Master Equation for a Laser Model

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Synopsis

A master equation of a Laser system is presented from a dynamical point of view.

It is assumed that the Laser system is coupled with other independent subsystems. They play a role as distinct reservoirs in our Laser system.

As an application of this approach, the master equation is obtained for a system that corresponds to a Laser model considered by Scully-Lamb. The equation obtained here includes the dynamical effect of an atomic system, i. e., atomic coherence.

§ 1. Introduction

Recently various statistical mechanical theories have been developed by many authors in order to explain the properties of Laser systems.

Especially, Haken et al.[1] presented a model in which the Laser system is in contact with several independent heat reservoirs of different temperatures, undergoing irreversible processes among them. In particular, Weidlich and Haake[2] showed the existence of the lasing state which might be considered as a revelation of the phase change in an open system by the method of a master equation similar to the Wangnness-Bloch equation of nuclear spin induction. In this theory the relaxation processes are introduced phenomenologically in the master equation, without taking into account its detailed derivation from a microscopic point of view.

On the other hand, Scully-Lamb[3] independently developed a quantal theory of Laser, which is an extention of the famous semiclassic theory by Lamb, and obtained the statistical properties of radiation field of Laser systems.

In § 2, one of the purposes of this paper is presented. Here, from a dynamical point of view, a generalized master equation was setforth for a system coupled with several independent subsystems. Thus the discussion in § 2 is analogous to Argyres' theory[4] for the relaxation of spin system, in which Wangness-Bloch equation was derived from a dynamical point of view. The second purpose of this paper is to present a generalized master equation corresponding to the Laser system considered by Scully-Lamb as described in § 3.

§ 2. Derivation of the master equation

Now let us consider the following system. The system with which we are mainly concerned shall be called A and is described by dynamical variables $\{a_i\}$.

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Systems B and C are those that represent independent reservoirs consisting of many degrees of freedom with variables $\{b_i\}$ and $\{c_i\}$. For example, the variables $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ can be taken to be Pauli spin operators or Bose operators depending on the system characteris tics, and the interaction between them can be simply described by the products of spin flip (or spin down) operators and annhilation (or creation) operators. Three systems A, B and C, compose a closed system. For our system, the Hamiltonian is written as

$$H = H_A + H_B + H_C + \lambda_1 H_{AB} + \lambda_2 H_{AC} \tag{1}$$

where H_A, H_B and H_C are unperturbed Hamiltonians of the systems A, B and C. H_{AB} and H_{AC} are the interaction Hamiltonians between them, with coupling constants, λ_1 and λ_2.

Hereafter, the generalized master equation for the system of interest A is derived by assuming that the subsystems B and C are in equilibrium only at an initial time. The equation of motion of the total density matrix reads as

$$\frac{d\rho}{dt} = -i\,[H, \rho] \qquad (\text{with } \hbar = 1) \tag{2}$$

In order to describe the system of interest A, a projection operator P is defined

$$P\rho = \mathop{Tr}_{B,C}[\rho]/\mathop{Tr}_{B,C}[1] \equiv \rho_A \tag{3}$$

Using the definition of this projection operator, one obtains exact equation for the system A,

$$\begin{aligned}\frac{d\rho_A}{dt} = -i\,[H_A, \rho_A] &- i\lambda_1 P\left[H_{AB}, \int_0^t ds \exp\left\{-i(1-P)Ls\right\}(1-P)L\rho_A(t-s)\right] \\ &- i\lambda_2 P\left[H_{AC}, \int_0^t ds \exp\left\{-i(1-P)Ls\right\}(1-P)L\rho_A(t-s)\right] \\ &- i\lambda_1 P\left[H_{AB}, \exp\left\{-i(1-P)Lt\right\}\rho'(0)\right] \\ &- i\lambda_2 P\left[H_{AC}, \exp\left\{-i(1-P)Lt\right\}\rho'(0)\right]\end{aligned} \tag{4}$$

where L is the Liouville operator and $\rho'(0) = \rho_A(0)\cdot\rho_B(0)\cdot\rho_C(0)$. Now assuming that the total density matrix satisfies following two initial conditions (i) $\rho(0) = \rho_A(0)\cdot\rho_B(0)\cdot\rho_C(0)$ and (ii) $\rho_B(0)$ and $\rho_C(0)$ are diagonal with respect to the representation which diagonalizes unperturebed Hamiltonian. The following approximate equation is derived when the lowest order terms in the coupling constants λ_1 and λ_2 are retained:

$$\begin{aligned}\frac{d\rho_A}{dt} = -i\,[H_A, \rho_A] &- \lambda_1^2 P\int_0^t ds\left[H_{AB}, \exp\left\{-iL_0 s\right\}\left[H_{AB}\rho_A(t-s)\right]\right] \\ &- \lambda_2^2 P\int_0^t ds\left[H_{AC}, \exp\left\{-iL_0 s\right\}\left[H_{AC}, \rho_A(t-s)\right]\right] \\ &- \lambda_1^2 P\int_0^t ds\left[H_{AB}, \exp\left\{-iL_0 s\right\}\left[H_{AB}, \exp\left\{-iL_0(t-s)\right\}\rho'(0)\right]\right]\end{aligned}$$

$$-\lambda_1^2 P\int_0^t ds\Big[H_{AC}, \text{epx}\{-iL_0 s\}\big[H_{AC}, \exp\{-iL_0(t-s)\}\rho'(0)\big]\Big]$$
$$+0(\lambda^3) \tag{5}$$

In deriving eq. (5) it is proved that the cross interaction terms vanish in exactitude in the approximation mentioned above, i. e.,

$$\lambda_1\lambda_2 P\int_0^t ds\Big[H_{AB}, \exp\{-iL_0 s\}\big[H_{AC}, \rho_A(t-s)\big]\Big]$$
$$=\lambda_1\lambda_2 P\int_0^t ds\Big[H_{AC}, \exp\{-iL_0 s\}\big[H_{AB}, \rho_A(t-s)\big]\Big]$$
$$=0$$

and

$$\lambda_1\lambda_2 P\int_0^t ds\Big[H_{AB}, \exp\{-iL_0 s\}\big[H_{AC}, \exp\{-iL_0(t-s)\}\rho'(0)\big]\Big]$$
$$=\lambda_1\lambda_2 P\int_0^t ds\Big[H_{AC}, \exp\{-iL_0 s\}\big[H_{AB}, \exp\{-iL_0(t-s)\}\rho'(0)\big]\Big]$$
$$=0 \tag{6}$$

In case the orders of two coupling constants are nearly equal. i. e. $O(\lambda_1)\approx O(\lambda_2)\equiv O(\lambda)$, eq. (5) becomes markovian in the van Hove limit by taking the time scale $\lambda^2 t=O(1)$, coresponding to the generalized Wangness-Bloch eq. In case of $O(\lambda_1)\gg O(\lambda_2)$ or $O(\lambda_1)\ll O(\lambda_2)$, however, eq. (5) has a non-markovian property. The discussion concerning this point is given in connection with the Scully-Lamb theory.

The extension of this method to the system with n-independent subsystems is straightforward. In this case the master equation reads:

$$\frac{d\rho_A}{dt}=-i\,[H_A, \rho_A]-\sum_{i=1}^{n}\lambda_i^2 P\int_0^t ds\Big[H_{AB_i}, \exp\{-iL_0 s\}\big[H_{AB_i}\,\rho_A(t-s)\big]\Big]$$
$$-\sum_{i=1}^{n}\lambda_i^2 P\int_0^t ds\Big[H_{AB_i}, \exp\{-iL_c s\}\big[H_{AB_i}, \exp\{-iL(t-s)\}\rho'(0)\big]\Big] \tag{7}$$

where i represents an independent subsystem. It is worthy of note that if the system A is a Laser system composed of active atoms and of a lasing mode of radiation field, and if subsystems $B_i\,(i=1, 2, \cdots, n)$ are reservoirs to the Laser system, then the eq. (7) is comparable to the basic equation of Weidlich-Haake's theory.

§ 3. Application to Scully-Lamb's Laser System

The system discussed in § 2 is not appropriate as a Lasser model, since by the assumption of diagonalization of the Hamiltonian H_A, the co-operative phenomena such as phase transitions do not occur in this system. To derive the equation for such a system that exhibits cooperative phenomena, it is further necessary to seperate the term $H_A\to H_A^{(I)}+H_A^{(II)}+\lambda_0 H_A^{(I)-(II)}\,(\lambda_0\gg\lambda_1, \lambda_2)$.

Although the direct treatment of such a case is interesting, we shall now overpass problems related to this case. It is intended in this paper only to apply the present approach to an simplified system of Laser considered by Scully-Lamb. As their theory describes a system coupled to two independent heat reservoirs with different temperatures, the treatment carrieed out in § 2 can be applied directly to a study of a Laser system.

It is well-known that their theory succeeded in deriving the basic properties of the radiation field of a Laser system such as the photon distribution $\rho_{n,n}$ and the spectral profile of the system.

In deriving their results, the following assumptions were made. The first is that the state of the radiation field will not change appreciably during the time in interacting with one atom and that the macroscopic changes of the field will be followed only through the interaction with n atoms, the macroscopic changes being simply obtained by multiplying n to the amount of the change due to one atom interaction. The second is that in calculating the changes representing the gain and the loss of the radiation field, it is possible to cut off the effect of perturbations due to atom-photon interaction in different orders independently.

Hereafter, it is shown that the generalized master equation for the radiation field, which corresponds to the theory of Scully-Lamb, is obtained and that the equation thus obtained is an extension of Scully-Lamb's result to the case when the dynamical effect of atomic system can not be ignored.

To exctend the above discussion, it is further necessary to choose the system variables in § 2 as follows:

H_A is the Hamiltonian of a lasing mode of the radiation field, in which we are interested. H_B and H_C are Hamiltonians of the atomic sub-systems consisting of two-level atoms. While the system C is cold, the system B is so hot that one can apply the concept of negative temperature when the temperature of the system is defined. H_{AB} and H_{AC} are the interaction Hamiltonians with each sub-system whose coupling constants are λ_1 and λ_2 respectively. In order to take the non-linear effect of atomic system, a consideration is made for the case where the orders of two coupling constants are quite different i. e., $O(\lambda_1) \gg O(\lambda_2)$.

Now, we shall consider the statistical properties of our system by applying the approach discussed in § 2 in case of $O(\lambda_1^2) \approx O(\lambda_2)$. Though other cases can be treated in a similar manner, we shall confine ourselves in this paper to the case

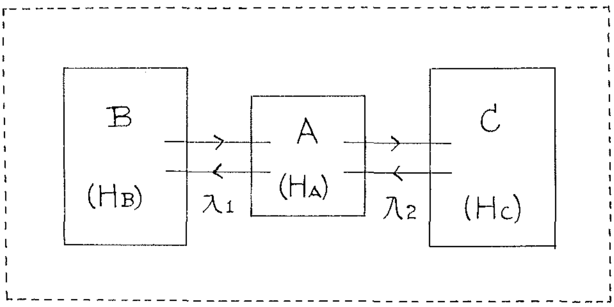

Fig. 1.

mentioned above.

Then, the eq. (5) in § 2 can be reduced to the following two cases:
case 1) $\lambda^2 t = \lambda_1^2 t = O(1)$

Here, eq. (5) reads in a following form by neglecting the terms $O(\lambda_2^2)$:

$$\frac{d\rho_A}{dt} = -i\,[H_A, P_A] - \lambda_1^2 P \int_0^t ds \Big[H_{AB}, \exp\{-iL_0 s\}\big[H_{AB}, \rho_A(t-s)\big]\Big]$$
$$-\lambda_1^2 P \int_0^t ds \Big[H_{AB}, \exp\{-iL_0 s\}\big[H_{AB}, \exp\{-iL_0(t-s)\}\rho'(0)\big]\Big] \tag{8}$$

case 2) $\lambda^2 t = \lambda_2^2 t = O(1)$

The terms of an order $O(\lambda_1^4)$ are retained and the eq. (5) reads as follows:

$$\frac{d\rho_A}{dt} = -i\,[H_A, \rho_A] - \lambda_1^2 P \int_0^t ds \Big[H_{AB}, \exp\{-iL_0 s\}\big[H_{AB}, \rho_A(t-s)\big]\Big]$$
$$-\lambda_2^2 P \int_0^t ds \Big[H_{AC}, \exp\{-iL_0 s\}\big[H_{AC}, \rho_A(t-s)\big]\Big]$$
$$-\lambda_1^2 P \int_0^t ds \Big[H_{AB}, \exp\{-iL_0 s\}\big[H_{AB}, \exp\{-iL_0(t-s)\}\rho'(0)\big]\Big]$$
$$-\lambda_2^2 P \int_0^t ds \Big[H_{AC}, \exp\{-iL_0 s\}\big[H_{AC}, \exp\{-iL_0(t-s)\}\rho'(0)\big]\Big]$$
$$+\lambda_1^4 P \int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \Big[H_{AB}, \exp\{-iL_0 s_2\}(1-P)\big[H_{AB}, \exp\{-iL_0(s_1-s_2)\}$$
$$\times(1-P)\big[H_{AB}, \exp\{-iL_0(s_1-s_2)\}\big[H_{AB}, \rho_A(t-s)\big]\cdots\Big]$$
$$+\lambda_1^4 P \int_0^t ds \int_0^s ds_1 \int_0^{s_1} ds_2 \Big[H_{AB}, \exp\{-iL_0 s_2\}(1-P)\big[H_{AB}, \exp\{-iL_0(s_1-s_2)\}$$
$$\times(1-P)\,H_{AB}, \exp\{-iL(s-s_1)\}\big[H_{AB}, \exp\{-iL_0(t-s)\}\rho'(0)\cdots\big]\Big] \tag{9}$$

In deriving the eq. (9), we used terms of even orders in each coupling constant such as λ_1^3, $\lambda_1\lambda_2^2$, etc, vanish. This master equation (9) corresponds to the one obtained by Scully-Lamb taking the atomic non-linearity up to 3-rd order. (cf. eq. (3. 31) of ref. 5)) However, this equation shows a non-markovian character, even if the van Hove limit, i. e., $\lambda^2 t$=fixed and $\lambda\to 0$, $t\to\infty$, is taken.

The non-markovian character of eq. (9) is not included in Scully-Lamb's results and may play an important role when the atomic coherence cannot be ignored, such as in solid state Laser systems.

§ 4. Discussions

Problems of *phase transitions in open systems* (*or dissipative structures*) have become one of the attractive facets in statistical physics.

It is considered that the phenomena to be described appropriately by the term "phase transition in open systems" span various fields of science. Their typical examples are :

Benard problem in fluid-dynamics, chemical oscillation in autocatalytic chemical reactions, excitation of nervous systems and building up of coherent oscillations in Laser systems.

Many papers[6~7] handling these problems have been presented, from a unified point of view, by the method of statistical thermophysics. Several authors have shown that the macroscopic phenomena generated far from equilibrium may be treated on the basis of the markovian stochastic process. It was not shown in their papers, however, why the dynamical system could be treated within the framework of the markovian process[8~10].

In order to overcome this question, the present author considered it to be important to start from a dynamical point of view. The result obtained in this preliminary report presents an example of the fact that the Fokker-Planck and its generalized approaches based on the markovian assumption seem to be not always appropriate to thes problems.

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