Generalized Sampling Theorem as an Interpolation Formula

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Abstract

A generalization of the sampling theorem is presented, taking into account the sampled values of a function and of its derivatives.

New examples of sampling formulae are also given, which include, respectively as special cases, the formulae given by Lagrange, Shannon, Someya, Takizawa, Kroll, Isomiti, Jagerman and Fogel, and Linden and Abramson. Some of the formulae given here, can be used effectively as interpolation formulae.

Zusammenfassung

Es wird eine Verallgemeinerung des Abtast-theorems (sampling theorems) präsentiert, wobei die abgetasteten Werte einer Funktion und ihrer Ableitungen berücksichtigt werden.

Gegeben wurden neue Beispiele der Abtastformeln, welche beziehungsweise als Spezielfälle, die Formeln von Lagrange, Shannon, Someya, Takizawa, Kroll, Isomiti, Jagerman-Fogel, und Linden-Abramson, enthalten. Einige von den hier angegebenen Formeln können als Interpolationsfolmeln erfolgreich benutzt werden.

Résumé

Une généralisation de la théorème à échantillonnage (sampling theorem) est présentée par la considération des valeurs échantillonnées d'une fonction et des ses dérivées.

Nouveaux exemples des formules à échantillonnage sont aussi donnés, qui généralisent respectivement les formules de Lagrange, Shannon, Someya, Takizawa, Kroll, Isomiti, Jagerman-Fogel, et Linden-Abramson. Quelques formules, qui sont ici données, sont peut-être utilisées effectivement comme les formules d'interpolation.

§ 1. Preliminaries

The generalization of the sampling theorem and the reconstruction of a band-limited function from its sampled values and sampled derivatives were made by Kohlenberg[1], Fogel[2], Jagerman and Fogel[3], Bond and Cahn[4], and Linden and Abramson[5]. The sampling theorem was also generalized by Balakrishnan[6] to the case of a continuous-parameter stochastic process. On the other hand, it was

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pointed out that the sampling intervals need not be uniformly distributed[7].

In previous papers[8)~12)], one of the present authors proposed a generalized sampling theorem taking into consideration the reciprocity relation of integral transforms, and gave new sampling formulae as examples. Here in this paper, the authors present another generalization of the sampling theorem, in such a way as to reconstruct a function from its sampled values and sampled derivatives. Also presented are some of the new sampling formulae, which include, respectively as special cases, the sampling formulae previously given by several authors.

The concept of *generalized frequency* is also introduced and it was shown that the generalized frequency can be effectively applied to express the condition for establishing the authors' generalized sampling theorem, which corresponds to the condition of a *band-limited* function in the case of Shannon's theorem.

Some of the sampling theorems given here, *e.g.* the generalized Lagrangian interpolation formulae, are also useful as extrapolation formulae.

§ 2. Generalization of Sampling Theorem

Shannon's sampling theorem, which is considered to be an interpolation formula, has the following two important properties: First, his theorem is quite *similar* to the expansion formula of a function in a system of *orthogonal* functions, in the sense that the system of sampling functions (*i.e.* sinc-functions) is not orthogonal but *almost* orthogonal in the interval of the sampled function. Secondly, his theorem contains the *sampled value* (*height*) of a function at *many sampling points*, and does not contain the sampled derivatives of the function. In this sense, the sampling formula has an aspect quite similar to Lagrangian interpolation formula. Shannon's formula, however, can not be effectively used as an extrapolation formula, because his formula is expanded in a series of sinc-functions, each of which is quite similar[21] to a δ-function.

On the other hand, the Taylor series approximates a function, by making use of the *sampled value* (*height*) as well as the *sampled derivatives* of the function at *a fixed point.*

In other words, Shannon's sampling theorem involves the sampled value (height) of a function at many sampling points, and is quite similar to its orthogonal expansion in sampling functions (sinc-functions). While, the Taylor series contains the sampled value (height) and sampled derivatives of a function at a sampling point. In this sense, Shannon's theorem can be considered to have taken into account the characters of the function over the whole domain of the sampled interval, such as in the case of determining the Fourier coefficients in a Fourier series. While, the Taylor series takes into accout the characters of a function at a fixed point in the sampling domain, such as its height, slope, curvature, etc.

In the present paper the authors have attempted to construct new sampling theorem, which involves the sampled values (heights) of a function as well as its sampled higher derivatives, in such a way that the present theorem unifies the properties of both Shannon's and Taylor's theorems. The authors' theorem can be reduced to Shannon's and Taylor's theorem, respectively as special cases.

Theorem 2-1. Generalized Sampling Theorem

An entire function $f(z)$ is expressed by:

$$f(z)=\sum_n\sum_{k=0}^{m_n}\sum_{j=0}^{m_n-k}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot(z-z_n)^{j+k}\cdot\frac{g(z)}{(z-z_n)^{m_n+1}} \tag{2-1}$$

$$=\sum_n\sum_{s=0}^{m_n}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^{s}\cdot\frac{g(z)}{(z-z_n)^{m_n+1}}, \tag{2-1'}$$

if the following conditions are satisfied:

(I) $g(z)$ is an entire function, (2-2)

(II) $g(z)$ has zeros of (m_n+1)-th order at points $z=z_n$ (n=integers), *i.e.*

$$\left.\begin{aligned}&g(z_n)=g'(z_n)=g''(z_n)=\cdots=g^{(m_n)}(z_n)=0\,,\\ \text{and}\quad &\\ &g^{(m_n+1)}(z_n)\neq 0\,,\end{aligned}\right\} \tag{2-3}$$

for m_n=non-negative integers*), which depend on n, and

(III) $$\lim_{z\to\infty}\frac{f(z)}{g(z)}=0\,. \tag{2-4}$$

Here, for the sake of brevity, we have put:

$$f_n^{(j)}=\left[\frac{d^j}{dz^j}f(z)\right]_{z=z_n}, \tag{2-5}$$

and

$$H_n^{(k)}=\left[\frac{d^k}{dz^k}\frac{(z-z_n)^{m_n+1}}{g(z)}\right]_{z=z_n}. \tag{2-6}$$

The summation over n means the summation over the whole set of *sampling points* $z=z_n$, which satisfy condition (II). The function $g(z)/(z-z_n)^{m_n+1}$ shall be called a *generalized sampling function.*

Proof.

We shall take a closed curve C in a complex plane, in such a manner that the curve C encloses some poles of the function $f(z)/g(z)$ within it, and no pole lies on the curve C itself (*cf.* Fig. 1).

Accordingly, by conditions (I) and (II), the function $f(z)/g(z)$ is a meromorphic function in the domain enclosed by the curve C. In this case, the Cauchy theorem reads:

$$\begin{aligned}\frac{f(z)}{g(z)}&=\frac{1}{2\pi i}\int_{\mathfrak{C}}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}\,d\zeta\\ &=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}\,d\zeta-\sum\operatorname{Res}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}\,,\end{aligned} \tag{2-7}$$

*) By $g'(z)$ we understand $dg(z)/dz$, while $g^{(0)}(z)$ is the function $g(z)$ itself.

for any z which lies inside the closed curve $\mathfrak{C}$. In the expression (2–7), the integration is taken counterclockwise along the closed contour C, and Res $F(\zeta)$ means the residue of function $F(\zeta)$ at the pole $\zeta=z_n$ (*cf.* Fig. 1). The summation intends to take the sum of residues all over the poles z_n which lie inside the curve C.

We shall deform the contour C into a circle of radius R, with its centre at the origin. If we tend R to infinity and take condition (III) into account, the first term of (2–7):

$$I \equiv \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}\,d\zeta\,, \qquad (2\text{–}8)$$

is seen to approach zero (*cf.* Appendix A). That is,

$$\lim_{R\to\infty} I = 0\,.$$

In the second term of (2–7), if we take the limiting process: $R\to\infty$, the summation covers all the poles at $\zeta=z_n$ of the meromorphic function $f(z)/g(z)$ in the whole complex plane. In this sense, we shall write the summation as $\sum\limits_n$.

We shall calculate the value of residue at the point $\zeta=z_n$, as follows:

a) If $f(z_n)\neq 0$, under conditions (I) and (II), the function $f(\zeta)/(\zeta-z)\,g(\zeta)$ is seen to have a pole of (m_n+1)-th order at the point $\zeta=z_n$. In this case, the residue of $f(\zeta)/(\zeta-z)\,g(\zeta)$ at the point $\zeta=z_n$ is given by the m_n-th coefficient in the Taylor expansion of $(\zeta-z_n)^{m_n+1}\cdot f(\zeta)/(\zeta-z)\,g(\zeta)$ around the point $\zeta=z_n$. Thus, we obtain:

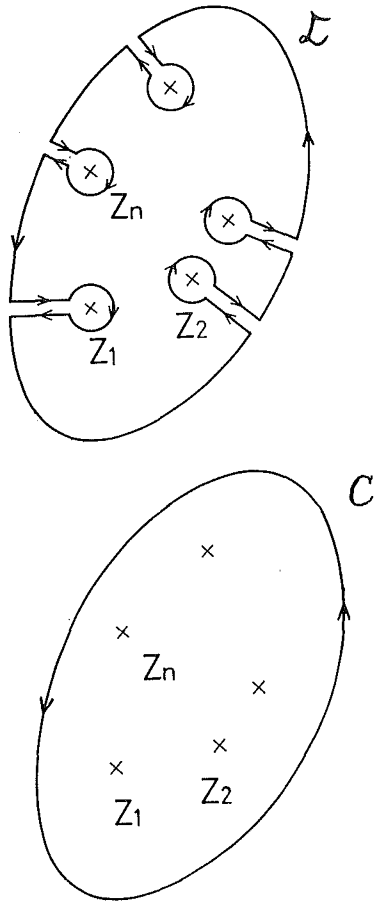

Fig. 1. Poles $\zeta=z_n$ and integration contours $\mathfrak{C}$ and C in expression (2–7).

$$\operatorname{Res}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)} = \frac{1}{m_n!}\left[\frac{d^{m_n}}{d\zeta^{m_n}}\,\frac{(\zeta-z_n)^{m_n+1}\cdot f(\zeta)}{(\zeta-z)\,g(\zeta)}\right]_{\zeta=z_n}. \qquad (2\text{–}9)$$

Taking

$$\left.\begin{aligned} h(\zeta) &= \frac{g(\zeta)}{(\zeta-z_n)^{m_n+1}}\,, \\ &\text{and} \\ H(\zeta) &= \frac{1}{h(\zeta)} = \frac{(\zeta-z_n)^{m_n+1}}{g(\zeta)}\,, \end{aligned}\right\} \qquad (2\text{–}10)$$

in the expression (2–9), we obtain:

$$-\sum_n \operatorname{Res}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}=-\sum_n\frac{1}{m_n!}\left[\frac{d^{m_n}}{d\zeta^{m_n}}\frac{f(\zeta)\,H(\zeta)}{\zeta-z}\right]_{\zeta=z_n}$$

$$=-\sum_n\frac{1}{m_n!}\sum_{k=0}^{m_n}{}_{m_n}C_k\left[\left(\frac{f(\zeta)}{\zeta-z}\right)^{(m_n-k)}\cdot H^{(k)}(\zeta)\right]_{\zeta=z_n}$$

$$=-\sum_n\frac{1}{m_n!}\sum_{k=0}^{m_n}{}_{m_n}C_k\sum_{j=0}^{m_n-k}{}_{m_n-k}C_j\left[\left(\frac{1}{\zeta-z}\right)^{(m_n-k-j)}\cdot f^{(j)}(\zeta)\cdot H^{(k)}(\zeta)\right]_{\zeta=z_n}$$

$$=-\sum_n\sum_{k=0}^{m_n}\sum_{j=0}^{m_n-k}\frac{1}{m_n!}\cdot{}_{m_n}C_k\cdot{}_{m_n-k}C_j\cdot\frac{(-1)^{m_n-k-j}(m_n-k-j)!}{(z_n-z)^{m_n-k-j+1}}\cdot f_n^{(j)}\cdot H_n^{(k)}$$

$$=\sum_n\sum_{k=0}^{m_n}\sum_{j=0}^{m_n-k}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot\frac{(z-z_n)^{j+k}}{(z-z_n)^{m_n+1}} \tag{2-11}$$

$$=\sum_n\sum_{s=0}^{m_n}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot\frac{(z-z_n)^{s}}{(z-z_n)^{m_n+1}}\,, \tag{2-11'}$$

where

$$f_n^{(j)}=\left[\frac{d^j}{d\zeta^j}f(\zeta)\right]_{\zeta=z_n},$$

and

$$H_n^{(k)}=\left[\frac{d^k}{d\zeta^k}H(\zeta)\right]_{\zeta=z_n}$$

$$=\frac{(-1)^k}{h_n}\cdot\begin{vmatrix}\frac{h_n'}{h_n}, & 1, & 0, & & & \\ \frac{h_n''}{h_n}, & {}_2C_1\frac{h_n'}{h_n}, & 1, & 0, & \mathbf{0} & \\ \vdots & \vdots & & & & \\ \frac{h_n^{(k-1)}}{h_n}, & {}_{k-1}C_1\frac{h_n^{k(-2)}}{h_n}, & {}_{k-1}C_2\frac{h_n^{(k-3)}}{h_n}, & \cdots, & 1 & \\ \frac{h_n^{(k)}}{h_n}, & {}_kC_1\frac{h_n^{(k-1)}}{h_n}, & {}_kC_2\frac{h_n^{(k-2)}}{h_n}, & \cdots, & {}_kC_{k-1}\frac{h_n'}{h_n}\end{vmatrix}, \tag{2-12}$$

for positive integers k, with

$${}_pC_q\frac{h_n^{(p-q)}}{h_n}=\frac{p!(m_n+1)!}{q!(m_n+1+p-q)!}\cdot\frac{g_n^{(m_n+1+p-q)}}{g_n^{(m_n+1)}}, \qquad (p,q=0,1,2,\cdots)$$

$$g_n^{(r)}=\left[\frac{d^r}{d\zeta^r}g(\zeta)\right]_{\zeta=z_n}, \qquad (r=0,1,2,\cdots)$$

and

$$h_n^{(r)}=\left[\frac{d^r}{d\zeta^r}h(\zeta)\right]_{\zeta=z_n}. \qquad (r=0,1,2,\cdots)$$

In Appendix B, the expression (2–12) is given explicitly in terms of g_n and derivatives $g_n^{(r)}$.

b) If the function $f(\zeta)$ has a zero of p_n-th order $(m_n-p_n\geqslant 0)$, at the point $\zeta=z_n$, *i.e.*

if

$$\left.\begin{aligned} &f(z_n)=f'(z_n)=f''(z_n)=\cdots=f^{(p_n-1)}(z_n)=0, \\ \text{and} \\ &f^{(p_n)}(z_n)\neq 0, \quad (p_n=\text{positive integers, which depend on } n) \end{aligned}\right\} \tag{2-13}$$

then the function $f(\zeta)/(\zeta-z)\,g(\zeta)$ has a pole of (m_n-p_n+1)-th order at the point $\zeta=z_n$. And the expression (2-9) needs modification. In this case, the residue at $\zeta=z_n$ is given by:

$$\operatorname{Res}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}=\frac{1}{(m_n-p_n)!}\left[\frac{d^{m_n-p_n}}{d\zeta^{m_n-p_n}}\frac{(\zeta-z_n)^{m_n-p_n+1}f(\zeta)}{(\zeta-z)\,g(\zeta)}\right]_{\zeta=z_n}. \tag{2-14}$$

Putting

$$\hat{f}(\zeta)=\frac{f(\zeta)}{(\zeta-z_n)^{p_n}}, \tag{2-15}$$

and considering that

$$\hat{f}_n\equiv\left[\hat{f}(\zeta)\right]_{\zeta=z_n}=\frac{1}{p_n!}\left[f^{(p_n)}(\zeta)\right]_{\zeta=z_n}\neq 0, \tag{2-16}$$

by means of (2-13), we have the following expression from (2-11):

$$-\sum_n \operatorname{Res}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}=\sum_n\sum_{k=0}^{m_n-p_n}\sum_{j=0}^{m_n-p_n-k}\frac{\hat{f}_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot\frac{(z-z_n)^{j+k}}{(z-z_n)^{m_n-p_n+1}}. \tag{2-17}$$

Now we have, from (2-13) and (2-15):

$$\hat{f}_n^{(j)}\equiv\left[\frac{d^j}{d\zeta^j}\hat{f}(\zeta)\right]_{\zeta=z_n}=\frac{j!}{(j+p_n)!}\cdot f_n^{(j+p_n)}. \quad (j=0, 1, 2, \cdots, m_n-p_n) \tag{2-18}$$

Putting (2-18) into (2-17), we obtain:

$$-\sum_n \operatorname{Res}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}=\sum_n\sum_{k=0}^{m_n-p_n}\sum_{j=0}^{m_n-p_n-k}\frac{f_n^{(j+p_n)}}{(j+p_n)!}\cdot\frac{H_n^{(k)}}{k!}\cdot\frac{(z-z_n)^{j+p_n+k}}{(z-z_n)^{m_n+1}} \tag{2-19}$$

$$=\sum_n\sum_{k=0}^{m_n-p_n}\sum_{j=p_n}^{m_n-k}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot\frac{(z-z_n)^{j+k}}{(z-z_n)^{m_n+1}}. \tag{2-19'}$$

If we refer to the condition (2-13) in (2-19'), we obtain:

$$-\sum_n \operatorname{Res}\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}=\sum_n\sum_{k=0}^{m_n-p_n}\sum_{j=0}^{m_n-k}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot\frac{(z-z_n)^{j+k}}{(z-z_n)^{m_n+1}} \tag{2-20}$$

$$=\sum_n\sum_{k=0}^{m_n}\sum_{j=0}^{m_n-k}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(k)}}{k!}\cdot\frac{(z-z_n)^{j+k}}{(z-z_n)^{m_n+1}} \tag{2-21}$$

$$=\sum_n\sum_{s=0}^{m_n}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot\frac{(z-z_n)^{s}}{(z-z_n)^{m_n+1}}, \tag{2-21'}$$

because all the $f_n^{(j)}$'s vanish for $j=0, 1, 2, \cdots, p_n-1$. Refering to (2-11), (2-11'), (2-21), and (2-21'), it can be seen that the expressions (2-11) and (2-11') hold for the case b) as well as for the case a).

Multiplying $g(z)$ to both sides of (2–7), and taking the calculations in Appendices A and B into account, we finally obtain the generalized sampling theorem (2–1).

Remarks.

Further, it should be mentioned that condition (II) can be replaced by the following condition (II′):

(II′) The function $f(z)/g(z)$ has poles of finite positive integral order (m_n-p_n+1) at points $z=z_n$ (with $n=$integers and $m_n=0, 1, 2, \cdots$) and the function $f(z)$ has zeros of p_n-th order $(m_n-p_n\geqslant 0)$ at the same points $z=z_n$ ($p_n=$non-negative integers, which depend on n). (2-22)

This condition (II′) comes directly from proof given in cases a) and b).

In Theorem 2–1, if all the m_n's are set equal to m, we have:

Corollary 2-1.

An entire function $f(z)$ is represented by:

$$f(z)=\sum_{n}\sum_{s=0}^{m}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^s\cdot\frac{g(z)}{(z-z_n)^{m+1}}, \tag{2–23}$$

if the following conditions are satisfied:

(I) $g(z)$ is an entire function,

(II) $g(z)$ has zeros of $(m+1)$-th order at points $z=z_n$ ($n=$integers), *i.e.*

$$\left.\begin{array}{l} g_n=g_n'=g_n''=\cdots=g_n^{(m)}=0\,, \\ \text{and} \\ g_n^{(m+1)}\neq 0\,, \end{array}\right\} \tag{2–24}$$

for m a non-negative integer, which does not depend on n,

(III) $$\lim_{z\to\infty}\frac{f(z)}{g(z)}=0\,. \tag{2–25}$$

If we take $m=0$ in (2–23), we obtain the sampling formula:

$$\begin{aligned} f(z)&=\sum_n f_n\cdot H_n\cdot\frac{g(z)}{z-z_n}=\sum_n f_n\cdot\frac{g(z)}{(z-z_n)\cdot h_n} \\ &=\sum_n f_n\cdot\frac{g(z)}{(z-z_n)\cdot g_n'}\,, \end{aligned} \tag{2–26}$$

which is nothing but the expression suggested by van der Pol[22] for the entire function $g(z)$ with simple roots at $z=z_n$, $f(z)$ being a band-limited function with a cut-off frequency, and the functions $f(z)$ and $g(z)$ are assumed to have no common root.

For the case $m\geqslant 1$ in (2–23), we take the function $g(z)$ practically as

$$g(z)=\phi^{m+1}(z)\,, \tag{2–27}$$

where $\phi(z)$ is an entire function with simple zeros at points $z=z_n$. Then the sampling formula (2–23) reads:

$$f(z)=\sum_n \sum_{s=0}^{m} \sum_{j=0}^{s} \frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot(z-z_n)^s\cdot\frac{\phi^{m+1}(z)}{(z-z_n)^{m+1}}, \tag{2-28}$$

where $H_n^{(k)}$'s are given by (B-11) in Appendix B, with

$$h_n=\frac{1}{(m+1)!}\,g_n^{(m+1)}=(\phi_n')^{m+1}, \tag{2-29}$$

and

$$\begin{aligned} {}_sC_r\,\frac{h_n^{(s-r)}}{h_n} &= \frac{s!\,(m+1)!}{r!\,(m+1+s-r)!}\cdot\frac{g_n^{(m+1+s-r)}}{g_n^{(m+1)}} \\ &= \frac{s!\,(m+1)!}{r!\,(\phi_n')^{m+1}}\cdot\sum_{\substack{p+q+u+\cdots=m+1\\ p+2q+3u+\cdots=m+1+s-r}}\frac{1}{p!\,q!\,u!\cdots}\cdot(\phi_n')^p\cdot\left(\frac{1}{2!}\phi_n''\right)^q\cdot\left(\frac{1}{3!}\phi_n'''\right)^u\cdots. \end{aligned} \tag{2-30}$$

From theorem (2-23), we find the following corollary:

Corollary 2-2.

If the entire function $g(z)$ can be expanded in the Taylor series at the points $z=z_n$ in the following form:

$$\left.\begin{aligned} &\text{(IV)}\qquad g(z)=A_{m+1}(z-z_n)^{m+1}+\sum_{s=1}^{\infty}A_{2m+1+s}(z-z_n)^{2m+1+s}, \\ &\text{with} \\ &\qquad A_{m+1}\neq 0, \end{aligned}\right\} \tag{2-31}$$

and if the conditions (I), (II), and (III) in Corollary 2-1, are satisfied, then the expression (2-23) can be reduced to:

$$f(z)=\sum_n \sum_{s=0}^{m} f_n^{(s)}\cdot\frac{(z-z_n)^s}{s!}\cdot\frac{1}{\dfrac{g_n^{(m+1)}}{(m+1)!}}\cdot\frac{g(z)}{(z-z_n)^{m+1}}. \tag{2-32}$$

Proof.

From (2-31) and (B-6)∼(B-10) in Appendix B, we have

$$\frac{h_n^{(k)}}{h_n}=0, \qquad (\text{for } k=1,2,3,\cdots,m) \tag{2-33}$$

with

$$h_n=\left[\frac{g(\zeta)}{(\zeta-z_n)^{m+1}}\right]_{\zeta=z_n}=A_{m+1}\neq 0. \tag{2-34}$$

The expressions (2-33), (2-34), and (B-11) in Appendix B, lead to

$$H_n^{(k)}=0, \qquad (\text{for } k=1,2,3,\cdots,m) \tag{2-35}$$

and

$$H_n^{(0)}=H_n=\frac{1}{h_n}=\frac{1}{A_{m+1}}=\frac{(m+1)!}{g_n^{(m+1)}}\neq 0. \tag{2-36}$$

We put (2-35) and (2-36) into (2-23), and obtain the sampling theorem (2-32).

This theorem (2-32) is a generalization of the formulae given by Fogel[2),] Jagerman-Fogel[3], and Linden-Abramson[5]. The example of theorem (2–32) will be given in the following paper.

The formulae (2–1), (2–1′), (2–23), (2–26), (2–28), and (2–32), are *generalized sampling theorems*, in the sense that they give the function $f(z)$ in terms of the sampled values f_n and sampled derivatives $f_n^{(j)}$ $(j=1, 2, 3, \cdots)$ of an entire function $f(z)$. The reconstruction of an entire function from its sampled values and sampled derivatives is also mentioned by Jagerman and Fogel[3], and Linden and Abramson[5]. The present theorems (2–1), (2–1′), (2–23), (2–26), (2–28), and (2–32), are shown to include the results obtained by these authors, respectively as special cases. Shannon's[14),15)] and Someya's[13)] sampling theorems as well as other sampling formulae[1)~12),16),17)] are included as special cases of the present theorem (2–26). These will be shown in the following paper in the form of various examples.

It should be mentioned here that expression (2–23) has somewhat different features as was given by Linden and Abramson[5], who took essentially as $g(z) = \sin^{m+1}(\pi z/h)$ and $z_n = nh$, $|h| \leqq (m+1)/2W$, with a constant W. The expression (2–32) holds only under the condition (IV) in (2–31).

§ 3. The properties of the Expansion-Formula of the Generalized Sampling Theorem

We shall take $\Phi(z)$ as the expansion-formula of the generalized sampling theorem (2–1′):

$$\Phi(z) \equiv \sum_n \phi_n(z) \tag{3–1}$$

$$= \sum_n \sum_{s=0}^{m_n} \sum_{j=0}^{s} \frac{f_n^{(j)}}{j!} \cdot \frac{H_n^{(s-j)}}{(s-j)!} \cdot \frac{(z-z_n)^s}{(z-z_n)^{m_n+1}} g(z), \tag{3–2}$$

and consider the properties of the function $\Phi(z)$. The function $\Phi(z)$ is of the following characters:

i) The generalized sampling function:

$$h(z) = \frac{g(z)}{(z-z_n)^{m_n+1}} = \frac{1}{H(z)}, \tag{3–3}$$

has a finite value at $z = z_n$, *i.e.*

$$\lim_{z \to z_n} h(z) = \frac{1}{(m_n+1)!} \cdot \left[\frac{d^{m_n+1}}{dz^{m_n+1}} g(z) \right]_{z=z_n} < +\infty . \tag{3–4}$$

This is seen directly from condition (II) in (2–3).

ii) From (3–1) and (3–2), we have:

$$\phi_n(z) H(z) = \sum_{s=0}^{m_n} \frac{(z-z_n)^s}{s!} \sum_{j=0}^{s} {}_sC_j \cdot f_n^{(j)} \cdot H_n^{(s-j)} \tag{3–5}$$

$$= \sum_{s=0}^{m_n} \frac{(z-z_n)^s}{s!} \cdot \left[\frac{d^s}{dz^s} \left\{ f(z) H(z) \right\} \right]_{z=z_n}, \tag{3–6}$$

with

$$\frac{1}{H(z)}=h(z)=\frac{g(z)}{(z-z_n)^{m_n+1}}.$$

Accordingly, the function $\phi_n(z)\,H(z)$ is considered to be expressed in a *truncated* Taylor series (*i. e.* with finite initial terms) of the function $f(z)\,H(z)$ around the point $z=z_n$.

iii) We shall consider the limiting cases: $z\to z_n$ and $z\to z_r(\neq z_n)$, respectively in the expression (3–6).
Differentiating both sides of (3–6), we have

$$\frac{d^k}{dz^k}\left\{\phi_n(z)\,H(z)\right\}=\sum_{s=k}^{m_n}\frac{(z-z_n)^{s-k}}{(s-k)!}\cdot\left[\frac{d^s}{dz^s}\left\{f(z)\,H(z)\right\}\right]_{z=z_n}, \tag{3–7}$$

for $0\leqslant k\leqslant m_n$.

For $z\to z_n$, we have $H(z_n)\neq 0$, and obtain from (3–7):

$$\left[\frac{d^k}{dz^k}\left\{\phi_n(z)\,H(z)\right\}\right]_{z=z_n}=\left[\frac{d^k}{dz^k}\left\{f(z)\,H(z)\right\}\right]_{z=z_n}. \tag{3–8}$$

The expression (3–8) leads to:

$$\phi_n^{(k)}(z_n)=f_n^{(k)},\qquad (\text{for } 0\leqslant k\leqslant m_n) \tag{3–9}$$

with

$$f_n^{(k)}=\left[\frac{d^k}{dz^k}f(z)\right]_{z=z_n}.$$

On the other hand, we have, from (3–6):

$$\phi_n(z)=h(z)\cdot\sum_{s=0}^{m_n}\frac{(z-z_n)^s}{s!}\cdot\left[\frac{d^s}{dz^s}\left\{f(z)\,H(z)\right\}\right]_{z=z_n}. \tag{3–10}$$

$$\equiv h(z)\cdot Q(z), \tag{3–10'}$$

and

$$\frac{d^k}{dz^k}\phi_n(z)=\sum_{p=0}^{k}{}_kC_p\cdot h^{(p)}(z)\cdot Q^{(k-p)}(z). \tag{3–11}$$

For $z\to z_r(\neq z_n)$, where z_r are zeros $\left(\text{of } (m_r+1)\text{-th order}\right)$ of the function $g(z)$, we obtain from (3–11):

$$\phi_n^{(k)}(z_r)=0,\quad \text{for}\quad 0\leqslant k\leqslant m_n\quad \text{and for all}\quad r\neq n \tag{3–12}$$

with

$$h^{(p)}(z_r)=0,\quad \text{for}\quad 0\leqslant p\leqslant k\leqslant m_n\quad \text{and for all}\quad r\neq n \tag{3–13}$$

because the function $h(z)$ has a zero of (m_r+1)-th order at the sampling point $z=z_r\,(\neq z_n)$.

From (3–9) and (3–12), we have

$$\phi_n^{(k)}(z_r)=f_n^{(k)}\cdot\delta_{n,r},\quad \text{for}\quad 0\leqslant k\leqslant m_n \tag{3–14}$$

with Kronecker's delta $\delta_{n,r}$.

From (3–14), we obtain:

$$\Phi^{(k)}(z_r) = \sum_n \phi_n^{(k)}(z_r) = \sum_n f_n^{(k)} \cdot \delta_{n,r}$$
$$= f_r^{(k)}. \quad \text{for} \quad 0 \leqslant k \leqslant m_r, \quad \text{and for any sampling points } z = z_r. \tag{3-15}$$

iv) If z lies in the convergence domain around the point $z = z_n$ of the Taylor series (3–6), when m_n goes to infinity, we have

$$\phi_n(z)\, H(z) = f(z)\, H(z)\,. \tag{3-16}$$

In this case, we have

$$\phi_n(z) = f(z)\,, \qquad (\text{for } m_n \to \infty) \tag{3-17}$$

in the convergence domain (around the point $z = z_n$) of the Taylor series (3–6).

§ 4. Remarks on Some Examples of the Generalized Sampling Formulae

A) For the case $m=0$ in (2–23), we obtain the sampling formulae (2–26) which were suggested by van der pol[22].

a) If we take $g(z)$ as a polynomial of degree s, then the expression (2–26) is nothing but Lagrange's interpolation formula. If $f(z)$ is a polynomial of degree $(s-1)$, then the right-hand side of (2–26) expresses exactly $f(z)$. Even if the function $f(z)$ is a polynomial of a higher degree, the expression (2–26) can be used as an approximation formula[23].

b) If we take $g(z)$ in (2–26) as $\sin(\alpha z + \beta)$, with two constants α and β, then we obtain Someya's sampling formula[13], which reduces to Shannon's sampling formula[14),15] for the case $\beta = 0$.

c) If the function $g(z)$ in (2–26) is selected to be equal to[18)~20),23),24]:

i) $g(z) = \cos(\alpha \cos^{-1} \beta z), \quad \alpha\beta \neq 0$

ii) $g(z) = z \sin \beta z - A \cos \beta z, \quad \beta \neq 0$

iii) $g(z) = z \cos \beta z - B \sin \beta z, \quad \beta \neq 0$

iv) $g(z) = J_\nu(\beta z), \quad \beta \neq 0$

v) $g(z) = z J_\nu'(\beta z) + h J_\nu(\beta z), \quad \beta \neq 0$

vi) $g(z) = Y_\nu(\alpha z)\, J_\nu(\beta z) - J_\nu(\alpha z)\, Y_\nu(\beta z), \quad \alpha\beta \neq 0$

and

vii) $g(z) = Re \, \exp\left[i(\alpha z^2 + \beta)\right], \qquad \alpha \neq 0$

respectively, with Bessel function $J_\nu(z)$, Neumann function $Y_\nu(z)$, and constants α, β, A, B, and h, then we obtain the sampling formulae previously given by several authors[1)~5),7)~12),16),17)].

B) For the case $m \geqslant 1$ in (2–23), the function $g(z)$ is practically taken to be

$$g(z) = \phi^{m+1}(z)\,, \tag{4-1}$$

where $\phi(z)$ is an entire function with simple zeros at points $z = z_n$. Then the sampling formula (2–23) is slightly simplified, with $H_n^{(k)}$'s given by (B–11) and with

$$h_n=\frac{1}{(m+1)!}\,g_n^{(m+1)}=(\psi_n')^{m+1}, \tag{4–2}$$

and

$$\begin{aligned}{}_sC_r\,\frac{h_n^{(s-r)}}{h_n}&=\frac{s!\,(m+1)!}{r!\,(m+1+s-r)!}\cdot\frac{g_n^{(m+1+s-r)}}{g_n^{(m+1)}}\\&=\frac{s!(m+1)!}{r!(\psi_n')^{m+1}}\cdot\sum_{\substack{p+q+u+\cdots=m+1\\p+2q+3u+\cdots=m+1+s-r}}\frac{1}{p!\,q!\,u!\cdots}\cdot(\psi_n')^p\cdot\left(\frac{1}{2!}\psi_n''\right)^q\cdot\left(\frac{1}{3!}\psi_n'''\right)^u\cdots\,.\end{aligned} \tag{4–3}$$

It is also convenient, for example, to take the function $\psi(z)$ in (4–1) as:

a) $\psi(z)=$ an orthogonal polynomial of degree s,

b) $\psi(z)=\sin(\alpha z+\beta)$, with two constants $\alpha\neq 0$ and β,

and

c) $\psi(z)$ will be taken respectively as the function $g(z)$ given in i)$\sim$vii) of c) in the case **A**).

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Appendix A

The evaluation of the contour integral (2–8) is made as follows:
We put

$$I=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)\,g(\zeta)}\,d\zeta, \tag{A–1}$$

and deform the curve C into a circle of radius R with its centre at the origin, then we obtain:

$$I=\frac{1}{2\pi i}\int_R\frac{f(Re^{i\theta})}{(Re^{i\theta}-z)\cdot g(Re^{i\theta})}\cdot iRe^{i\theta}\cdot d\theta\,. \tag{A–2}$$

Making the radius R infinitely large in the expression (A–2), we obtain:

$$\begin{aligned}\lim_{R\to\infty}|I|&\leqslant\lim_{R\to\infty}\frac{1}{2\pi}\cdot\max_{\zeta\in Re^{i\theta}}\left|\frac{f(\zeta)}{g(\zeta)}\right|\cdot\frac{R}{R-|z|}\cdot\int_{\theta=0}^{2\pi}d\theta\\&=\lim_{R\to\infty}\max_{\zeta\in Re^{i\theta}}\left|\frac{f(\zeta)}{g(\zeta)}\right|\cdot\frac{R}{R-|z|}\,.\end{aligned} \tag{A–3}$$

From condition (III), we have:

$$\lim_{R\to\infty}|I|\leqslant 0\,. \tag{A–4}$$

Thus, the integral (2–8) is proved to vanish as R becomes infinitely large:

$$\lim_{R\to\infty} I = 0\,. \tag{A-5}$$

Appendix B

We shall calculate the expression (2–12) by making use of $g(z_n)$ and its derivatives. From the expression (2–10), we have the identity:

$$h(\zeta)\cdot H(\zeta)=1\,, \tag{B-1}$$

with

$$h(\zeta)=\frac{g(\zeta)}{(\zeta-z_n)^{m_n+1}}\,.$$

Differentiating both sides of this expression with respect to ζ, we obtain successively:

$$\left.\begin{aligned}
&h'(\zeta)\cdot H(\zeta)+h(\zeta)\cdot H'(\zeta) &&=0,\\
&h''(\zeta)\cdot H(\zeta)+{}_2C_1h'(\zeta)\cdot H'(\zeta)+h(\zeta)\cdot H''(\zeta) &&=0,\\
&\quad\vdots &&\ \ \vdots\\
\text{and}\quad & &&\\
&h^{(k)}(\zeta)\cdot H(\zeta)+{}_kC_1h^{(k-1)}(\zeta)\cdot H'(\zeta)+{}_kC_2h^{(k-2)}(\zeta)\cdot H''(\zeta) &&\\
&\qquad+\cdots+{}_kC_{k-1}h'(\zeta)\cdot H^{(k-1)}(\zeta)+h(\zeta)\cdot H^{(k)}(\zeta)=0. &&
\end{aligned}\right\} \tag{B-2}$$

A system of the above $(k+1)$ equations (B–1) and (B–2) can be solved with respect to $(k+1)$ unknown functions: $H(\zeta), H'(\zeta), H''(\zeta), \cdots,$ and $H^{(k)}(\zeta)$. We have:

$$H^{(s)}(\zeta)=\frac{D_s(\zeta)}{\Delta_s(\zeta)}\,,\qquad (s=0,1,2,\cdots,k) \tag{B-3}$$

with

$$\Delta_s(\zeta)=\begin{vmatrix}
h(\zeta), & 0, & & & \\
h'(\zeta), & h(\zeta), & \ddots & \mathbf{0} & \\
\vdots & \vdots & \ddots & & \\
h^{(s-1)}(\zeta), & {}_{s-1}C_1\,h^{(s-2)}(\zeta), & \cdots, & h(\zeta), & 0\\
h^{(s)}(\zeta), & {}_sC_1\,h^{(s-1)}(\zeta), & \cdots, & {}_sC_{s-1}h'(\zeta), & h(\zeta)
\end{vmatrix}=h^{s+1}(\zeta)\,, \tag{B-4}$$

and

$$D_s(\zeta)=\begin{vmatrix}
h(\zeta), & 0, & & & ,1\\
h'(\zeta), & h(\zeta), & 0, & \mathbf{0} & ,0\\
\vdots & \vdots & \ddots & & \vdots\\
h^{(s-1)}(\zeta), & {}_{s-1}C_1\,h^{(s-2)}(\zeta), & \cdots, & h(\zeta), & 0\\
h^{(s)}(\zeta), & {}_sC_1\,h^{(s-1)}(\zeta), & \cdots, & {}_sC_{s-1}\,h'(\zeta), & 0
\end{vmatrix}. \tag{B-5}$$

We shall refer to the expression (2–10) and the condition (II) in (2–3), *i.e.*, $g(\zeta)$ can be expanded in Taylor series around the point $\zeta=z_n$, as follows:

$$g(\zeta)=\sum_{j=1}^{\infty}A_{m_n+j}\cdot(\zeta-z_n)^{m_n+j}\,, \tag{B-6}$$

with

$$A_r = \frac{1}{r!}\left[\frac{d^r}{d\zeta^r}\, g(\zeta)\right]_{\zeta=z_n}. \qquad (r=\text{positive integers}) \tag{B-7}$$

Then we can see at once:

$$\begin{aligned} h^{(k)}(\zeta) &= \frac{d^k}{d\zeta^k}\, h(\zeta) = \frac{d^k}{d\zeta^k}\, \frac{g(\zeta)}{(\zeta-z_n)^{m_n+1}} \\ &= \sum_{s=0}^{\infty} A_{m_n+1+k+s} \cdot \frac{(k+s)!}{s!} \cdot (\zeta-z_n)^s, \end{aligned} \tag{B-8}$$

and

$$\frac{h_n^{(k)}}{h_n} = \frac{k!\,(m_n+1)!}{(m_n+1+k)!} \cdot \frac{g_n^{(m_n+1+k)}}{g_n^{(m_n+1)}}, \tag{B-9}$$

with

$$\left.\begin{aligned} &h_n^{(k)} = \left[\frac{d^k}{d\zeta^k}\, h(\zeta)\right]_{\zeta=z_n}, \\ &\text{and} \\ &g_n^{(k)} = \left[\frac{d^k}{d\zeta^k}\, g(\zeta)\right]_{\zeta=z_n}. \qquad (k=\text{positive integers}) \end{aligned}\right\} \tag{B-10}$$

From (B-3), (B-4), (B-5), and (B-9), we obtain:

$$H_n^{(k)} = \begin{vmatrix} 1, & 0, & 0, & & & \dfrac{1}{h_n} \\ \dfrac{h_n'}{h_n}, & 1, & 0, & & \mathbf{0} & 0 \\ \dfrac{h_n''}{h_n}, & {}_2C_1\dfrac{h_n'}{h_n}, & 1, & & & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \dfrac{h_n^{(k-1)}}{h_n}, & {}_{k-1}C_1\dfrac{h_n^{(k-2)}}{h_n}, & {}_{k-1}C_2\dfrac{h_n^{(k-3)}}{h_n}, & \cdots, & 1, & 0 \\ \dfrac{h_n^{(k)}}{h_n}, & {}_kC_1\dfrac{h_n^{(k-1)}}{h_n}, & {}_kC_2\dfrac{h_n^{(k-2)}}{h_n}, & \cdots, & {}_kC_{k-1}\dfrac{h_n'}{h_n}, & 0 \end{vmatrix}$$

$$= \frac{(-1)^k}{h_n} \begin{vmatrix} \dfrac{h_n'}{h_n}, & 1, & & \mathbf{0} & \\ \dfrac{h_n''}{h_n}, & {}_2C_1\dfrac{h_n'}{h_n}, & 1, & & \\ \vdots & \vdots & \vdots & \ddots & \\ \dfrac{h_n^{(k-1)}}{h_n}, & {}_{k-1}C_1\dfrac{h_n^{(k-2)}}{h_n}, & {}_{k-1}C_2\dfrac{h_n^{(k-3)}}{h_n}, & \cdots, & 1 \\ \dfrac{h_n^{(k)}}{h_n}, & {}_kC_1\dfrac{h_n^{(k-1)}}{h_n}, & {}_kC_2\dfrac{h_n^{(k-2)}}{h_n}, & \cdots, & {}_kC_{k-1}\dfrac{h_n'}{h_n} \end{vmatrix}, \tag{B-11}$$

with

$$H_n^{(k)} = \left[\frac{d^k}{d\zeta^k} H(\zeta)\right]_{\zeta=z_n}, \tag{B-12}$$

$$_sC_r \frac{h_n^{(s-r)}}{h_n} = \frac{s!}{r!} \cdot \frac{(m_n+1)!}{(m_n+1+s-r)!} \cdot \frac{g_n^{(m_n+1+s-r)}}{g_n^{(m_n+1)}}, \tag{B-13}$$

and

$$h_n = \frac{1}{(m_n+1)!} g_n^{(m_n+1)} = \frac{1}{H_n} = \frac{1}{H_n^{(0)}}, \tag{B-14}$$

for positive integers k, s, and r.

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