New Formulae of the Generalized Sampling Theorem I

— Formulae making use of the Sampled Zero-th Order Derivatives —

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Abstract

New sampling formulae are presented, based on the generalized sampling theorem by Takizawa and Isigaki. Some examples given in the present paper include the sampling formulae, by means of which one can reconstruct a continuous function from its sampled values and sampled derivatives. The sampling formulae stated here can be effectively applied as interpolation or extrapolation formulae.

Zusammenfassung

Neue Abtastformeln werden in der vorliegenden Arbeit auf der Grundlage des von Takizawa und Isigaki abgeleiteten verallgemeinerten Abtasttheorems angegeben. Die hier behandelten Beispiele enthalten solche Abtastformeln, mit denen man eine kontinuierliche Funktion durch Ihre abgetasteten Werte und abgetasteten Ableitungen wieder konstruieren kann. Die hier beschriebenen Abtastformeln können als Interpolations- oder Extrapolationsformeln erfolgreich angewandt werden.

§ 0. Preliminaries

The generalization of the sampling theorem and the reconstruction of a band-limited function from its sampled values and sampled derivatives were made by Kohlenberg[1], Fogel[2], Jagerman and Fogel[3], Bond and Cahn[4], and Linden and Abramson[5]. The sampling theorem was also generalized by Balakrishnan[6] to the case of a continuous-parameter stochastic process. On the other hand, it was pointed out that the sampling intervals need not be uniformly distributed[7].

In previous papers[8)~12], one of the present authors (T.) proposed a generalized sampling theorem taking the reciprocity relation of integral transforms into account, and gave new sampling formulae as examples. Takizawa and Isigaki[13] also presented another generalization of sampling theorem to reconstruct a continuous function from its sampled values and sampled derivatives. They also suggested some of the new sampling formulae as special cases of their theorem. Recently Takizawa[14] made some comments on their generalized sampling theorem, and introduced the notion of the generalized frequency and generalized band-

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limited spectrum function, in order to discuss the condition on which their theorems were based. Truncation error of the sampling expansion was also estimated by him.

In this paper, the authors present several examples of the generalized sampling theorem, based on the expressions given by Takizawa and Isigaki[13)], and Takizawa[14)]. Expressions in the references 13) and 14) shall be quoted with authors' initials, *e. g.*, (TI–2–1), (T–1–1) etc., respectively.

We shall begin with the consideration of the examples of the generalized sampling theorem (TI–2–1), (TI–2–1′), (T–1–1), (T–1–1′), and (T–1–1″), for the case of small $m_n \geqslant 0$.

§ 1. Examples of Sampling Formulae making use of the Zero-th Order Derivatives of Sampled Function

We shall refer to expressions (TI–2–1), (TI–2–1′), or (T–1–1)∼(T–1–1″), mainly for the case $m_n=0$, which corresponds to the case for the sampling formula making use of the sampled values of an entire function. For $m=0$ in (TI–2–27) or (T–2–5), we obtain the sampling formula (TI–2–23) or (T–2–9), which reduces to the expression suggested by van der Pol[15)], *i. e.*, (TI–2–26) or (T–2–9).

1) If we take an orthogonal polynomial $\phi(z\,;\,s)$ of degree s, in the interval: $a \leqslant z \leqslant b$, with its s zeros z_n $(n=1, 2, 3, \cdots, s)$, *i. e.*, $\phi(z_n\,;\,s)=0$, and

$$\phi(z\,;\,s) \in \left\{\phi(z\,;\,k)\,;\int_a^b W(z)\cdot\phi(z\,;\,m)\cdot\phi(z\,;\,r)\cdot dz=\delta_{m,r}\right\}, \tag{1–1}$$

with the weight function $W(z)$, then expression (T–2–9) approximates $f(z)$ by

$$\overline{f(z)}=\sum_{n=1}^{s} f(z_n)\cdot\frac{\phi(z\,;\,s)}{(z-z_n)\cdot\phi'(z_n\,;\,s)}+K\cdot\phi(z\,;\,s), \tag{1–2}$$

the first term of which is a polynomial of degree $(s-1)$.

Function $\overline{f(z)}$ in (1–2) with $K=0$, coincides with $f(z)$ at the s points z_n $(n==1, 2, 3, \cdots, s)$, and therefore is identically equal to $f(z)$, if $f(z)$ is a polynomial of degree $(s-1)$. In this case, $K=\lim_{z\to\infty} f(z)/\phi(z\,;\,s)$ vanishes. If $f(z)$ is a polynomial of degree s, then the second term of the right-hand side in (1–2) is necessary to express precisely the function $f(z)$, where $K=\lim_{z\to\infty} f(z)/\phi(z\,;\,s)$ does not vanish but remains finite.

Of course, if $f(z)$ is of higher degree, function $\overline{f(z)}$ in (1–2) with $K=0$, does not represent $f(z)$ exactly, but only approximately. And this formula is known as *Lagrange's interpolation formula*.

If $f(z)$ is a polynomial of degree as high as $(2s–1)$, it is also shown[16)] that

$$\int_a^b W(z)\cdot f(z)\cdot dz=\int_a^b W(z)\cdot\overline{f(z)}\cdot dz, \tag{1–3}$$

with $W(z)$ the weight function associated with $\phi(z\,;\,s)$. Because $f(z)-\overline{f(z)}$ is a polynomial of degree at most $(2s–1)$ and certainly has among its zeros those of

$\phi(z\,;\,s)$, since

$$f(z_n)=\overline{f(z_n)}\,,\qquad \text{for}\qquad \phi(z_n\,;\,s)=0\,. \tag{1-4}$$

Therefore, we can write

$$f(z)-\overline{f(z)}=\phi(z\,;\,s)\cdot r(z)\,, \tag{1-5}$$

where $r(z)$ is a polynomial of degree at most $(s-1)$. Then

$$\int_a^b W(z)\cdot f(z)\cdot dz-\int_a^b W(z)\cdot\overline{f(z)}\cdot dz=$$
$$=\int_a^b W(z)\cdot\phi(z\,;\,s)\cdot r(z)\cdot dz=0\,, \tag{1-6}$$

because $\phi(z\,;\,s)$ is orthogonal to every polynomial of lower degree.

We put expression (1–2) into (1–3), and obtain

$$\int_a^b W(z)\cdot f(z)\cdot dz=\sum_{n=1}^{s}\lambda_n\cdot f(z_n)\,, \tag{1-7}$$

with constants:

$$\lambda_n=\int_a^b W(z)\cdot\frac{\phi(z\,;\,s)}{(z-z_n)\cdot\phi'(z_n\,;\,s)}\cdot dz\,,\qquad (n=1,2,3,\cdots,s) \tag{1-8}$$

which are often called the *Christoffel numbers*.

Formula (1–7) states that if $f(z)$ is a polynomial of at most degree $(2s-1)$, then the integral can be evaluated if one knows the value of $f(z)$ at no more than s points. This formula can be used to evaluate integrals approximately when $f(z)$ is not a polynomial of at most degree $(2s-1)$, and is nothing but *Gauss' quadrature formula.*

In some practical approximations of Lagrange's formula, one often takes:

$$g(z)=\prod_{k=1}^{s}(z-\eta_k)\,,\qquad \text{(all the } \eta_k\text{'s are distinct)} \tag{1-9}$$

in (T–2–9), and obtains the sampling formula

$$\overline{f(z)}=\sum_{m=1}^{s}f(\eta_m)\cdot L_m(z)+K\cdot g(z)\,, \tag{1-10}$$

where

$$L_m(z)=\prod_{\substack{k=1\\k\neq m}}^{s}\frac{z-\eta_k}{\eta_m-\eta_k}\,, \tag{1-11}$$

with

$$L_m(\eta_p)=\delta_{m,p}\,.\qquad (1\leqslant p\leqslant s) \tag{1-12}$$

As an example of (1–10), we shall take:

$$f(z)=z^N,\qquad (N=\text{natural number not exceeding } s) \tag{1-13}$$

with $g(z)$ expressed by (1–9), then we obtain:

$$\frac{z^N}{g(z)}=\sum_{m=1}^{s}\frac{\eta_m^N}{(z-\eta_m)\cdot g'(\eta_m)}+K_N\,,\qquad \left(\text{with } g'(\eta_m)=\prod_{\substack{k=1\\k\neq m}}^{s}(\eta_m-\eta_k)\right)\tag{1–14}$$

with $K_N=1$ for $N=s$, and $K_N=0$ for $1\leqslant N\leqslant s-1$, because of condition (III′) in (T–2–2), or (III″) in (T–2–4).

Expression (1–14) with $z=0$, reads as follows:

$$\sum_{m=1}^{s}\frac{\eta_m^N}{g'(\eta_m)}=0\,,\qquad (\text{for } N=0,1,2,\cdots,s-2)\tag{1–15}$$

and

$$\sum_{m=1}^{s}\frac{\eta_m^{s-1}}{g'(\eta_m)}=1\,.\tag{1–16}$$

If expression (1–9) has double zeros, the sampling formula (1–10) is to be modified by means of (TI–2–1) or (T–2–3). The detailed expression shall be given in another paper[28].

2) If we put:

$$g(z)=\cos(\alpha\cdot\cos^{-1}\beta z)\,,\qquad (\alpha\cdot\beta\neq 0)\tag{1–17}$$

in (T–2–9), with constants α and β, we can approximate $f(z)$ by:

$$\overline{f(z)}=\sum_{n}(-1)^n\cdot f(z_n)\cdot\sin\left(\frac{2n+1}{2\alpha}\pi\right)\cdot$$
$$\cdot\frac{\cos(\alpha\cdot\cos^{-1}\beta z)}{\alpha\beta\cdot(z-z_n)}+K\cdot\cos(\alpha\cdot\cos^{-1}\beta z)\,,\tag{1–18}$$

with

$$\cos(\alpha\cdot\cos^{-1}\beta z_n)=0\,,\tag{1–19}$$

i. e.

$$\cos^{-1}\beta z_n=\frac{(2n+1)\pi}{2\alpha}\,.\qquad (n=\text{integers})\tag{1–20}$$

If function $f(z)$ is a polynomial of degree $(s-1)$, we have:

$$f(z)=\frac{1}{s}\cdot\sum_{n=0}^{s-1}(-1)^n\cdot f(z_n)\cdot\sin\left(\frac{2n+1}{2s}\pi\right)\cdot$$
$$\cdot\frac{\cos(s\cdot\cos^{-1}\beta z)}{\beta\cdot(z-z_n)}\,,\qquad (s=\text{positive integer})\tag{1–21}$$

referring to condition (III″) in (T–2–4), where z_n $(n=0,1,2,\cdots,s-1)$ are zeros of $\cos(s\cdot\cos^{-1}\beta z)$, *i. e.*, $z_n=(1/\beta)\cdot\cos((2n+1)\pi/2s)$.

In expression (1–18), if we put $\beta z=\cos\theta$ and $\beta z_n=\cos\theta_n$, we have

$$\bar{f}\left(\frac{\cos\theta}{\beta}\right)=\frac{1}{s}\cdot\sum_{n=0}^{s-1}(-1)^n\cdot f\left(\frac{\cos\theta_n}{\beta}\right)\cdot\sin\theta_n\cdot$$
$$\cdot\frac{\cos(s\theta)}{\cos\theta-\cos\theta_n}+K\cdot\cos(s\theta)\,.\tag{1–22}$$

Accordingly we obtain:

$$\int_0^{\pi} \bar{f}\left(\frac{\cos\theta}{\beta}\right)\cdot d\theta = \frac{1}{s}\cdot\sum_{n=0}^{s-1}(-1)^n\cdot f\left(\frac{\cos\theta_n}{\beta}\right)\cdot\sin\theta_n\cdot$$
$$\cdot\oint_0^{\pi}\frac{\cos(s\theta)}{\cos\theta-\cos\theta_n}\,d\theta + K\cdot\frac{\sin(s\pi)}{s}, \tag{1-23}$$

where the integration $\oint_0^{\pi} d\theta$ is to be understood to take Cauchy's principal values at $\theta=\theta_n=(2n+1)\pi/2s$. Hence we obtain:

$$\int_0^{\pi} \bar{f}\left(\frac{\cos\theta}{\beta}\right)\cdot d\theta = \frac{\pi}{s}\cdot\sum_{n=0}^{s-1}(-1)^n\cdot f\left(\frac{\cos\theta_n}{\beta}\right)\cdot\sin(s\theta_n),$$
$$(s=\text{positive integer}) \tag{1-24}$$

by making use of the integral formula:

$$\oint_0^{\pi}\frac{\cos(s\theta)}{\cos\theta-\cos\theta^*}\,d\theta = \pi\,\frac{\sin(s\theta^*)}{\sin\theta^*}. \qquad (s=\text{non-negative integer})$$

Expression (1–24) corresponds to the approximate quadrature formula given by Multhopp[17] and Kondô[18].

3) If we take:

$$g(z)=\sin(\beta z+\gamma), \qquad (\beta\neq 0) \tag{1-25}$$

in (T–2–9), with constants β and γ, we obtain *Someya's sampling formula*,[19] referring to (T–4–10′):

$$f(z)=\sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi-\gamma}{\beta}\right)\cdot\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}, \tag{1-26}$$

with sampling function $\sin(\beta z+\gamma-n\pi)/(\beta z+\gamma-n\pi)$, provided that $f(z)\cdot\exp[-|\beta|\cdot\cdot|y|]$ vanishes for large $|y|$, with $z=x+iy$. Function $(\sin z)/z$ is often called sinc z.

If we take $f(z)=$constant and $g(z)=\sin(\beta z+\gamma)$ in (T–2–9), then condition (III″) in (T–2–4) is expressed by:

$$\lim_{R\to\infty}\frac{1}{2\pi i}\oint_R\frac{f(\zeta)}{(\zeta-z)\cdot\sin(\beta\zeta+\gamma)}\,d\zeta=0, \tag{1-27}$$

from inequality (T–4–10′). In (1–27) the integration contour is taken to be a circle of radius R with its centre at the origin. Hence, from (1–26) and (1–27) we obtain:

$$1=\sum_{n=-\infty}^{+\infty}\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}. \qquad (\beta\neq 0) \tag{1-28}$$

It is also known that condition (III) in (TI–2–4), (TI–2–25), or (T–1–4), is satisfied, provided that the following inequality[3),14)] holds:

$$\frac{1}{|\beta|}<\frac{1}{W}, \tag{1-29}$$

with W the *maximum frequency* of a *band-limited function* $f(z)$, and that the total variation of the Fourier-Stieltjes spectrum of $f(z)$ is bounded. If we take

$\gamma=0$ in (1–26), then under condition (1–29) we have *Shannon's formula*[20),21)]:

$$f(z)=\sum_{n=-\infty}^{+\infty} f\left(\frac{n\pi}{\beta}\right)\cdot\frac{\sin(\beta z-n\pi)}{\beta z-n\pi}. \qquad (\beta\neq 0) \tag{1–30}$$

From (1–26) and (1–27), we have:

$$\cos(\alpha z)=\sum_{n=-\infty}^{+\infty}\cos\left(\frac{n\pi-\gamma}{\beta}\cdot\alpha\right)\cdot\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}, \qquad (\beta\neq 0,\ |\alpha|<|\beta|) \tag{1–31}$$

and

$$\sin(\alpha z)=\sum_{n=-\infty}^{+\infty}\sin\left(\frac{n\pi-\gamma}{\beta}\cdot\alpha\right)\cdot\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}, \qquad (\beta\neq 0,\ |\alpha|<|\beta|) \tag{1–32}$$

provided that $|\alpha|-|\beta|<0$ (*cf.* (T–4–10′)), *i. e.*

$$\frac{1}{|\beta|}<\frac{1}{|\alpha|}. \tag{1–33}$$

4) If we take $g(z)$ as a product of two entire functions $g_1(z)$ and $g_2(z)$:

$$g(z)=g_1(z)\cdot g_2(z), \tag{1–34}$$

we obtain, from (T–2–9), the following expansion formula for an entire function $f(z)$, under condition (III″) in (T–2–4):

$$f(z)=\sum_n f(z_n)\cdot\frac{g_1(z)\cdot g_2(z)}{(z-z_n)\cdot g_1'(z_n)\cdot g_2(z_n)}+ \\ +\sum_m f(z_m)\cdot\frac{g_1(z)\cdot g_2(z)}{(z-z_m)\cdot g_1(z_m)\cdot g_2'(z_m)}+K\cdot g_1(z)\cdot g_2(z), \tag{1–35}$$

with simple zeros z_n and z_m, respectively of functions $g_1(z)$ and $g_2(z)$. Thus we have $g'(z_n)=g_1'(z_n)\cdot g_2(z_n)$, and $g'(z_m)=g_1(z_m)\cdot g_2'(z_m)$.

As an example of (1–35), we shall take:

$$g(z)=\sin(\beta z+\gamma)\cdot\prod_{k=1}^{s}(z-\eta_k), \quad (\beta\neq 0,\ \text{all the } \eta_k\text{'s are distinct}) \tag{1–36}$$

in (T–2–9). This gives the following sampling formula:

$$f(z)=\sum_{n=-\infty}^{+\infty} f(z_n)\cdot\frac{\sin(\beta z+\gamma-n\pi)}{\beta z+\gamma-n\pi}\cdot\prod_{k=1}^{s}\frac{z-\eta_k}{z_n-\eta_k}+ \\ +\sum_{m=1}^{s} f(\eta_m)\cdot\frac{\sin(\beta z+\gamma)}{\sin(\beta\eta_m+\gamma)}\cdot\prod_{\substack{k=1\\k\neq m}}^{s}\frac{z-\eta_k}{\eta_m-\eta_k}, \qquad (\beta\neq 0) \tag{1–37}$$

with $z_n=(n\pi-\gamma)/\beta$ ($n=$integers), all the η_m's ($m=1, 2, 3, \cdots, s$) and z_n's ($n=$integers) being distinct, *i. e.*, $\eta_m\neq(n\pi-\gamma)/\beta$, provided that $f(z)$ is entire and that $\lim_{z\to\infty} f(z)/g(z)=0$.

5) If we take

$$g(z)=z\cdot\sin(\beta z)-A\cdot\cos(\beta z), \qquad (\beta\cdot A\neq 0) \tag{1–38}$$

or

$$g(z)=z\cdot\cos(\beta z)-B\cdot\sin(\beta z), \qquad (\beta\cdot B\neq 0) \tag{1–39}$$

in (T–2–9), with constants A, B, and β, then we obtain respectively :

$$f(z)=2\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} f(\lambda_n)\cdot\frac{\cos(\beta\lambda_n)}{2\beta\lambda_n+\sin(2\beta\lambda_n)}\cdot\frac{z\cdot\sin(\beta z)-A\cdot\cos(\beta z)}{z-\lambda_n}, \tag{1–40}$$

or

$$f(z)=-2\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} f(\mu_n)\cdot\frac{\sin(\beta\mu_n)}{2\beta\mu_n-\sin(2\beta\mu_n)}\cdot$$
$$\cdot\frac{z\cdot\cos(\beta z)-B\cdot\sin(\beta z)}{z-\mu_n}+f(0)\cdot\frac{z\cdot\cos(\beta z)-B\cdot\sin(\beta z)}{z\cdot(1-\beta\cdot B)}, \tag{1–41}$$

where λ_n and μ_n (n=integers) are roots of the following equations, being arranged in ascending order of magnitude :

$$\lambda_n\cdot\sin(\beta\lambda_n)-A\cdot\cos(\beta\lambda_n)=0\,, \tag{1–42}$$

and

$$\mu_n\cdot\cos(\beta\mu_n)-B\cdot\sin(\beta\mu_n)=0\,. \qquad (\beta\cdot B\neq 1) \tag{1–43}$$

The roots λ_n's and μ_n's are taken to be positive for $n>0$, negative for $n<0$, and we put $\lambda_0=\mu_0=0$. λ_0 is not a zero of $g(z)$ in (1–38) for $A\neq 0$, while $\mu_0=0$ is a simple zero of $g(z)$ in (1–39) for $B\neq 0$.

For an even function $f(z)$ in (1–40), we have[10)~12),22),23)] :

$$f(z)=4\sum_{n=1}^{+\infty}\lambda_n\cdot f(\lambda_n)\cdot\frac{\cos(\beta\lambda_n)}{2\beta\lambda_n+\sin(2\beta\lambda_n)}\cdot\frac{z\cdot\sin(\beta z)-A\cdot\cos(\beta z)}{z^2-\lambda_n^2}, \tag{1–44}$$

with positive roots λ_n $(n=1, 2, 3, \cdots)$ of equation (1–42).

For an odd function $f(z)$ in (1–41), we have[10)~12),22),23)] :

$$f(z)=-4\sum_{n=1}^{+\infty}\mu_n\cdot f(\mu_n)\cdot\frac{\sin(\beta\mu_n)}{2\beta\mu_n-\sin(2\beta\mu_n)}\cdot\frac{z\cdot\cos(\beta z)-B\cdot\sin(\beta z)}{z^2-\mu_n^2}, \tag{1–45}$$

with positive roots μ_n $(n=1, 2, 3, \cdots)$ of equation (1–43).

For the limiting cases : $A\to 0$ in (1–38) and $B\to 0$ in (1–39), we obtain respectively the following formulae from (TI–2–1), (T–1–1), or (T–2–10),

$$f(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{\beta}{n\pi}\cdot f\left(\frac{n\pi}{\beta}\right)\cdot\frac{z\cdot\sin(\beta z-n\pi)}{\beta z-n\pi}+$$
$$+\left\{f(0)+z\cdot f'(0)\right\}\cdot\frac{\sin(\beta z)}{\beta z}, \qquad (\beta\neq 0) \tag{1–46}$$

and

$$f(z)=\sum_{n=-\infty}^{+\infty}(-1)^{n+1}\cdot\frac{2\beta}{(2n+1)\pi}\cdot f\left(\frac{2n+1}{2\beta}\pi\right)\cdot$$
$$\cdot\frac{z\cdot\cos(\beta z)}{\beta z-(2n+1)\pi/2}+f(0)\cdot\cos(\beta z), \qquad (\beta\neq 0) \tag{1–47}$$

which are somewhat different from Shannon's formula (1–30). Let us note that $z\cdot\sin(\beta z)$ has a zero of second order, and $z\cdot\cos(\beta z)$ has a simple zero, respectively at the origin.

Examples of (1–46) and (1–47) read as follows:

$$\sin(\alpha z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{\beta}{n\pi}\cdot\sin\left(\frac{n\pi\alpha}{\beta}\right)\cdot\frac{z\cdot\sin(\beta z-n\pi)}{\beta z-n\pi}+$$
$$+\frac{\alpha}{\beta}\cdot\sin(\beta z),\qquad(\beta\neq 0,\ |\alpha|\leqslant|\beta|)\tag{1–48}$$

$$\cos(\alpha z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{\beta}{n\pi}\cdot\cos\left(\frac{n\pi\alpha}{\beta}\right)\cdot\frac{z\cdot\sin(\beta z-n\pi)}{\beta z-n\pi}+$$
$$+\frac{\sin(\beta z)}{\beta z},\qquad(\beta\neq 0,\ |\alpha|\leqslant|\beta|)\tag{1–49}$$

$$J_\nu(\alpha z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{\beta}{n\pi}\cdot J_\nu\left(\frac{n\pi\alpha}{\beta}\right)\cdot\frac{z\cdot\sin(\beta z-n\pi)}{\beta z-n\pi}+$$
$$+\left\{J_\nu(0)+\frac{\alpha z}{2}\cdot\left(J_{\nu-1}(0)-J_{\nu+1}(0)\right)\right\}\cdot\frac{\sin(\beta z)}{\beta z},\qquad(\beta\neq 0,\ |\alpha|\leqslant|\beta|)\tag{1–50}$$

$$\sin(\alpha z)=\sum_{n=-\infty}^{+\infty}(-1)^{n+1}\cdot\frac{2\beta}{(2n+1)\pi}\cdot\sin\left(\frac{2n+1}{2\beta}\pi\alpha\right)\cdot$$
$$\cdot\frac{z\cdot\cos(\beta z)}{\beta z-(2n+1)\pi/2},\qquad(\beta\neq 0,\ |\alpha|\leqslant|\beta|)\tag{1–51}$$

$$\cos(\alpha z)=\sum_{n=-\infty}^{+\infty}(-1)^{n+1}\cdot\frac{2\beta}{(2n+1)\pi}\cdot\cos\left(\frac{2n+1}{2\beta}\pi\alpha\right)\cdot$$
$$\cdot\frac{z\cdot\cos(\beta z)}{\beta z-(2n+1)\pi/2}+\cos(\beta z),\qquad(\beta\neq 0,\ |\alpha|\leqslant|\beta|)\tag{1–52}$$

$$J_\nu(\alpha z)=\sum_{n=-\infty}^{+\infty}(-1)^{n+1}\cdot\frac{2\beta}{(2n+1)\pi}\cdot J_\nu\left(\frac{2n+1}{2\beta}\pi\alpha\right)\cdot$$
$$\cdot\frac{z\cdot\cos(\beta z)}{\beta z-(2n+1)\pi/2}+J_\nu(0)\cdot\cos(\beta z).\qquad(\beta\neq 0,\ |\alpha|\leqslant|\beta|)\tag{1–53}$$

The expression corresponding to (1–50) for $\nu=0$ and $\alpha=\beta=1$ was given by Wheelon[24].

6) If we take Bessel function of integral order ν for $g(z)$ in (T–1–1):

$$g(z)=J_\nu(\beta z),\qquad(\beta\neq 0)\tag{1–54}$$

with a constant β, we obtain, by means of (T–2–3), (T–2–4), and (T–4–10′),

$$f(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}f(z_n)\cdot\frac{1}{J_{\nu-1}(\beta z_n)}\cdot\frac{J_\nu(\beta z)}{\beta(z-z_n)}+E(z),\tag{1–55}$$

$$E(z)=\sum_{s=0}^{|\nu|-1}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_0^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{J_\nu(\beta z)}{z^{|\nu|}},\tag{1–56}$$

provided that $\lim_{z\to\infty}f(z)/J_\nu(\beta z)=0$. In (1–55) and (1–56), $f_0^{(j)}=f^{(j)}(z_0)=f^{(j)}(0)$,

$H_0^{(s-j)}=H^{(s-j)}(z_0)=H^{(s-j)}(0)$ with $g(z)=J_\nu(\beta z)$, and βz_n are zeros of Bessel function $J_\nu(z)$, *i. e.*

$$J_\nu(\beta z_n)=0\,, \qquad (n=\text{integers}) \tag{1–57}$$

where $z_0=0$ is a zero of $|\nu|$-th order at the origin (for $\nu\neq 0$). In case $\nu=0$, we have no z_0 and function $E(z)$ reduces to a null function.

If either a) $f(z)$ is an even function and ν is an even integer, or b) $f(z)$ is odd, with ν odd, we obtain[10)~12)] from (1–55)~(1–56):

$$f(z)=2\sum_{n=1}^{+\infty} f(z_n)\cdot\frac{\beta z_n}{J_{\nu-1}(\beta z_n)}\cdot\frac{J_\nu(\beta z)}{(\beta z)^2-(\beta z_n)^2}+E(z)\,, \tag{1–58}$$

with positive roots $z_n(n=1,2,3,\cdots)$ of equation (1–57).

When we put $f(z)=J_\mu(\alpha z)$ in (1–58), we obtain

$$J_\mu(\alpha z)=2\sum_{n=1}^{+\infty}\frac{\beta z_n\cdot J_\mu(\alpha z_n)}{J_{\nu-1}(\beta z_n)}\cdot\frac{J_\nu(\beta z)}{(\beta z)^2-(\beta z_n)^2}+E(z)\,.\quad (0\neq|\alpha|<|\beta|) \tag{1–58$'$}$$

It is interesting to take $g(z)=z\cdot J_\nu(\beta z)$ in (T–1–1), instead of (1–54):

$$g(z)=z\cdot J_\nu(\beta z)\,, \qquad (\nu=\text{integer}) \tag{1–54–1}$$

and we have

$$g'(z)=\beta z\cdot J'_\nu(\beta z)+J_\nu(\beta z)\,. \tag{1–54–2}$$

For (1–54–1), under condition $\lim_{z\to\infty} f(z)/\{z\cdot J_\nu(\beta z)\}=0$, the sampling formula reads:

$$f(z)=\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty} f(z_n)\cdot\frac{1}{\beta z_n\cdot J_{\nu-1}(\beta z_n)}\cdot\frac{z\cdot J_\nu(\beta z)}{z-z_n}+F(z)\,, \tag{1–55–1}$$

$$F(z)=\sum_{s=0}^{|\nu|}\sum_{j=0}^{s}\frac{f_0^{(j)}}{j!}\cdot\frac{H_0^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{J_\nu(\beta z)}{z^{|\nu|}}\,, \tag{1–56–1}$$

where $f_0^{(j)}=f^{(j)}(z_0)=f^{(j)}(0)$, $H_0^{(s-j)}=H^{(s-j)}(z_0)=H^{(s-j)}(0)$, with $g(z)=z\cdot J_\nu(\beta z)$, and βz_n are zeros of Bessel function $J_\nu(z)$, *i. e.*

$$J_\nu(\beta z_n)=0\,. \qquad (n=\text{integers}) \tag{1–57–1}$$

For $\nu\neq 0$, $z_0=0$ is a zero of $(|\nu|+1)$-th order at the origin. In case $\nu=0$, we have a simple zero $z_0=0$ at the origin, and $F(z)$ reduces to $f(0)\cdot J_0(\beta z)/J_0(0)=f(0)\cdot J_0(\beta z)$.

If either a) $f(z)$ is an even function and ν is an even integer, or b) $f(z)$ is odd, with ν odd, we obtain from (1–55–1),

$$f(z)=2\sum_{n=1}^{+\infty} f(z_n)\cdot\frac{1}{\beta z_n\cdot J_{\nu-1}(\beta z_n)}\cdot\frac{z^2\cdot J_\nu(\beta z)}{z^2-z_n^2}+F(z)\,. \tag{1–58–1}$$

If either a) $f(z)$ is even, with ν odd, or b) $f(z)$ is odd, with ν even, we have from (1–55–1),

$$f(z)=2\sum_{n=1}^{+\infty} f(z_n)\cdot\frac{1}{\beta\cdot J_{\nu-1}(\beta z_n)}\cdot\frac{z\cdot J_\nu(\beta z)}{z^2-z_n^2}+F(z)\,. \tag{1–58–1$'$}$$

In case $\mu=\nu=0$, expression (1–58′) reduces to the following one[27], if we put $z=0$:

$$1=2\sum_{n=1}^{+\infty}\frac{J_0(\alpha z_n)}{\beta z_n\cdot J_1(\beta z_n)}\,. \qquad (0\neq|\alpha|<|\beta|) \tag{1–58″}$$

While, in case $f(z)=\text{const}$ and $\nu=0$, (1–58–1) reads:

$$1=-2\sum_{n=1}^{+\infty}\frac{1}{\beta z_n\cdot J_1(\beta z_n)}\cdot\frac{z^2\cdot J_0(\beta z)}{z^2-z_n^2}+J_0(\beta z)\,. \tag{1–58–1″}$$

7) More generally, we shall take[25]:

$$g(z)=z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)\,, \qquad (\beta\cdot h\neq 0) \tag{1–59}$$

in (T–1–1), with an integer ν and constants β and h. The right-hand side of (1–59) reduces to $g'(z)/\beta$ in (1–54–2), when we put $h=1/\beta$. For (1–59) we obtain, referring to (T–1–9), (T–2–4), and (T–4–10′),

$$f(z)=-\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}f(\lambda_n)\cdot\frac{\beta\lambda_n}{(h^2+\lambda_n^2)\,\beta^2-\nu^2}\cdot$$

$$\cdot\frac{1}{J_\nu(\beta\lambda_n)}\cdot\frac{z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)}{z-\lambda_n}+G(z)\,, \tag{1–60}$$

$$G(z)=\sum_{s=0}^{|\nu|-1}\sum_{j=0}^{s}\frac{f_n^{(j)}}{j!}\cdot\frac{H_n^{(s-j)}}{(s-j)!}\cdot z^s\cdot\frac{z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)}{z^{|\nu|}}\,, \tag{1–61}$$

provided that $\lim_{z\to\infty} f(z)/\{z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)\}=0$. In (1–60), the values λ_n (n=integers) are roots of the following equation:

$$\lambda_n\cdot J'_\nu(\beta\lambda_n)+h\cdot J_\nu(\beta\lambda_n)=0\,, \tag{1–62}$$

with $\lambda_0=0$ a zero of $|\nu|$-th order at the origin (for $\nu\neq 0$). In case $\nu=0$, $\lambda_0=0$ is not a zero at the origin, and $G(z)$ reduces to a null function.

If either a) $f(z)$ is even, with ν an even integer, or b) $f(z)$ is odd, with ν odd, then we obtain [10)~12),22),23] from (1–60),

$$f(z)=-2\sum_{n=1}^{+\infty}f(\lambda_n)\cdot\frac{\beta\lambda_n^2}{(h^2+\lambda_n^2)\,\beta^2-\nu^2}\cdot$$

$$\cdot\frac{1}{J_\nu(\beta\lambda_n)}\cdot\frac{z\cdot J'_\nu(\beta z)+h\cdot J_\nu(\beta z)}{z^2-\lambda_n^2}+G(z)\,. \tag{1–63}$$

If we tend h to infinity in expressions (1–59)～(1–62), we have the expressions which are essentially the same as (1–54)～(1–57).

The formulae corresponding to (1–44), (1–45), and (1–63), were also given by Kroll[22] and Isomiti[23], based on the integral transforms[10)~12)] of the sampled function.

8) Further, let us take[25] a linear combination $T_\mu(x, z)$ of Bessel function $J_\mu(z)$ and Neumann function $Y_\mu(z)$ of order μ:

$$T_\mu(x, z)=Y_\nu(x)\cdot J_\mu(z)-J_\nu(x)\cdot Y_\mu(z)\,. \tag{1–64}$$

We shall put:

$$g(z) = T_\nu(\alpha z, \beta z), \qquad (\nu = \text{integers}, \ \alpha \neq \beta, \ \alpha\beta \neq 0) \tag{1-65}$$

in (T-2-9), with constants α and β. Then we obtain[10)~12)], from (TI-2-4) or (T-1-4) with (T-4-10′),

$$f(z) = -\frac{\pi}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n \cdot J_\nu(\alpha\lambda_n) \cdot J_\nu(\beta\lambda_n)}{J_\nu^2(\alpha\lambda_n) - J_\nu^2(\beta\lambda_n)} \cdot \frac{T_\nu(\alpha z, \beta z)}{z - \lambda_n}, \tag{1-66}$$

provided that $\lim_{z\to\infty} f(z)/T_\nu(\alpha z, \beta z) = 0$. The values λ_n (n = integers) are roots of the following equation:

$$T_\nu(\alpha\lambda_n, \beta\lambda_n) = 0, \tag{1-67}$$

i. e.

$$Y_\nu(\alpha\lambda_n) \cdot J_\nu(\beta\lambda_n) = J_\nu(\alpha\lambda_n) \cdot Y_\nu(\beta\lambda_n), \tag{1-68}$$

being arranged in ascending order of magnitude for increasing n, where $\lambda_0 = 0$ is not a zero of $T_\nu(\alpha z, \beta z)$. We referred to expression (1-68) and the Wronskian of cylinder functions $J_\nu(z)$ and $Y_\nu(z)$ in order to obtain the sampling formula (1-66).

Function $T_\nu(\alpha z, \beta z)$ (ν = integer) is an even function, because of $J_\nu(-z) = (-1)^\nu \cdot J_\nu(z)$ and $Y_\nu(-z) = (-1)^\nu \cdot Y_\nu(z)$ for an integer ν. So, if $f(z)$ is an even function, expression (1-66) reduces to[10)~12)]:

$$f(z) = -\pi \cdot \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n^2 \cdot J_\nu(\alpha\lambda_n) \cdot J_\nu(\beta\lambda_n)}{J_\nu^2(\alpha\lambda_n) - J_\nu^2(\beta\lambda_n)} \cdot \frac{T_\nu(\alpha z, \beta z)}{z^2 - \lambda_n^2}, \tag{1-69}$$

with positive roots λ_n ($n = 1, 2, 3, \cdots$) of equation (1-67).
If function $f(z)$ is odd, expression (1-66) reads:

$$f(z) = -\pi \cdot \sum_{n=1}^{+\infty} f(\lambda_n) \cdot \frac{\lambda_n \cdot J_\nu(\alpha\lambda_n) \cdot J_\nu(\beta\lambda_n)}{J_\nu^2(\alpha\lambda_n) - J_\nu^2(\beta\lambda_n)} \cdot \frac{z \cdot T_\nu(\alpha z, \beta z)}{z^2 - \lambda_n^2}. \tag{1-70}$$

If we replace $f(z)$ by $f(z) \cdot B_\nu(z)$ in (1-66), we obtain a sampling formula for $f(z) \cdot B_\nu(z)$, which reads:

$$f(z) \cdot B_\nu(z) = -\frac{\pi}{2} \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} f(\lambda_n) \cdot B_\nu(\lambda_n) \cdot \cdot \frac{\lambda_n \cdot J_\nu(\alpha\lambda_n) \cdot J_\nu(\beta\lambda_n)}{J_\nu^2(\alpha\lambda_n) - J_\nu^2(\beta\lambda_n)} \cdot \frac{T_\nu(\alpha z, \beta z)}{z - \lambda_n}, \tag{1-71}$$

with a given function $B_\nu(z)$, under condition:

$$\lim_{z\to\infty} f(z) \cdot B_\nu(z) / T_\nu(\alpha z, \beta z) = 0, \tag{1-72}$$

which corresponds to condition (III) in (TI-2-4) or (T-1-4).

If we take

$$B_\nu(z) = J_\nu^2(z) + Y_\nu^2(z), \tag{1-73}$$

in (1-71), we obtain a sampling formula previously given by one of the authors[10)~12)].

9) If we take

$$g(z)=\sin(\beta z^2+\gamma), \qquad (\beta\cdot\gamma\neq 0) \tag{1-74}$$

in (T-2-9), we obtain:

$$f(z)=\sum_n f(z_n)\cdot\frac{\sin(\beta z^2+\gamma-n\pi)}{2\beta z_n\cdot(z-z_n)}, \tag{1-75}$$

where $z_n=\pm\sqrt{(n\pi-\gamma)/\beta}$ (n=integers), and the summation over n covers all the values of z_n, *i.e.*, positive and negative square roots. In case $\gamma=0$, $\sin(\beta z^2+\gamma)$ has a double zero at the origin, and expression (1-75) needs modification.

10) If we take $g(z)$ to be a product of two entire functions, such as $\sin(\beta z+ +\gamma)\cdot J_\nu(pz)$, (a Lagrangean polynomial)$\times J_\nu(pz)$, etc., we obtain a sampling formula from (T-2-9) by similar calculations as in (1-35).

Truncation error of the sampling expansion (T-2-9) can be easily estimated. The bound for truncation error[14),26)] of (T-2-9) in case $m_n=0$ was already given by (T-2-13) or (T-6-7).

Concluding Remarks

In this paper were given several new examples of sampling formulae, based on the generalized sampling theorem (TI-2-1), (T-1-1), or (T-2-3). Formulae which make use of the sampled zero-th order derivatives (*i.e.*, the values of the sampled function itself), were mainly treated. Our formulae coincide in many cases with those derived from formula (T-2-9) with $K=0$, as was suggested by van der Pol[15)]. While, it seems to the authors that formulae such as (1-10), (1-14), (1-18), (1-24), (1-35), (1-37), (1-46)~(1-55), (1-58), (1-60), (1-63) etc., were newly presented in this paper. Examples of the generalized sampling formulae which make use of the sampled higher order derivatives, will follow in another paper[28)].

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