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Maximal Circuit-free and Cutset-free Subgraphs

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Abstract

A maximal circuit-free subgraph and a maximal cutset-free subgraph have important meanings in a graph. That is to say, a maximal circuit-free subgraph of a graph is a tree and a maximal cutset-free subgraph of a graph is a cotree. In this paper, the properties of a maximal circuit-free and cutset-free subgraph are investigated. And it is shown that a maximal circuit-free and cutset-free subgraph h_m of a graph is a certain hybrid tree and the number of edges of a hybrid tree containing the maximum number of edges changes according to the partition (E_y, E_z) of the set of edges E and its minimum value m_f is equal to the number of edges of h_m. The number m_f coincides with the topological degrees of freedom of the graph.

I. Introduction

Circuits (or loops) and cutsets are basic concepts in graph theory. Some fundamental concepts of a graph are defined by using these basic concepts. The rank of a graph is defined to be the number of linearly independent cutsets in the graph. The nullity of a graph is the number of linearly independent circuits in the graph. A tree on a graph is a maximal circuit-free*1 subgraph and a cotree on a graph is a maximal cutset-free subgraph[2),3)] and the number of edges of a tree or a cotree equals the rank or the nullity of the graph, respectively. These show that a maximal circuit-free subgraph and a maximal cutset-free subgraph have important meanings in a graph. Then, what meaning does a maximal circuit-free and cutset-free subgraph have? In this paper, the properties of a maximal circuit-free and cutset-free subgraph are investigated and the relations among the subgraphs and the hybrid trees are presented.

II. Preliminaries

In order to clarify the relations among a maximal circuit-free and cutset-free subgraph and hybrid trees, first we shall present the definition of hybrid trees and some properties which will be necessary for the following discussions.

G is an undirected and connected graph. Let E be the set of edges of G and partition E into two subsets E_y and E_z such that $E_y \cup E_z = E$ and $E_y \cap E_z = \phi$.

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*1 A subgraph of G is said to be circuit-free or cutset-free in G if it contains no circuits or no cutsets respectively.

Definition 1: ε_z and $\bar{\varepsilon}_z$ ($\bar{\varepsilon}_z = E_z - \varepsilon_z$) are subsets of E_z in such a manner that the subgraphs[*2] ε_z and $\bar{\varepsilon}_z$ are cutset-free and circuit-free in G respectively. ε_y and $\bar{\varepsilon}_y$ ($\bar{\varepsilon}_y = E_y - \varepsilon_y$) are subsets of E_y in such a way that the subgraphs ε_y and $\bar{\varepsilon}_y$ are circuit-free and cutset-free respectively in the graph G_y derived from G by open-circuiting[*3] all the edges of ε_z and short-circuiting all the edges of $\bar{\varepsilon}_z$. Then the subgraph $\varepsilon_y \cup \varepsilon_z$ is called a hybrid tree in G with respect to the partition (E_y, E_z) of E[*4].

In this definition, if $E_y \neq \phi$ and $E_z = \phi$, then since $G = G_y$ and ε_y and $\bar{\varepsilon}_y$ are circuit-free and cutset-free in G_y respectively, the hybrid tree ε_y is a tree of G[*5]. If $E_y = \phi$ and $E_z \neq \phi$, then since $G_y = \phi$ and ε_z and $\bar{\varepsilon}_z$ are cutset-free and circuit-free in G respectively, the hybrid tree ε_z is a cotree of G. This means that a hybrid tree of G becomes a tree or a cotree of G according to a partition of E. In a hybrid tree $\varepsilon_y \cup \varepsilon_z$, the subgraphs ε_y and ε_z form a part of a tree and a cotree in G respectively. So it may safely be said that a hybrid tree has properties of both a tree and a cotree.

Let HT, T and CT be the sets of hybrid trees, trees and cotrees in G respectively. It is clear from the definition of hybrid trees that $\varepsilon_y \cup \bar{\varepsilon}_z$ and $\bar{\varepsilon}_y \cup \varepsilon_z$ (which is the complement of $\varepsilon_y \cup \bar{\varepsilon}_z$) are circuit-free and cutset-free in G respectively and thus $\varepsilon_y \cup \bar{\varepsilon}_z$ and $\bar{\varepsilon}_y \cup \varepsilon_z$ are a tree and a cotree of G respectively. Hence we obtain,

$$T = \left\{ t \mid t = ht \oplus E_z,\ ht \in HT \right\} \tag{1}$$

$$CT = \left\{ \bar{t} \mid \bar{t} = ht \oplus E_y,\ ht \in HT \right\} \tag{2}$$

where $\oplus$ denotes the ring sum. These show the relations among HT, T and CT and when $E_y \neq \phi$ and $E_z = \phi$, $T = HT$ and when $E_y = \phi$ and $E_z = \phi$, $CT = HT$.

The number of edges in a tree and a cotree of G is identical with the rank γ and the nullity μ of G respectively. The number of edges in a hybrid tree of G, however, is not always equal to γ or μ and every hybrid tree of G does not contain the same number of edges. Let hH_k be a subset of HT in such a way that

$$hT_k = \left\{ ht \,\middle|\, |ht| = k,\ ht \in HT \right\} \tag{3}$$

Theorem 1: Let HT be the set of hybrid trees of G with respect to the partition (E_y, E_z) of E. Then HT is classified as follows,

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*2 The subgraph specified by the subset of edges ε is defined as the subgraph whoes set of edges is ε and whoes set of vertices is the set of all the vertices that are incident with the members of ε.

Hereafter, the subgraph with the suffix "y" or "z" denotes that of E_y or E_z, respectively.

*3 The term "open-circuiting an edge" is used here in the meaning of removing the edge and the term "short-circuiting an edge" is used in the meaning of identifying the two end vertices of the edge and then removing it.

*4 If $\varepsilon_z = \phi$, $\bar{\varepsilon}_z \neq \phi$ and G_y consists of some self-loops only, then the hybrid tree is denoted by ϕ.

*5 The subgraphs g_1 and g_2 ($g_2 = E - g_1$) of G are a tree and a cotree respectively, if g_1 and g_2 are circuit-free and cutset-free in G, respectively.

$$HT=\{hT_m,\ hT_{m-2},\ \cdots,\ hT_{l+2},\ hT_l\} \tag{4}$$

where $hT_k \neq \phi$, $m \geq k \geq l$ and m, k and l are nonnegative integers.

Proof: Let $ht_k \in hT_k$ be a hybrid tree in such a way that $ht_k=\varepsilon_y \cup \varepsilon_z$, $\bar{\varepsilon}_z=E_z-\varepsilon_z$ and $\bar{\varepsilon}_y=E_y-\varepsilon_y$. If $|\varepsilon_z|=j$, then $|\varepsilon_y|=k-j$. The rank of the graph G_y derived from G by open-circuiting all the edges of ε_z and short-circuiting all the edges of $\bar{\varepsilon}_z$ is $k-j$, since ε_y is a tree of G_y. If E_z is partitioned into two subsets ε_z' and $\bar{\varepsilon}_z'$ are such that $|\varepsilon_z'|=j-1$ and ε_z' and $\bar{\varepsilon}_z'$ are cutset-free and circuit-free in G respectively, the rank of G_y' derived from G by open-circuiting all the edges of ε_z' and short-circuiting all the edges of $\bar{\varepsilon}_z'$ is $k-j-1$. Let ε_y' be a subset of E_y in such a way that ε_y' and $\bar{\varepsilon}_y'$ $(=E_y-\varepsilon_y')$ are circuit-free and cutset-free in G_y' respectively. Then $|\varepsilon_y'|=k-j-1$ and the subgraph $\varepsilon_y' \cup \varepsilon_z'$ is a hybrid tree and $|\varepsilon_y' \cup \varepsilon_z'|=k-2$. If $\varepsilon_z'=j+1$, then $|\varepsilon_y' \cup \varepsilon_z'|=k+2$. This means that in any partition of E_z, hybrid trees whose number of edges are $k+1$ or $k-1$ are not obtained.

In order to prove that $hT_k \neq \phi$ when $hT_{k-2} \neq \phi$ and $hT_{k+2} \neq \phi$, let us assume $hT_k=\phi$. Let $ht' \in hT_{k-2}$, $ht'' \in hT_{k+2}$, $ht'=\varepsilon_y' \cup \varepsilon_z'$, $ht''=\varepsilon_y'' \cup \varepsilon_z''$, $\bar{\varepsilon}_z'=E_z-\varepsilon_z'$ and $\bar{\varepsilon}_z''=E_z-\varepsilon_z''$. From the assumption, for any edge $e_z \in \varepsilon_z''$, the subgraph $e_z \cup \bar{\varepsilon}_z''$ contains a circuit. Let $\bar{\varepsilon}'=\bar{\varepsilon}_z'' \cap \bar{\varepsilon}_z'$ and $\bar{\varepsilon}''=\bar{\varepsilon}_z''-\bar{\varepsilon}'$, then $\bar{\varepsilon}_z'$ consists of $\bar{\varepsilon}'$ and $|\bar{\varepsilon}''|+2$ edges of ε_z''. Since the subgraph $\bar{\varepsilon}_z'$ contains no circuits, the subgraph $\bar{\varepsilon}_z' \cup \bar{\varepsilon}''$ contains at most $|\bar{\varepsilon}''|$ circuits. The subgraph $\bar{\varepsilon}_z' \cup \bar{\varepsilon}''$ however contains $|\bar{\varepsilon}''|+2$ circuits, because the subgraph $\bar{\varepsilon}_z' \cup \bar{\varepsilon}''$ consists of $\bar{\varepsilon}_z''$ and $|\bar{\varepsilon}''|+2$ edges of ε_z''. This contradicts the assumption. Q. E. D.

The values of m and l in the classification (4) of HT vary with the partition $(E_y,\ E_z)$ of$E$. If the graph G is separable and E is partitioned into two subsets E_y and E_z such that E_z consists of all the edges of some separable components and $E_y=E-E_z$, then $m=l$. If the graph G is nonseparable, then $m=l$ if and only if HT coincides with T or CT then $m=l=\gamma$ or μ.

III. Maximal Circuit-Free and Cutset-Free Subgraphs

Let H be the set of subgraphs of G satisfying the following condition: the graph derived from G by short-circuiting some edges of a subgraph $h \in H$ and open-circuiting all the other edges of h contains no circuits but self-loops. Then there exists the minimum subgraph satisfying the above condition. Let $H_m \subset H$ be the set of the minimum subgraphs. Then we obtain the following theorem.

Theorem 2: Let h_m be a maximal circuit-free and cutset-free subgraph of G. Then $h_m \in H_m$.

Proof. For any edge $e_i \in (E-h_m)$, $(i=1,\ 2,\ \cdots)$, the subgraph $h_m \cup e_i$ contains a circuit or a cutset of G. Let ε_c be the set of all the edges contained in such circuits and ε_1 be the subset of h_m in such a way that $\varepsilon_1 \subset \varepsilon_c$ and $\varepsilon_2\,(=h_m-\varepsilon_1) \not\subset \varepsilon_c$. Clearly, the graph derived by short-circuiting all the edges of ε_1 and open-circuiting all the edges of ε_2 contains no circuits but self-loops, namely, $h_m \in H$.

Consider a subgraph g of G which is circuit-free and cutset-free and is not maximal. Then there exists at least one edge e_j of $E-g$ in such a way that

$g \cup e_j$ contains neither circuits nor cutsets of G. Therefore, g is not an element of H. This means that if a subgraph $h \in H$ is circuit-free and cutset-free, it must be a maximal circuit-free and cutset-free subgraph of G. In order to show that $h_m \in H_m$, we assume that $h_m \notin H_m$ and g_h is an element of H_m in such a way that $g_h = g_1 \cup g_2$, $g_1 \cap g_2 = \phi$ and the graph derived from G by short-circuiting all the edges of g_1 and open-circuiting all the edges of g_2 contains no circuits but only self-loops. From the above explanations, g_h must contain circuits or cutsets. Assuming that circuits are contained in g_h and let L_p be one of them. Then, L_p contains at least two edges of g_2 because if L_p consists of only edges of g_1 or one edge of g_2 and edges of g_1, the subgraph $g_h - e_p$, $(e_p \in L_p)$ belongs to H and g_h is not a minimum subgraph. Let $L_{p2} = L_p \cap g_2$ and G_p be the graph derived from G by short-circuiting all the edges of g_1 and open-circuiting all the edges of $g_2 - L_{p2}$. In G_p, L_{p2} forms circuits and since $g_h \in H_m$, there exists an edge e_p of L_{p2} in such a way that e_p and the edges of $\bar{g}_h$ $(=E - g_h)$ construct a circuit L_s consisting of more than three edges. Let $e_s \in L_s$, $(e_s \notin L_{p2})$ and $g_2' = (g_2 - e_p) \cup e_s$. Then, $g_1 \cup g_2'$ belongs to H and does not contain the circuit L_p. Applying this procedure ot all the circuits of g_h, we obtain the subgraph that contains no circuits and consists of the same number of edges asg_h. Similarly, when g_h contains cutsets or both circuits and cutsets, we obtain the subgraph g_h' from g_h in such a way that $|g_h'| = |g_h|$, $g_h \in H$ and g_h' contains neither circuits nor cutsets. This contradicts the assumption that $h_m \notin H_m$.

Q. E. D.

This theorem shows that a maximal circuit-free and cutset-free subgraph is one of the minimum subgraphs in H. As an example, let us consider a graph G of Fig. 1. Let $h_m = \{e_1, e_4, e_7\}$, then h_m is a maximal circuit-free and cutset-free subgraph and the graph derived from G by open-circuiting the edge e_7 and short-circuiting the edges e_1 and e_4 contains no circuits but only has sely-loops.

A tree t and a cotree $\bar{t}$ are elements of H, since the graph derived from G by short-circuiting all the edges of t consists only of self-loops, and the graph derived from G by open-circuiting all the edges of $\bar{t}$ contains no circuits. We obtain the following lemma concerning hybrid trees.

Lemma 1: Let ht be a hybrid tree of G with respect to the partition (E_y, E_z) of E. Then $ht = \varepsilon_y \cup \varepsilon_z$ is an element of hT_m, if and only if the graph G' derived from G by open-circuiting all the edges of ε_z and short-circuiting all the edges of ε_y contains no circuits but only has self-loops.

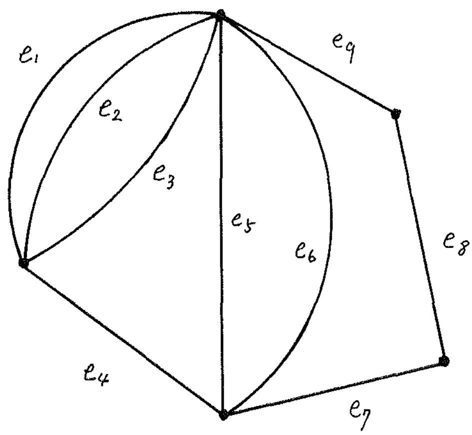

Fig. 1. *A graph G.*

Proof: Let $ht_m \in hT_m$, $ht_m = \varepsilon_y \cup \varepsilon_z$, $\bar{\varepsilon}_y = E_y - \varepsilon_y$ and $\bar{\varepsilon}_z = E_z - \varepsilon_z$. From the definition of hybrid trees, ε_z must be a maximal cutset-free subgrape of E_z and then the subgraphs $\bar{\varepsilon}_z$ and $\bar{\varepsilon}_z \cup E_y$ contain no circuits. Let G' be the graph derived from G by open-circuiting all the edges of ε_z. In the graph G', the subgraph $\varepsilon_y \cup \bar{\varepsilon}_z$ contains no circuits and the endpoints of an edge in $\bar{\varepsilon}_y$ are contained in the subgraph ε_y. Therefore, the graph derived from G' by short-circuiting all the edges of ε_y contains no circuits but only has self-loops.

Let $ht_k \in HT$, $ht_k \notin hT_m$, $ht_k = \varepsilon'_y \cup \varepsilon'_z$, $\bar{\varepsilon}'_y = E_y - \varepsilon'_y$ and $\bar{\varepsilon}'_z = E_z - \varepsilon'_z$. Since ε'_z is not a maximal cutset-free subgraph of E_z in G and $\bar{\varepsilon}'_z$ contains no circuits of G, there exists some circuits consisting of edges of $\bar{\varepsilon}'_z$ and E_y. Let one of them be L'. Since $\bar{\varepsilon}'_z \cup \varepsilon'_y$ contains no circuits, L' contains edges of $\bar{\varepsilon}'_y$. Therefore, the graph G' derived from G by open-circuiting all the edges of ε_z and short-circuiting all the edges of ε_y contains some circuits which are not self-loops.

Q. E. D.

This lemma shows that only the element $ht_m = \varepsilon_y \cup \varepsilon_z$ of hT_m in HT has the properties which are such that the graph derived from G by short-circuiting all the edges of ε_y and open-circuiting all the edges of ε_z contains no circuits but self-loops. From lemma 1 and theorem 2, we obtain the following theorem.

Theorem 3 : The minimum value m_f of m in the classification (4) of HT equals the number of edges contained in a maximal circuit-free and cutset-free subgraph in G.

Proof: Let ht_m be a hybrid tree of hT_{m_f} with respect to the partition (E^f_y, E^f_z) of E which gives the minimum value m_f of m. Then from lemma 1 and theorem 2, ht_m belongs to H and $|ht_m| \geq |h_m|$, where h_m is a maximal circuit-free and cutset-free subgraph. Therefore, we have only to prove that h_m can be a hybrid tree in G. For any edge e_i of $\bar{h}_m (=E-h_m)$, the subgraph $h_m \cup e_i$ contains a circuit or a cutset. Let E_y be the set of all the edges contained in such circuits and E_z be $E-E_y$. And let $h_m = \varepsilon_y \cup \varepsilon_z$, $\varepsilon_y \subset E_y$, $\varepsilon_z \subset E_z$, $\bar{\varepsilon}_y = E_y - \varepsilon_y$ and $\bar{\varepsilon}_z = E_z - \varepsilon_z$. Then $\bar{\varepsilon}_z$ contains no circuits and since $h_m \cup e_i$, $(e_i \in \bar{\varepsilon}_z)$ contains a cutset of G, $\bar{\varepsilon}_z$ contains no circuits. In the graph G_y derived from G by short-circuiting all the edges of $\bar{\varepsilon}_z$ and open-circuiting all the edges of ε_z, ε_y contains no circuits and since $h_m \cup e_i$, $(e_i \in \bar{\varepsilon}_y)$ contains a circuit of G, $\bar{\varepsilon}_y$ contains no cutsets. This means that h_m is a hybrid tree of G with respect to the partition (E_y, E_z).

Q. E. D.

As an example, let us consider the graph G of Fig. 1 again. In the graph G, a subgraph $h_m = \{e_1, e_4, e_7\}$ is a maximal circuit-free and cutset-free one. Let $E_z = \{e_7, e_8, e_9\}$ and $E_y = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Then $h_m = \varepsilon_y \cup \varepsilon_z$, $(\varepsilon_z = \{e_7\}, \varepsilon_y = \{e_1, e_4\})$ is a hybrid tree and $|h_m| (=3)$ is a minimum value of m.

In the proof of theorem 3, a method of the partition of E minimizing "m" was shown. The method, however, is not unique. Because if the subgraph $h_m \cup e_i$, $(e_i \in \bar{h}_m)$ contains both a circuit and a cutset of G, e_i may be an element of either E_y or E_z.

Since a maximal circuit-free and cutset-free subgraph h_m in G contains neither circuits nor cutsets, h_m can be a subset of both a tree and a cotree of G.

Thus h_m may be also regarded as the non-commom branches (or chords) of a pair of maximally distant trees of G[4),5)]. And the number of edges in a maximal circuit-free and cutset-free subgraph in G coincides with the topological degrees of freedom of G.

IV. Conclusion

The properties of a maximal circuit-free and cutset-free subgraph are investigated and the relationhsips among the subgraph and hybrid trees are presented. And it is shown that the number of edges of a hybrid tree containing the maximum number of edges changes according to the partition (E_y, E_z) of E and its minimum value is equal to the number of edges in a maximal circuit-free and cutset-free subgraph of G. (Note that the number of edges in a tree or a cotree of G is constant and is equal to the number of edges in a maximal circuit-free subgraph or a maximal cutset-free subgraph of G, respectively). These results are important not only as a basic theory of a graph, but also as an application to network theory, since the number of edges of a maximal circuit-free and cutset-free subgraph gives the degrees of freedom of the physical system (graph).

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