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LAMINAR WAKE OF A FLAT PLATE IN A UNIFORM SHEAR FLOW

by

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SUMMARY

The characteristics of a wake behind a body in flow is important in various aspects: the momentum of fluid in the wake has a clue to evaluate the force acting on a body, and a shielding effect of a body is also a knowledge required for an interest of practical applications. The studies on a wake have generally been pursued in cases of a uniform flow.

The present paper describes a detailed analysis of a far wake behind a plate mounted in a uniform shear flow, and an approximate analysis of a near wake is also attempted to show the development of the wake as a whole.

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1. Introduction

The wake behind a body is usually confined in a rather narrow and long region downstream when the Reynolds number is sufficiently large. Therefore, the velocity components u' and v' in the wake behind a two-dimensional body and the Cartesian coordinates x', y' may be governed by $u' \gg v'$ and $\partial/\partial y' \gg \partial/\partial x'$ as in the same manner of the boundary layer studies.

A wake is usually divided into two categories--near wake and far wake. The nature of a near wake is generally complicated and is usually hard to theoretically analyze. However, only the case of a flat plate mounted parallel to a uniform flow is treated by Goldstein(1) and Stewartson(2), for the flow does not separate at the trailing edge. On the other hand, a far wake is rather easy to analytically treat its flow characteristics, since the boundary layer approximation and a linearizing approximation of Oseen type could be applied. The analysis of a far wake is worthwhile, because the force acting on a body immersed in a flow could be defined irrespective of the shape of a body and of the flow in its vicinity.

The flow we most frequently encounter in practice is not necessarily uniform. However, the flow treated by most of the former studies seems to have been rather simple cases of a uniform flow. The present paper is interested in a laminar wake behind a plate in a simple shear flow, as the most basic and simple case of a non-uniform oncoming flow. Detailed analysis is given for the far wake, and an approximate solution for the near wake is also attempted to show its process of developing to the far wake. The analysis is further extended to the drag of the plate.

2. Formulation of the problem

When a representative physical length ℓ and a representative velocity U_0 are properly selected as reference quantities, a two-dimensional flow field can conveniently be described by the non-dimensional coordinates x, y and flow parameters. The dimensionless stream function ψ ($=\psi'/U_0\ell$) in this case may be defined by $u=\partial\psi/\partial y$ and $v=-\partial\psi/\partial x$, so that the equation of continuity will simultaneously be satisfied. The Navier-Stokes equation for a steady two-dimensional flow is

$$(\partial\psi/\partial y)(\partial\Delta\psi/\partial x)-(\partial\psi/\partial x)(\partial\Delta\psi/\partial y)=\Delta\Delta\psi/R \tag{1}$$

where R is the Reynolds number defined by $U_0\ell/\nu$ and $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$. As in the case of a boundary layer studies, the width of a laminar far wake can be assumed to be proportinal to $\ell R^{-\frac{1}{2}}$.

The velocity of the oncoming shear flow $u'=U_0+\omega y'$ shown in Fig. 1 gives the velocity distribution for the present analysis in dimensionless form:

$$u_\infty = 1+Ky \tag{2}$$

where ω is the velocity gradient involved in the approaching flow and $K=\omega\ell/U_0$. When the width of the wake is denoted by δ' on the physical plane, U_0, the oncoming velocity along x'-axis in this case must be much larger than $\omega\delta'$ so far as the effect of ω could be treated as a small perturbation. On the other hand, the boundary layer approximation technique assumes that $\delta'/x' \ll 1$ or $\delta' \sim x'(U_0x'/\nu)^{-\frac{1}{2}}$, which means $(U_0x'/\nu)^{1/2} \gg 1$. Therefore, the validity of the present analysis is limited in the following range:

$$R^{-\frac{1}{2}} \ll x^{\frac{1}{2}} \ll (KR^{-\frac{1}{2}})^{-1} \tag{3}$$

Since the Reynolds number for a far wake could be understood to be sufficiently large, $|y'| \sim O(\ell R^{-\frac{1}{2}})$. Therefore, the order of the magnitude of a new parameter to be introduced inside the far wake defining by

$$Y = R^{\frac{1}{2}}y \tag{4}$$

must be $Y \sim O(1)$ in the far wake. Then, the matched asymptotic expansion after Van Dyke's analysis (3) gives

$$\psi(x,y;R) \sim \Psi_1(x,y)+R^{-\frac{1}{2}}\Psi_2(x,y)+\cdots \quad \text{(outside the wake)} \tag{5}$$

$$\psi(x,y;R) \sim R^{-\frac{1}{2}}\psi_1(x,Y)+R^{-1}\psi_2(x,Y)+\cdots \quad \text{(inside the wake)} \tag{6}$$

The comparison of the terms of the same order with respect to R in the relations obtainable by substituting eq.(5) and eq.(6) into eq.(1) gives

$$(\partial\Psi_1/\partial y)(\partial\Delta\Psi_1/\partial x)-(\partial\Psi_1/\partial x)(\partial\Delta\Psi_1/\partial y)=0 \tag{7}$$

$$\frac{\partial\Psi_1}{\partial y}\frac{\partial\Delta\Psi_2}{\partial x}+\frac{\partial\Psi_2}{\partial y}\frac{\partial\Delta\Psi_1}{\partial x}-\frac{\partial\Psi_1}{\partial x}\frac{\partial\Delta\Psi_2}{\partial y}-\frac{\partial\Psi_2}{\partial x}\frac{\partial\Delta\Psi_1}{\partial y}=0 \tag{8}$$

$$\partial\mathcal{L}_1(\psi_1)/\partial Y=0 \tag{9.a}$$

$$\partial\mathcal{L}_2(\psi_1,\psi_2)/\partial Y=0 \tag{9.b}$$

where,

$$\mathcal{L}_1(\psi_1)=\frac{\partial\psi_1}{\partial Y}\frac{\partial^2\psi_1}{\partial x\partial Y}-\frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_1}{\partial Y^2}-\frac{\partial^3\psi_1}{\partial Y^3} \tag{10}$$

$$\mathcal{L}_2(\psi_1,\psi_2)=\frac{\partial\psi_1}{\partial Y}\frac{\partial^2\psi_2}{\partial x\partial Y}+\frac{\partial\psi_2}{\partial Y}\frac{\partial^2\psi_1}{\partial x\partial Y}-\frac{\partial\psi_1}{\partial x}\frac{\partial^2\psi_2}{\partial Y^2}-\frac{\partial\psi_2}{\partial x}\frac{\partial^2\psi_1}{\partial Y^2}-\frac{\partial^3\psi_2}{\partial Y^3} \tag{11}$$

Since $\Psi_1(x,y)$ implies the stream function of the flow excluding the disturbance of the wake, the solution for Ψ_1 will readily be obtained from eq.(2):

$$\Psi_1(x,y)=y+(K/2)y^2 \tag{12}$$

Substituting eq.(12) into eq.(8), one has $\partial\Delta\Psi_2/\partial x=0$.

Thus,
$$\Delta \Psi_2(x,y)=0 \tag{13}$$
because of the Helmholtz's conservation of vorticity law along a streamline outside the wake, since the relation $\partial\Delta\Psi_2/\partial x=0$ does not include any contribution of viscosity.

The velocity components outside and inside the wake may be expanded in the same manner as for the stream functions in eqs.(5) and (6):

$$u(x,y;R)=U_1(x,y)+R^{-\frac{1}{2}}U_2(x,y)+\cdots \ (14) \qquad v(x,y;R)=V_1(x,y)+R^{-\frac{1}{2}}V_2(x,y)+\cdots \ (15)$$
$$u(x,y;R)=u_1(x,Y)+R^{-\frac{1}{2}}u_2(x,Y)+\cdots(16) \qquad v(x,y;R)=R^{-\frac{1}{2}}v_1(x,Y)+R^{-1}v_2(x,Y)+\cdots \ (17)$$

Since the flow inside the wake behind a body in a uniform flow should be symmetrical with respect to $Y=0$,
$$\partial u_1/\partial Y=0 \quad \text{for} \quad Y=0 \tag{18}$$

Further, so far as the velocities given by eqs.(14) and (16) for outside and inside the wake could be continuous, the boundary condition at $Y\rightarrow\pm\infty$ gives $u_1(x,Y)\rightarrow U_1(x,0)$ and $u_2(x,Y)\rightarrow KY+U_2(x,0)$. Thus, $\psi_1(x,Y)\rightarrow YU_1(x,0)+X_1(x)$ and $\psi_2(x,Y)\rightarrow(\frac{1}{2})KY^2+YU_2(x,0)+X_2(x)$ for $Y\rightarrow\pm\infty$. On the other hand, eq.(9) gives $\mathcal{L}_1(\psi_1)=F_1(x)$ and $\mathcal{L}_2(\psi_1,\psi_2)=F_2(x)$. The functions of x, F_1 and F_2 can now be evaluated by eqs.(10) and (11) with the boundary condition for $Y\rightarrow\pm\infty$: $F_1(x)=0$ and $F_2(x)=-K(dX_1/dx)+dU_2(x,0)/dx$. However, the function $X_1(x)$ in F_2 must still be determined. Since $X_1(x)=\psi_1(x,Y)-YU_1(x,0)$, $X_1=-\int_0^\infty[1-u_1(x,Y)]dY$ for $Y>0$, and $X_1=\int_0^\infty[1-u_1(x,Y)]dY$ for $Y<0$. Thus,
$$X_1=-\delta_1^*(x)\,\mathrm{sgn}\,Y \tag{19}$$
where, $\delta_1^*(x)=\int_0^\infty[1-u_1(x,Y)]dY$. Therefore, the relations $\mathcal{L}_1=F_1(x)$, $\mathcal{L}_2=F_2(x)$ and the equation of continuity give the equations of motion and the continuity relationship for the flow inside the wake:
$$u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial Y}-\frac{\partial^2 u_1}{\partial Y^2}=0 \tag{20.a}$$
$$\partial u_1/\partial x+\partial v_1/\partial Y=0 \tag{20.b}$$
$$u_1\frac{\partial u_2}{\partial x}+u_2\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_2}{\partial Y}+v_2\frac{\partial u_1}{\partial Y}-\frac{\partial^2 u_2}{\partial Y^2}=K\frac{d\delta_1^*}{dx}\mathrm{sgn}\,Y+\frac{d}{dx}U_2(x,0) \tag{21.a}$$
$$\partial u_2/\partial x+\partial v_2/\partial Y=0 \tag{21.b}$$

3. Velocity distributions in the wake

Eqs.(20) and (21) must be solved with respect to u_1, v_1 and u_2, v_2 to obtain the velocities in the wake. u_1 and v_1 are the velocity components in the far wake behind a plate in uniform flow of $u_\infty=1$, which are already obtained by Goldstein(4) and Stewartson(5). Namely,
$$u_1=1-(w_1^{(1)}+w_1^{(2)}+\cdots) \tag{22}$$
$$v_1=v_1^{(1)}+v_1^{(2)}+\cdots \tag{23}$$
where,
$$w_1^{(1)}=Ax^{-\frac{1}{2}}e^{-\frac{1}{2}\eta^2} \tag{24}$$
$$v_1^{(1)}=-\frac{A}{\sqrt{2}}x^{-1}\eta e^{-\frac{1}{2}\eta^2} \tag{25}$$
$$w_1^{(2)}=\frac{A^2}{2}x^{-1}[e^{-\eta^2}+(\frac{\pi}{2})^{\frac{1}{2}}\eta e^{-\frac{1}{2}\eta^2}\mathrm{erf}(\frac{\eta}{\sqrt{2}})] \tag{26}$$
$$v_1^{(2)}=\frac{A^2}{(2)^{3/2}}x^{-\frac{3}{2}}[(\frac{\pi}{2})^{\frac{1}{2}}(1-\eta^2)e^{-\frac{1}{2}\eta^2}\mathrm{erf}(\frac{\eta}{\sqrt{2}})-\eta e^{-\eta^2}-\sqrt{\pi}\,\mathrm{erf}\,\eta] \tag{27}$$
The function η involved in these equations is given by
$$\eta=Y/(2x)^{\frac{1}{2}} \quad \text{and} \quad \mathrm{erf}\,\eta=\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}d\eta \tag{28}$$
The other terms $w_1^{(3)}, v_1^{(3)}$ etc. that should appear in eqs.(22) and (23) are also obtainable, but they are omitted here as trivial terms. The next step is to solve eq.(21) to obtain u_2 and v_2. Evidently, the first term in the right-hand side of eq.(21.a) represents the effect of vorticity involved in the approaching flow, and the second term $dU_2(x,0)/dx$ means the inclination of flow against the original direction because of the displacement effect of the wake itself. Since eq.(21.a) is linear in respect of u_2 and v_2, the solutions of the two sets of equations obtainable by separating the two terms in the right-hand side can be superposed to integrate eq.(21.a):

$$\mathcal{L}(u_1,v_1,u_2^{(s)},v_2^{(s)})=K\frac{d\delta_1^*}{dx}\mathrm{sgn}\,Y \tag{29.a}$$
$$\partial u_2^{(s)}/\partial x+\partial v_2^{(s)}/\partial Y=0 \tag{29.b}$$
$$\mathcal{L}(u_1,v_1,u_2^{(d)},v_2^{(d)})=\frac{dU_2(x,0)}{dx} \tag{30.a}$$
$$\partial u_2^{(d)}/\partial x+\partial v_2^{(d)}/\partial Y=0 \tag{30.b}$$
$$u_2=u_2^{(s)}+u_2^{(d)}, \qquad v_2=v_2^{(s)}+v_2^{(d)} \tag{31}$$

The boundary conditions for eqs.(29) and (30) are:

for $Y \Rightarrow \infty$: $u_2^{(s)} \Rightarrow KY$, $u_2^{(d)} \Rightarrow U_2(x,0)$ and for $Y=0$: $\partial u_2^{(d)}/\partial Y=0$, $v_2^{(d)}=0$ (32)

(A) Solution of eqs.(29.a) and (29.b)
Eqs.(29.a) and (29.b) can be rewritten by putting $u_2^{(s)}=KY-w_2^{(s)}$:

$$\frac{\partial w_2^{(s)}}{\partial x}-\frac{\partial^2 w_2^{(s)}}{\partial Y^2}=K\left(v_1-Y\frac{\partial w_1}{\partial x}-\mathrm{sgn}\,Y\frac{d}{dx}\int_0^{+\infty}w_1 dY\right)+w_1\frac{\partial w_2^{(s)}}{\partial x}+w_2^{(s)}\frac{\partial w_1}{\partial x}-v_1\frac{\partial w_2^{(s)}}{\partial Y}-v_2^{(s)}\frac{\partial w_1}{\partial Y} \tag{33.a}$$

$$(\partial w_2^{(s)}/\partial x)-(\partial v_2^{(s)}/\partial Y)=0 \tag{33.b}$$

Therefore, the solution of these equations should give the values of $u_2^{(s)}$ and $v_2^{(s)}$. Let expand $w_2^{(s)}$ and $v_2^{(s)}$ in the following manner by understanding that $|w_2^{(1,s)}|>|w_2^{(2,s)}|>|w_2^{(3,s)}|\dots$ and $|v_2^{(1,s)}|>|v_2^{(2,s)}|>|v_2^{(3,s)}|\dots$ when x is sufficiently large:

$$w_2^{(s)}=w_2^{(1,s)}+w_2^{(2,s)}+w_2^{(3,s)}+\cdots$$
$$v_2^{(s)}=v_2^{(1,s)}+v_2^{(2,s)}+v_2^{(3,s)}+\cdots \tag{34}$$

Then, the solution of the equations derivable by substituting $w_2^{(s)}=w_2^{(1,s)}$ and $v_2^{(s)}=v_2^{(1,s)}$ into eqs.(33) gives the first approximation of $w_2^{(s)}$ and $v_2^{(s)}$. And the next step is to substitute $w_2^{(s)}=w_2^{(1,s)}+w_2^{(2,s)}$ and $v_2^{(s)}=v_2^{(1,s)}+v_2^{(2,s)}$ again into eqs.(33) to improve the accuracy of the second approximate values of $w_2^{(s)}$ and $v_2^{(s)}$ by understanding the values of $w_2^{(1,s)}$ and $v_2^{(1,s)}$ are known in the former first step and vice versa. Namely, the substitution of $w_2^{(s)}=w_2^{(1,s)}$ and $v_2^{(s)}=v_2^{(1,s)}$ into eqs.(33) gives the following relations when the same order respecting to x is compared:

$$\frac{\partial w_2^{(1,s)}}{\partial x}-\frac{\partial^2 w_2^{(1,s)}}{\partial Y^2}=K\left(v_1^{(1)}-Y\frac{\partial w_1^{(1)}}{\partial x}-\mathrm{sgn}\,Y\frac{d}{dx}\int_0^{+\infty}w_1^{(1)}dY\right) \tag{35.a}$$

$$\frac{\partial w_2^{(1,s)}}{\partial x}-\frac{\partial v_2^{(1,s)}}{\partial Y}=0 \tag{35.b}$$

In the same way, other sets of equations can be derived by putting $w_2^{(s)}=w_2^{(1,s)}+w_2^{(2,s)}$, $v_2^{(s)}=v_2^{(1,s)}+v_2^{(2,s)}$ and so on to determine $w_2^{(2,s)}, w_2^{(3,s)}\dots$ and $v_2^{(2,s)}, v_2^{(3,s)}\dots$. Eq.(24) gives $\int_0^{\infty}w_1\,dY=\sqrt{\pi}A$ or $d(\int_0^{\infty}w_1\,dY)/dx=0$. Therefore, introduction of this relation to the right-hand side of eq.(35.a) together with eq.(26) results to give

$$\frac{\partial w_2^{(1,s)}}{\partial x}-\frac{\partial^2 w_2^{(1,s)}}{\partial Y^2}=-\frac{KA}{\sqrt{2}}x^{-1}\eta^3 e^{-\frac{1}{2}\eta^2} \tag{36}$$

When this equation is rearranged by putting

$$w_2^{(1,s)}=KAF_1(\eta) \tag{37}$$

one has

$$F_1''+\eta F_1'=\sqrt{2}\,\eta^3 e^{-\frac{1}{2}\eta^2} \tag{38}$$

Thus,

$$F_1=-(2)^{-\frac{3}{2}}(\eta^3+3\eta)e^{-\frac{1}{2}\eta^2}+(3\sqrt{2}/4+a_1)(\pi/2)^{\frac{1}{2}}\mathrm{erf}(\eta/\sqrt{2})+a_2$$

a_1, a_2 being integration constants, which can be determined by the boundary condition in eq.(32). Namely, $a_1=-3\sqrt{2}/4$ and $a_2=0$, since $F_1 \Rightarrow 0$ when $\eta \Rightarrow \pm\infty$. Therefore, eq.(37) becomes

$$w_2^{(1,s)}=-(2)^{-\frac{3}{2}}KA(\eta^3+3\eta)e^{-\frac{1}{2}\eta^2} \tag{39}$$

When $\partial w_2^{(1,s)}/\partial x=\partial v_2^{(1,s)}/\partial Y$ obtainable from eq.(35.b) is substituted into eq.(36), one has

$$\frac{\partial v_2^{(1,s)}}{\partial Y}=(2x)^{-\frac{1}{2}}\frac{\partial^2 w_2^{(1,s)}}{\partial Y^2}-KAx^{-\frac{1}{2}}\eta^3 e^{-\frac{1}{2}\eta^2}$$

with which $v_2^{(1,s)}$ can be computed by introducing eq.(39) to this equation:

$$v_2^{(1,s)}=(KA/4)x^{-\frac{1}{2}}(\eta^4+4\eta^2+5)e^{-\frac{1}{2}\eta^2}+b_1 \tag{40}$$

The integration constant b_1 which may be a function of x in general can be determined by the principle of rapid decay of vorticity(6). Namely, the vorticity $\zeta_2^{(1,s)}=\partial v_2^{(1,s)}/\partial x+\partial w_2^{(1,s)}/\partial Y$ that would be introduced by $w_2^{(1,s)}$ and $v_2^{(1,s)}$ at $\eta \Rightarrow \pm\infty$ should show a nature of an exponential decay to zero. When $v_2^{(1,s)}$ is written in the form of $v_2^{(1,s)}=KAx^{-\frac{1}{2}}G_1(\eta)+b_1$ after eq.(40), $\zeta_2^{(1,s)}$ becomes

$$\zeta_2^{(1,s)}=(\partial v_2^{(1,s)}/\partial x)+(\partial w_2^{(1,s)}/\partial Y)=\frac{KA}{\sqrt{2}}x^{-\frac{1}{2}}F_1'-\frac{KA}{2}x^{-\frac{3}{2}}(\eta G_1)'+\frac{db_1}{dx}$$

Since the first two terms in the right-hand side of this equation evidently show an exponential decay for $\eta \Rightarrow \pm\infty$, db_1/dx must be equal to zero. Therefore, the boundary condition $v_2^{(1,s)} \Rightarrow 0$ at $x \Rightarrow \infty$ gives $b_1=0$. Thus,

$$v_2^{(1,s)}(x,0)=(5/4)KAx^{-\frac{1}{2}} \tag{41}$$

In the same way, $w_2^{(2,s)}$, $w_2^{(3,s)}$, ... and $v_2^{(2,s)}$, $v_2^{(3,s)}$, ... can be computed, though the process is rather lengthy:

$$w_2^{(2,s)}=-\frac{KA^2}{\sqrt{2}}\left[\frac{1}{4}(\eta^3+3\eta)e^{-\eta^2}-\sqrt{\pi}(1-e^{-\frac{1}{2}\eta^2})\mathrm{sgn}\,\eta-\sqrt{\pi}\,\mathrm{erf}\,\eta+\frac{1}{4}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\eta^4+2\eta^2-5)e^{-\frac{1}{2}\eta^2}\mathrm{erf}\left(\frac{\eta}{\sqrt{2}}\right)+Ce^{-\frac{1}{2}\eta^2}\right]x^{-1}$$

$$v_2^{(2,s)}=-(1/\sqrt{2})x^{-\frac{1}{2}}\eta\, w_2^{(2,s)}$$

$$w_2^{(3,s)}=\frac{KA^3}{16\sqrt{6}}x^{-1}\ln x(\eta^5+\alpha\eta)e^{-\frac{1}{2}\eta^2}+O(x^{-1}),\quad v_2^{(3,s)}=-\frac{KA^3}{36\sqrt{6}}x^{-\frac{3}{2}}\ln x[\eta^6-\eta^4+(\alpha-4)\eta^2-\alpha-8]e^{-\frac{1}{2}\eta^2}+O(x^{-\frac{3}{2}})$$

where, α is a constant that should be determined to ensure an exponential decay of the term which has an order of magnitude $O(x^{-2})$ at $x \Rightarrow \infty$.

(B) Solution of eqs.(30.a) and (30.b)

It is already found by Imai's work(7) that the flow outside the wake of a two-dimensional symmetrical body can be described by the imaginary part of a complex potential f(z):

$$\psi(x, y ; R) = I[f(z)] \tag{41'}$$

where, $f(z) = z + (AR^{-\frac{1}{2}}/\sqrt{\pi})\ln z - i(\pi/2)^{\frac{1}{2}}A^2R^{-\frac{1}{2}}z^{-\frac{1}{2}} + [(1/2)(3/\pi)^{\frac{1}{2}}A^3R^{-\frac{1}{2}} - (2/\pi)A^2R^{-1}]z^{-1}\ln z + \cdots$

The terms of $O(R^{-\frac{1}{2}})$ and $O(R^{-1})$ respectively mean the displacement of the wake introduced by u_1, v_1 and that introduced by u_2, v_2. The contribution of $u_2^{(d)}, v_2^{(d)}$ to the flow outside the wake must have a higher order of magnitude than $O(R^{-\frac{1}{2}})$, if there exists any. Therefore, the effect of vorticity could be obviated so long as the terms up to $O(R^{-\frac{1}{2}})$ are considered in the complex potential. Thus, the relations in eqs.(5) and (41) lead to

$$\Psi_2(x, y) = I\left[\frac{A}{\sqrt{\pi}}\ln z - i\left(\frac{\pi}{2}\right)^{\frac{1}{2}}A^2z^{-\frac{1}{2}} + \frac{1}{2}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}A^3z^{-1}\ln z + \cdots\right]$$

Further, the real part of the complex velocity derivable from $f(z)$ must be equal to $U_2(x,y)$ in eq.(14):

$$U_2(x, 0) = \frac{A}{\sqrt{\pi}}x^{-1} - \frac{1}{2}\left(\frac{3}{\pi}\right)^{\frac{1}{2}}A^3x^{-2}\ln x + \cdots$$

The first term in the right-hand side in this equation will be taken as the dominant term here. As in the former case of $u_2^{(d)}$, one puts $u_2^{(d)} = (A/\sqrt{\pi})x^{-1} - w_2^{(d)}$. Substitution of this equation into eqs. (30.a) and (30.b) gives the equations to be solved to determine $w_2^{(d)}$ and $v_2^{(d)}$. Expanding these two terms into $w_2^{(d)} = w_2^{(1,d)} + \cdots$ and $v_2^{(d)} = v_2^{(1,d)} + \cdots$, one has a series of a set of relations to refine the accuracy of $w_2^{(d)}$ and $v_2^{(d)}$. The results of analysis for $w_2^{(1,d)}$ and $v_2^{(1,d)}$ are as follows:

$$w_2^{(1,d)} = -\frac{A^2}{2\sqrt{\pi}}x^{-\frac{3}{2}}\ln x(\eta^2 - 1)e^{-\frac{1}{2}\eta^2} + O(x^{-\frac{3}{2}})$$

$$v_2^{(1,d)} = \frac{A^2}{\sqrt{8\pi}}x^{-2}\ln x(\eta^3 - 3\eta)e^{-\frac{1}{2}\eta^2} + O(x^{-2})$$

4. Drag

Taking two perpendicular sections to flow A_1B_1 and A_2B_2 sufficiently upstream and downstream from a body, one considers the control volume of a rectangular shape $A_1B_1B_2A_2A_1$. The positions of A_1, B_1, A_2, B_2 are understood to be sufficiently afar off the x'-axis. The mass m_1 and the momentum M_1 flowing into the control volume under consideration through the section denoted by A_1B_1 are $m_1 = \int_{-\infty}^{+\infty}\rho u'_\infty dy'$ and $M_1 = \int_{-\infty}^{+\infty}\rho u'^2_\infty dy'$. On the other hand, the mass m_2 and the momentum M_2 flowing out from the section downstream denoted by A_2B_2 are $m_2 = \int_{-\infty}^{+\infty}\rho(u'_\infty - w')dy'$ and $M_2 = \int_{-\infty}^{+\infty}\rho(u'_\infty - w')^2dy'$. Since there is a difference of $m_1 - m_2 = \int_{-\infty}^{+\infty}\rho w'dy'$ between these two masses, this amount of mass must flow out through the sides of the control volume. The mass $(m_1 - m_2)$ in this case may be understood to have an average velocity of U_0 in x'-direction, as the approaching velocity to the plate u'_∞ includes a simple shear flow. Namely, the momentum in x'-direction that would be carried out from control volume must be $M_3 = \int_{-\infty}^{+\infty}\rho U_0 w'dy'$. Therefore, D, the drag of the plate becomes $D = M_1 - M_2 - M_3$. In other words, the drag coefficient C_D takes the following form:

$$C_D = \int_{-\infty}^{+\infty}(w + 2Kyw - w^2)dy$$

where, $C_D = D/(\rho U_0^2\ell)$. Substituting the expression for w already obtained, one has the following relation for the limiting case of $x \to \infty$ after integration:

$$\begin{aligned} C_D R^{1/2} &= \sqrt{2}A\int_{-\infty}^{+\infty}e^{-\frac{1}{2}\eta^2}d\eta + \sqrt{2}KR^{-\frac{1}{2}}A^2\int_{-\infty}^{+\infty}F_2(\eta)d\eta \\ &= 2\sqrt{\pi}A - 2(\pi/2)^{\frac{1}{2}}KR^{-\frac{1}{2}}A^2C \end{aligned} \tag{42}$$

This is the equation by which the two constants A and C remained so far can be evaluated. As will readily be seen in eq.(42), C may be related with the drag to be introduced by the vorticity included in the main flow, since the contribution of K vanishes when C becomes zero. C_D, the drag coefficient of a flat plate is already obtained:

$$A = (1/\sqrt{4\pi})R^{\frac{1}{2}}C_D^{(1)} \tag{43}$$

This relation can easily be obtained from eq.(42) by putting $C_D = C_D^{(1)}$ for $K = 0$. Therefore, substituting eq.(43) into eq.(42), one has

$$C_D = C_D^{(1)} - (8\pi)^{-\frac{1}{2}}C_D^{(1)2}KC \tag{44}$$

In the case of a flat plate of a finite length ℓ mounted in a simple shear flow, C could be judged to be zero, since the effect of shear would probably cancel each other along the upper and lower surfaces. In this case, $A = 2\lambda/\sqrt{\pi}$, where $\lambda = 0.33206$.

5. An approximate analysis of near wake

Stewartson(2) showed that the boundary layer technique is not applicable within a circular region of radius $O(\ell R^{-\frac{3}{4}})$ with its center at the trailing edge of the plate. However, Goldstein's solution of a near wake, obtained by analytically extending the Blasius solution, could be applied except in the immediate vicinity of the trailing edge, because the region above stated would be very narrow when Reynolds number is sufficiently large.

By the use of an analytical result on a laminar boundary layer along a plate in a simple shear flow Murray(8) and Van Dyke(9), etc., an attempt is possible to analytically obtain the velocity profiles in the near wake of a flat plate by means of the similar manner with Goldstein's method. Eqs.(20) and (21) are again the governing relations by accepting the velocity components at the trailing edge already given by Murray and Van Dyke, etc.

The boundary condition at the outer edge of the wake gives $u_2(x,Y) \to KY + U_2(x,0)$ for $Y \to \pm\infty$, as was once used in the far wake analysis. Since the value of $U_2(x,0)$, the displacement effect of the wake on the flow inside the wake, is not evaluated as yet, one puts $u_2(x,Y) \to KY$ by ommitting the contribution of $U_2(x,0)$ for the present approximate treatment of the flow in near wake. Further, the values of $u_2(x,0)$, $v_2(x,0)$ and $u_2(x,0)$, $v_2(x,\infty)$ must be given to analyse the flow in near wake, none of which has not yet been determined. Therefore, the value of these terms must first be reasonably assumed for the time being. First, one assumes that $u_2(x,0)=0$ as will be given from the far wake analysis. As the second step, one further assumes either one of $v_2(x,0)$ or $v_2(x,\infty)=0$. Under these two sets of assumptions, (a) $u_2(x,0)=0$, $v_2(x,0)=0$ and (b) $u_2(x,0)=0$, $v_2(x,\infty)=0$, numerical solutions of the governing equations given by eqs.(20.a,b) and eqs.(21.a,b) are now possible to obtain. The numerical computations were performed by the finite differece method given by Schlichting(10), and the results of which for these two sets of assumptions are compared in Figs.2 and 3. Fig.2 shows the velocity defect along the center line of the wake ($Y=0$) as a function of x_1, the distance from the trailing edge. As is indicated in this figure, this result is for the case of a uniform flow. The result already obtained by Goldstein(1) and the result of the far wake analysis are also plotted for reference. A close agreement between the present numerical computations for the near wake and the far wake analysis can be expected, since a reasonably smooth connection between these two curves is evident. Further, one may be able to judge that the above assumptions employed for the analysis of near wake were not unfair, because the result of the present near wake analysis agrees very well with that of Goldstein when $KR^{-\frac{1}{2}}=0$. The distribution of u, the streamwise velocity component in near wake is plotted in Fig.3 for the case of $KR^{-\frac{1}{2}}=0.05$ at several sections in the wake. It may be worthwhile to mention that the effect of the two assumptions for $v_2(x,0)=0$ and $v_2(x,\infty)=0$ is so small that it can hardly be differenciated in this plot. The results of far wake analysis for two sections of $x_1=0.096$ and 3.072 are also indicated with broken line for reference.

6. Discussion of the results

From the above analysis one can write the velocity component in x-direction in a laminar far wake:

$$u(x,Y)=u_1(x,Y)+R^{-\frac{1}{2}}[u_2^{(s)}(x,Y)+u_2^{(d)}(x,Y)]+---- \tag{45.a}$$

where,

$$u_1(x,Y)=1-Ax^{-\frac{1}{2}}f_{11}(\eta)-A^2x^{-1}f_{12}(\eta)-A^3x^{-\frac{3}{2}}\ln x\, f_{13}(\eta)+O(x^{-\frac{3}{2}}) \tag{45.b}$$

$$u_2^{(s)}(x,Y)=K[Y-Af_{21}^{(s)}(\eta)-A^2x^{-\frac{1}{2}}f_{22}^{(s)}(\eta)-A^3x^{-1}\ln x\, f_{23}^{(s)}(\eta)+O(x^{-1})] \tag{45.c}$$

$$u_2^{(d)}(x,Y)=(A/\sqrt{\pi})x^{-1}-A^2x^{-\frac{3}{2}}\ln x+O(x^{-\frac{3}{2}}) \tag{45.d}$$

The value of velocity components at Y = 0 will readily be evaluated from eqs.(45) and equation of continuity:

$$u(x,0)=1-(A-\tfrac{A^2}{\sqrt{2}}KR^{-\frac{1}{2}}C)x^{-\frac{1}{2}}-(\tfrac{A^2}{2}-\tfrac{A}{\sqrt{\pi}}R^{-\frac{1}{2}})x^{-1}-(\tfrac{A^3}{8\sqrt{3}}+\tfrac{A^2}{2\sqrt{\pi}}R^{-\frac{1}{2}})x^{-\frac{3}{2}}\ln x+O(x^{-\frac{3}{2}})$$

$$v(x,0)=(A/4)KR^{-1}x^{-\frac{1}{2}}+(\alpha A^3/36\sqrt{6})KR^{-1}x^{-\frac{3}{2}}\ln x+O(x^{-\frac{3}{2}})$$

As will be detected in eqs.(45.c) and (45.d), the decay of the effect of vorticity involved in the approaching flow is much less compared to that of displacement effect of the wake itself at $x \to \infty$. Namely, speaking about the second order terms, the effect of the vorticity in main flow is dominant in the far wake.

It is difficult to exactly determine the origin of x in the analysis of a far wake, since the flow in the vicinity of a body itself is not exactly clear. However, it is known that neither the mid-point of a body nor the trailing edge of the body could be the origin. Goldstein(4) showed that the velocity distributions in near wake and far wake behind a flat plate in a uniform flow can smoothly be connected when the origin of x is taken at 0.52 upstream from the trailing edge, by comparing the velocity along x-axis at the mid-line of the wake. Namely, x_1, the dimensionless distance from the trailing edge could be expressed by $x = x_1 + 0.52$. Since the value of C could be judged to be approximately zero in the case of a simple shear flow, the velocity component at $Y = 0$ may coincide with the case of a uniform flow. Thus, the origin of x may be taken at the same point with the Goldstein's case. Actually, the result of the numerical computation of the velocity distribution in the far wake obtainable in this manner agrees well with that performed in near wake for $x>5$ as will be seen in Fig.3. A feature of development from near wake to far wake can also be seen. A close examination of these two velocity distributions reveals that the processes of development to the far wake are different each other—the velocity distribution at the higher-velocity side approaches faster to that of the far wake compared with the lower-velocity side. The nature of w, the velocity defect in a laminar wake behind a flat plate, is shown in Fig.4. It can clearly be seen that the asymmetrical feature of the velocity distribution becomes evident as x increases: the width of the wake extends to the lower-velocity side rather than to the higher velocity side, and the position to show the maximum velocity defect is shifted to the lower-velocity side.

It is usually a conventional way to employ $\delta_{1/2}$, the width of the wake defined by the distance between two points on the velocity distribution curve for 0.5 w_{max}. The feature of $Y_{1/2}^{+}$ and $Y_{1/2}^{-}$ which were read in this manner on the velocity distribution curves in Fig.4 is plotted in Fig.5. The values of $\delta_{1/2}$ for the two cases of a uniform flow and the present shear flow are compared on logarithmic plot in Fig.6.

7. Conclusion

The velocity distributions in laminar far wake behind a flat plate mounted in a simple shear flow were analytically treated in details. The effect of vorticity involved in the oncoming flow over the plate was treated as a small perturbation of the case of a uniform flow to compute a reasonable velocity distribution within a range of $(U_0\ell/\nu)^{-\frac{1}{2}} \ll (x'/\ell)^{\frac{1}{2}} \ll (U_0\ell/\nu)^{\frac{1}{2}}(\omega\ell/U_0)^{-1}$. An attempt was also made to compute the velocity distribution in a laminar near wake under acceptable assupmtions to demonstrate the development of flow from a near wake to a far wake.

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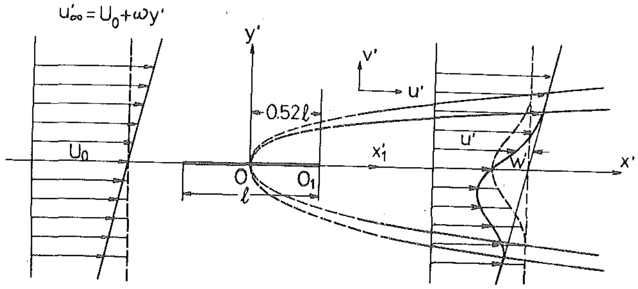

Fig. 1 Definition sketch and coordinate system.

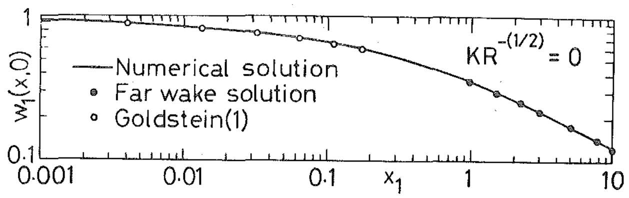

Fig. 2 Comparison of the streamwise velocity component along the center-line of the wake.

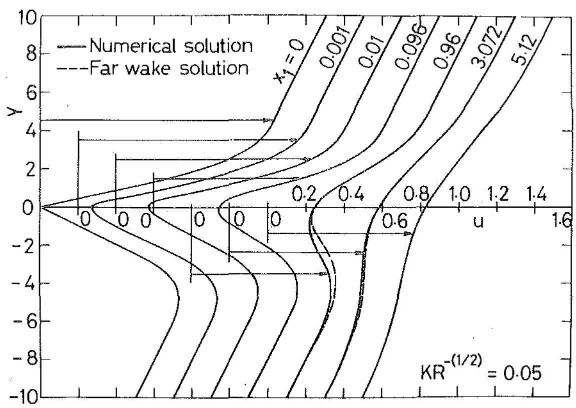

Fig. 3 Distribution of streamwise velocity component in the wake.

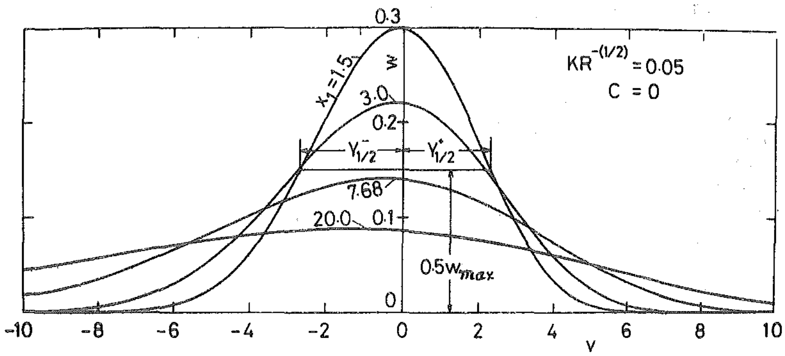

Fig. 4 Velocity defect in the wake.

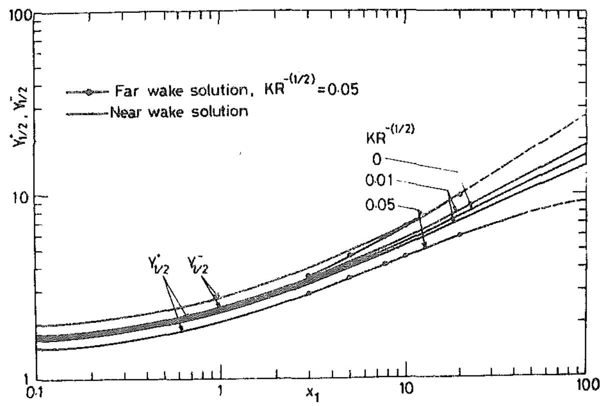

Fig. 5 Variation of $Y_{1/2}^+$ and $Y_{1/2}^-$ of the wake.

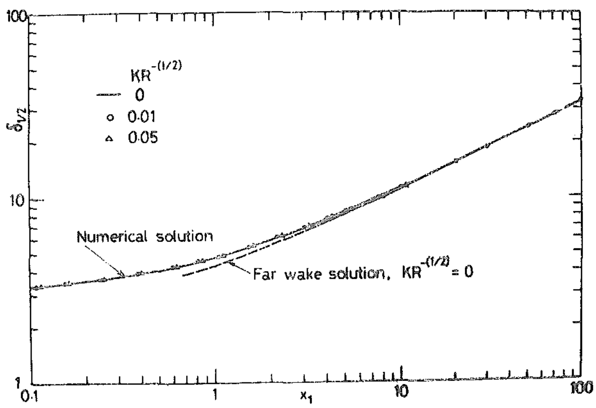

Fig. 6 Variation of the width of wake.