (9) becomes, simply,

$$\mu''+\frac{2}{r_0^2}\mu=\frac{\pm 2}{9X_0^2}. \tag{10}$$

If it is assumed that $\mu'(0)=0$ and $\mu(0)=\pm 1$, the solution of Eq. (10) is given by

$$\mu=\pm\frac{r_0^2}{9X_0^2}\pm\left(1-\frac{r_0^2}{9X_0^2}\right)\cos\frac{\sqrt{2}\,s}{r_0}. \tag{11}$$

The upper sign applies to the region where $\xi>0$ and the lower to the region in which $\xi<0$.

If the perveance of the beam is adjusted so that

$$r_0=3X_0, \tag{12}$$

this equation shows that $\mu=\pm 1$, independent of s. This means that there is no rippling of the beam.

3.2 The Critical Perveance

Now, translate the relation $r_0=3X_0$ directly into an expression of the perveance. If the beam is of average thickness T_a and width W, the planar diode formula of Child-Langmuir can be written

$$I_0 = \frac{4}{9}\varepsilon_0\sqrt{\frac{2e}{m}}\,\frac{WT_a}{X_0^2}V_a^{3/2}. \tag{13}$$

Then, it is easily shown that the critical perveance, $P \equiv I_0/V_a^{3/2}$ for no rippling, is given by the relation

$$(P)_{\mu=1} \equiv P_c = 4\varepsilon_0\sqrt{\frac{2e}{m}}\,\frac{WT_a}{r_0^2},$$

$$\approx 21\times10^{-6}\frac{WT_a}{r_0^2}. \tag{14}$$

This critical value of perveance exists for which no beam rippling occurs. It should be noted that the critical perveance does not occur for zero space-charge, but rather for a particular value of the perveance. From Eqs. (13) and (14) it follows that

$$P = \frac{r_0^2}{9X_0^2}P_C. \tag{15}$$

From Eqs. (5) and (13), the charge density can be written as

$$\rho = \frac{-4\varepsilon_0 V_a}{9X_0^2}. \tag{16}$$

Application of Eq. (12) to Eq. (16) yields

$$(\rho)_{\mu=1} = \frac{-4\varepsilon_0 V_a}{r_0^2} \approx \frac{-21\times10^{-6}}{\sqrt{2e/m}}\,\frac{V_a}{r_0^2}. \tag{17}$$

This is the charge density for which no radial variations of the beam occur and which reduces to the critical charge density discussed briefly by Nunn and Rowe[4)].

The relation obtained above will serve as a guide in evaluating the effect of changing the dimensions of the beam and also as a valuable guide in comparing one type of focusing structure with another.

3.3 The Nunn-Rowe Space-Charge Parameter

The Nunn-Rowe space-charge parameter is defined by

$$Q \equiv \frac{e\rho}{\Omega_0^2\varepsilon_0 m}, \tag{18}$$

where Ω_0 is the unperturbed spatial angular velocity of an optical-axis electron. This parameter represents the effects of all modes except the synchronous mode in CEF-type traveling-wave devices[4]. This parameter can be written as

$$Q = -\frac{\omega_p^2}{\Omega_0^2} \approx -\frac{\omega_q^2}{\Omega_0^2} \equiv -\beta_q^2\,, \tag{19}$$

where $\omega_p \equiv (e|\rho|/\varepsilon_0 m)^{1/2}$ is the plasma angular frequency, ω_q is the reduced plasma frequency and β_q is the reduced plasma propagation constant[1),5),6)]. It was assumed that the plasma reduction factor is approximately equal to unity. From Eqs. (16), (18) and (19), it follows that

$$Q = \frac{-2r_0^2}{9X_0^2}\,, \tag{20}$$

and

$$\beta_q = \frac{\sqrt{2}\,r_0}{3X_0}\,. \tag{21}$$

Application of Eq. (12) to Eq. (20) gives

$$(Q)_{\mu=1} = -2\,. \tag{22}$$

This value of Nunn-Rowe parameter means that there is no rippling in the focusing system. From Eqs. (19) and (22) it follows that

$$(\beta_q)_{\mu=1} = \sqrt{2}\,. \tag{23}$$

These results serve as a guide in evaluating the amplification mechanism and the diocotron effect in CEF-type traveling-wave devices.

3.4 The Radial Resonant Frequency

From Eqs. (19) and (23), the following can be obtained

$$\omega_q = \sqrt{2}\,\Omega_0\,. \tag{24}$$

The right side on Eq. (22), $\sqrt{2}\,\Omega_0$, is an angular frequency of a single harmonic motion in a radial direction. Namely, the edge-of-the-beam electron motion in radial and linear tangential perturbations is a combination of circular motion with an angular velocity Ω_0 and a simple harmonic motion in a radial direction of angular frequency $\sqrt{2}\,\Omega_0$, as shown in Eq. (11). The spatial angle corresponding to one period in radial and linear tangential perturbing influences is the characteristic rippling angle[6)]. Thus, Eq. (24) shows that for the condition of critical perveance, the radial resonant frequency of the beam is equal to the beam plasma frequency. From Eq. (21) the beam plasma frequency in general cases can be shown as

$$\omega_q = \frac{\sqrt{2}\,r_0\Omega_0}{3X_0}\,. \tag{25}$$

It is also of interest to express the beam motion in terms of the edge-of-the-beam electron wavelength λ_b,

$$\lambda_b = \frac{2\pi r_0 \Omega_0}{\sqrt{2}\,\Omega_0} = \sqrt{2}\,\pi r_0 . \tag{26}$$

Equation (26) shows that the wavelength is independent of the applied voltage.

3.5 Electrons within the Beam

The motion of electrons within the electron ribbon was studied by solutions of the paraxial-ray equation (4) and $F_{SC} = -eE_\xi = e\rho\xi/\varepsilon_0$. For convenience, it is assumed throughout this section that the electron ribbon remains on the $\xi = 0$ plane, so that the parameter ξ describes the displacement of an electron from the center, r_0. A form of the paraxial-ray equation describing the motion of electrons within the beam ($2\xi < T_a$) in CEF-type focusing systems becomes

$$\xi'' + (2+Q)\frac{\xi}{r_0^2} = 0 . \tag{27}$$

If we set $\xi(0) = \xi_0$ and $\xi'(0) = 0$, the solution of Eq. (27) in the range $-2 < Q \leqq 0$ is

$$\xi = \xi_0 \cos\left(\sqrt{2+Q}\,\frac{s}{r_0}\right) . \tag{28}$$

Since $s = r_0 \Omega_0 t$, the radial resonance angular frequency of electrons within the beam is $\sqrt{2+Q}\,\Omega_0$, which depends upon the beam space-charge density, namely the electron plasma frequency. The value of $\sqrt{2+Q}\,\Omega_0$ may range between zero (maximum charge density, corresponding to the value of critical perveance) and $\sqrt{2}\,\Omega_0$ of the electron ribbon (negligible space charge). Namely, the electron beam and the electrons within the beam may have different resonant resonant frequencies. Correspondingly, the wave length λ_e will assume values between infinity and the wavelength of the edge-of-the-beam electron λ_b, because the wavelength λ_e can be given by

$$\lambda_e = \frac{2\pi r_0 \Omega_0}{\sqrt{2+Q}\,\Omega_0} = \frac{2\pi r_0}{\sqrt{2+Q}} . \tag{29}$$

3.6 Beam Stiffness

The beam stiffness, S, can be used to describe the properties of a focusing system. Thus the parameter is defined as

$$S^2 = -(\partial\ddot{\xi}/\partial\xi) \tag{30}$$

at the optical-axis plane. The dots appearing above ξ signify the total time derivation of ξ. From Eqs. (28) and (30), the beam stiffness is given by

$$S^2 = (2+Q)\Omega_0^2$$
$$\approx 2\left(\Omega_0^2 - \frac{1}{2}\omega_q^2\right). \qquad (31)$$

Equation (31) shows that the stiffness S is equal to the radial resonance angular frequency of electrons within the beam. The second term in the right member arises from the presence of a space-charge in the beam and this means that the effect of space charge reduces—ever so slightly—the beam stiffness. It is noted that the beam stiffness and the radial resonance frequency can be obtained by other considerations. (see Appendix)

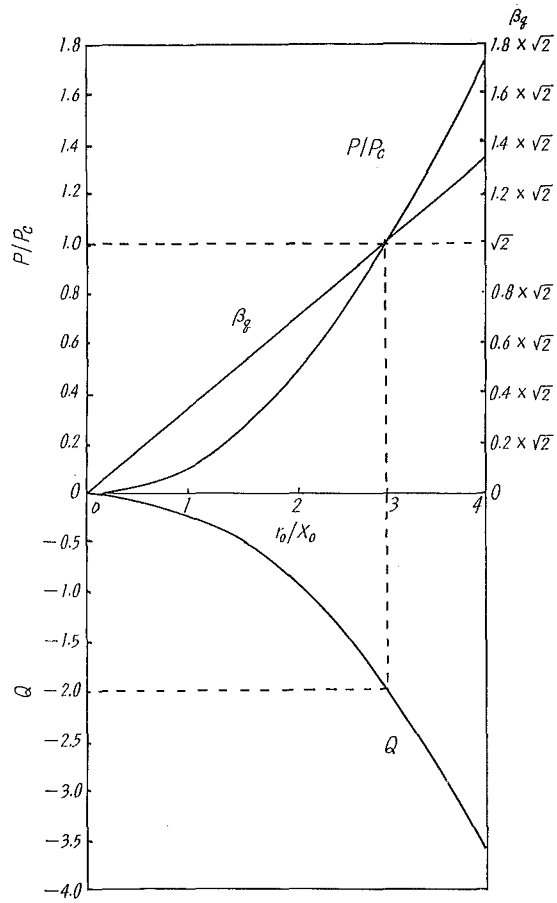

Fig. 4. Q, β_q and P/P_c vs r_0/X_0

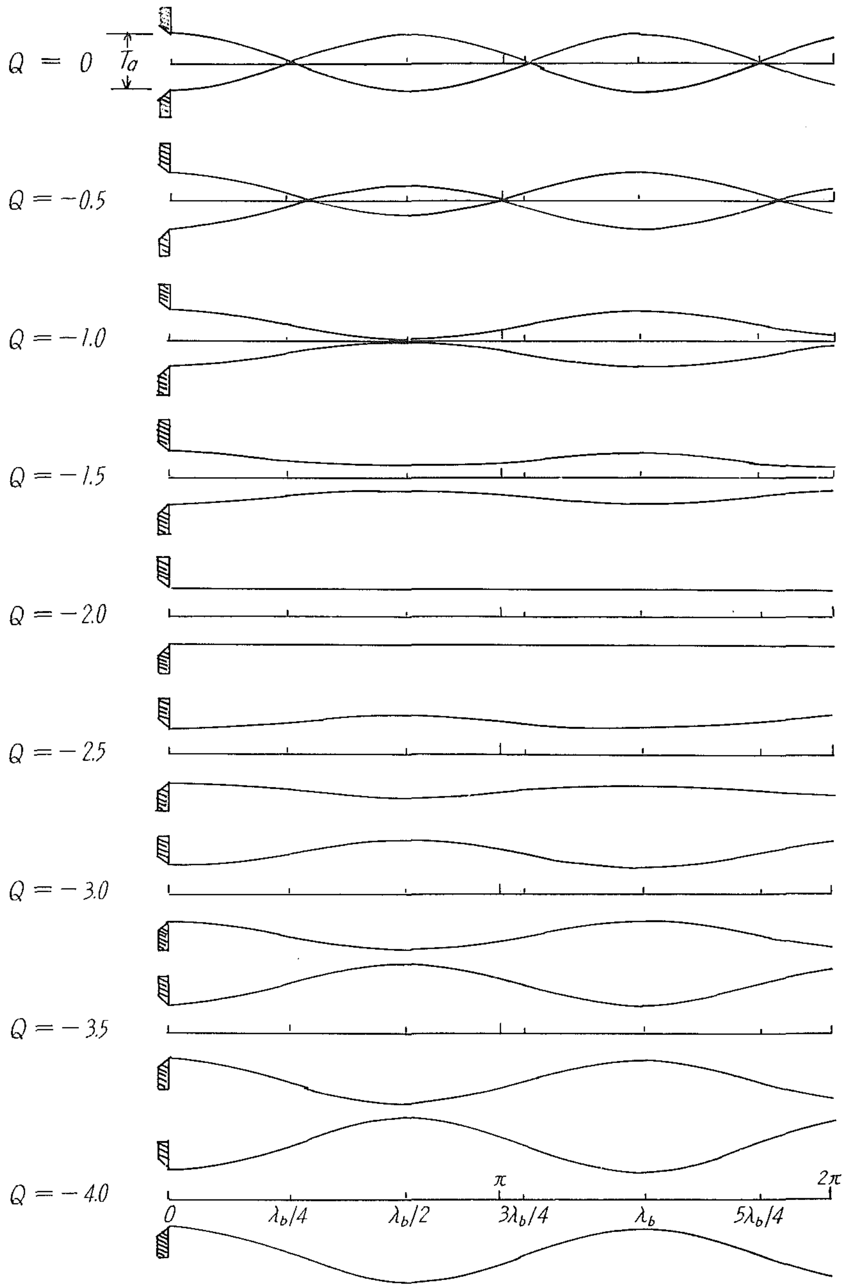

Fig. 5. Sketch showing paths of edge-of-the-beam electrons for various values of Q.

4. Examples of Rippling

Now, consider the rippling as a function of the position measured along the optical axis with Q as a parameter. From Eqs. (15), (20) and (21) parameters Q, β_q and P/P_c can be shown as a function of r_0/X_0 in Fig. 4. Therefore it is easy to show paths of electrons that enter a focusing structure at beam edge. Figure 5 shows a typical representation of the edge-of-the-beam electron paths obtained in the CEF-type focusing structure.

Inspection of Figs. 4 and 5 shows that in the range where $-2<Q\leqq 0$ (corresponds to the condition for which $\omega_q<\sqrt{2}\,\Omega_0$) the maximum amplitude of beam ripple is equal to the slit width T_a and in the case where $Q=-2$ (corresponds to the condition for which $\omega_q=\sqrt{2}\,\Omega_0$) there is no rippling and in the region where $Q<-2$ (corresponds to the condition for which $\omega_q>\sqrt{2}\,\Omega_0$) the minimum beam thickness is equal to the slit width T_a.

Another way of saying the same thing is that for $P<P_c$ the maximum beam thickness is equal to the width of the final gun slit and for $P>P_c$ the minimum beam thickness is equal to the slit width and for $P=P_c$, no rippling occurs.

5. Conclusions

The condition of no rippling of thin CEF-type electron beams was calculated. The derivation was based on Waters' paraxial-ray equation for two-dimensional systems having a curved optical axis.

The general characteristic of rippling could be summarized as follows;

1) There is no rippling of the electron beam when the value of Nunn-Rowe space-charge parameter is equal to -2.
2) In the case of no rippling, the beam plasma frequency is equal to the radial resonant frequency of the edge-of-the-beam electron.
3) In the case of no rippling, the value of reduced plasma propagetion constant is equal to $\sqrt{2}$.
4) The edge-of-the-beam electron wavelength λ_b is given by $\sqrt{2}\,\pi r_0$ and the wavelength on the electron motion within the beam λ_e is shown as $2\pi r_0/\sqrt{2+Q}$.
5) The beam stiffness is equal to the radial resonant angular frequency of electrons within the beam $\sqrt{2+Q}\,\Omega_0$.
6) The calculated paths of the edge-of-the-beam electron show that for $-2<Q\leqq 0$ the maximum beam thickness is equal to the width of the final gun slit and for $Q<-2$ the minimum beam thickness is equal to the slit width and for $Q=-2$ no rippling occurs.

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Appendix

The equations for motion for a single electron in a CEF-type focusing system (see Fig. 1) as given by

$$mr\dot{\theta}^2 - eE_r = m\ddot{r}\,, \qquad (A\cdot 1)$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = -erE_\theta\,, \qquad (A\cdot 2)$$

where E_r is the static radial electric field intensity, E_θ is the static azimuthal electric field intensity, r is the radial coordinate variable and θ is the azimuthal coordinate variable[6]. The dots appearing above the quantities in the foregoing equations signify the total time derivations of these terms. In the CEF-type system, the static electric field possesses only a radial component and E_θ vanishes. The angular momentum of the electron, $mr^2\dot{\theta}$ is constant. These equations then can be simplified to

$$\ddot{r}-r\dot{\theta}^2=-eE_r/m\,, \qquad (A\cdot 3)$$

$$mr^2\dot{\theta}=M\,, \qquad (A\cdot 4)$$

where M is the particle angular momentum.

The from of Poisson's equation for a thin electron ribbon in the presence of a steady radial electric field can be written

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right)=-\frac{\rho}{\varepsilon_0}\,. \qquad (A\cdot 5)$$

The equilibrium condition, under which the radial electric field force just balances the centrifugal force of an electron undergoing circular motion at radius r, may be written

$$\frac{mv_{\theta e}^2}{r}-eE_r=0\,, \qquad (A\cdot 6)$$

where $v_{\theta e}$ is the equilibrium linear tangential velocity of the electron at radius r. From the conservation-of-energy principle, $v_{\theta e}$ can be given by

$$v_{\theta e}^2=\frac{2eV}{m}\,, \qquad (A\cdot 7)$$

where V is the potential at the equilibrium radius.

Integrating Eq. (A·5) with respect to r, under the assumption that the volume charge density may be considered independent of radial variations, the potential becomes

$$V=-\frac{\rho}{4\varepsilon_0}r^2+C_1\ln r+C_2\,,$$

where C_1 and C_2 are constants of integration. It is assumed that $V=V_a$ and $r_0(dV/dr)=-2V_a$ at $r=r_0$, the potential becomes

$$V=V_a+\left(2V_a-\frac{\rho r_0^2}{2\varepsilon_0}\right)\ln\left(\frac{r_0}{r}\right)+\frac{\rho}{4\varepsilon_0}(r_0^2-r^2)\,. \qquad (A\cdot 8)$$

The the radial electric field can be given by

$$E_r=\left(2V_a-\frac{\rho r_0^2}{2\varepsilon_0}\right)\frac{1}{r}+\frac{\rho r}{2\varepsilon_0}\,. \qquad (A\cdot 9)$$

If the electron is assumed to enter the system at radius r_0 with an equilibrium velocity, but with a small radial-velocity component, then the angular momentum must be

$$mr^2\dot{\theta} = mr_0 v_{\zeta e}\,, \tag{A·10}$$

where the right member is the initial angular momentum of the electron at r_0. Finally, Eqs. (A·9) and (A·10) may be substituted into Eq. (A·3) to obtain

$$\ddot{r} = \left[\frac{v_{\theta e}^2}{r} + \frac{\rho e r}{2m\varepsilon_0}\right]\left[\left(\frac{r_0}{r}\right)^2 - 1\right]. \tag{A·11}$$

Now, it can be assumed that the perturbing influences are small compared to the steady motion of the electron. Thus we define

$$r = r_0 + r_1\,, \tag{A·12}$$

$$\dot{\theta} = \Omega_0 + \Omega_1\,, \tag{A·13}$$

where r_1 is the radial perturbation function which depends on time and Ω_1 is the azimuthal velocity perturbation function. From Eqs. (A·11) and (A·12) and the small perturbation assumption that r_1/r_0 is small compared to unity, the equation of motion simplifies to

$$\ddot{r}_1 + (2+Q)\Omega_0^2 r_1 = 0\,, \tag{A·14}$$

where

$$v_{\theta e} = r_0 \Omega_0\,. \tag{A·15}$$

Now, select as the initial conditions

$$r_1 = r_{10} \quad \text{and} \quad \dot{r}_1 = 0 \qquad \text{at} \quad t = 0 \tag{A·16}$$

Then the solution of Eq. (A·14) in the range $-2 < Q \leqq 0$ becomes

$$r_1 = r_{10} \cos(\sqrt{2+Q}\,\Omega_0)t\,. \tag{A·17}$$

Since variables r_1 and r_{10} correspond to ξ and ξ_0 of Eq. (28), respectively and $t = s/r_0\Omega_0$, we finally obtain

$$\xi = \xi_0 \cos(\sqrt{2+Q}\; s/r_0)\,. \tag{25}$$

From Eqs. (A·14) and (27), the beam stiffness S becomes

$$S^2 = (2+Q)\Omega_0^2\,. \tag{28}$$

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