Dynamical Singularities of Two-Layered Flow at the Outlet of an Open Channel

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Abstract

Dynamical singularities are at times experienced in a two-layered flow, particularly at the outlet of an open channel. A river mouth is a typical example, where the fresh water accelerates its velocity and decreases its depth, and as a result, the interfacial Frounde number must be unity. Equations of motion and continuity describing the two-layeres flow are transformed into a certain partial differential equation, under the assumption of a lighter fluid irrotational, without friction and mixing. The interfacial Froude number is a dominant parameter over the entire motion of the fresh water. When this number is replaced by the Mach number, the equation changes into a form exactly the same as the transonic flow in aerodynamics. In other words, there is a similarity between the two-layered flow and the transonic flow. From this viewpoint, various features are reconsidered concerning a two-layered flow. It brings a better understanding upon the singularities shown by the two-layered flow at the outlet.

1. Introduction

This paper describes singular properties, which are found in a two-layered flow in the neighborhood of the outlet of an open channel, when a lighter fluid flows out of a channel onto a stagnant heavier fluid which occupies a broad outer area. The singularities are explainable from a theory which is quite similar to that of a transonic flow in aerodynamics.

For instance, if we take the fresh water, it flows out from a river mouth, with gradual mixing with the sea water below, while the sea water intrudes inwards along the river bed and forms a "salt wedge." In this case, it has been understood so far that the interfacial Froude number F_i, which is defined by that $F_i = q/\sqrt{\varepsilon g h}$, must be unity at the river mouth, where q and h denote the velocity and the depth of the fresh water, respectively, g the gravitational acceleration and ε the density-difference parameter $(\rho_2 - \rho_1)/\rho_2$, in which ρ_1 and ρ_2 are densities of the fresh water and the sea water.

There are some theories concerning the reason why F_i takes unity at the river mouth, and also some experimental and field data as proof thereof. However, with regard to a horizontal distribution of numerical values of F_i around the river mouth, its importance has not been recognized as yet, because of the insufficiency and the difficulty of theory and experiment.

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The present author has come to recognize that the dynamics of the two-layered flow is essentially the same as that of the transonic flow, and from this viewpoint, he has attempted to reconsider this problem.

2. Fundamental equations

To describe a vivid two-layered flow, equations must include the effects of eddy viscosity, entrainment, interfacial resistance, etc., but only an extremely simplified set of equations is employed here, to treat the behavior of the outflow which is only transient.

Assumptions that the sea water is stagnant, the fresh water immiscible, and the vertical component of velocity negligible, gives rise to the following equations.

$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}=-\frac{1}{\rho_1}\frac{\partial p_1}{\partial x} \tag{1}$$

$$\frac{\partial v_1}{\partial t}+u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y}=-\frac{1}{\rho_1}\frac{\partial p_1}{\partial y} \tag{2}$$

$$0=-g-\frac{1}{\rho_1}\frac{\partial p_1}{\partial z} \tag{3}$$

$$0=-\frac{1}{\rho_2}\frac{\partial p_2}{\partial x} \tag{4}$$

$$0=-\frac{1}{\rho_2}\frac{\partial p_2}{\partial y} \tag{5}$$

$$0=-g-\frac{1}{\rho_2}\frac{\partial p_2}{\partial z} \tag{6}$$

$$\frac{\partial h_1}{\partial t}+\frac{\partial}{\partial x}(h_1u_1)+\frac{\partial}{\partial y}(h_1v_1)=0 \tag{7}$$

Subscripts 1 and 2 represent the fresh water and the sea water. Eqs. (1)-(3) belong to the fresh water, Eqs. (4)-(6) the salt water, and Eq. (7) is the equation of continuity of the fresh water. Symbols u, v, p and h are velocity components in the x and y axes, pressure and depth, respectively. Those are illustrated in Fig. 1.

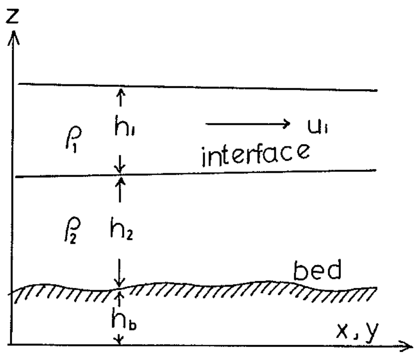

Fig. 1. Diagram of two-layered flow.

If the atmospheric pressure is given as p_0, the integration of Eq. (3) leads to

$$p_1 = p_0 + \rho_1 g(h_b + h_2 + h_1 - z) \tag{8}$$

From this, the pressure at the interface is

$$p_{\rm int} = p_0 + \rho_1 g h_1 \tag{9}$$

Integrating Eq. (6) vertically, and putting $p_2 = p_{int}$ at the interface, we obtain

$$p_2 = p_0 + \rho_1 g h_1 + \rho_2 g(h_b + h_2 - z) \tag{10}$$

Substituting Eq. (10) into Eqs. (4) and (5), we get

$$\frac{\partial p_2}{\partial x} = \rho_1 g \frac{\partial h_1}{\partial x} + \rho_2 g \frac{\partial}{\partial x}(h_b + h_2) = 0 \tag{11}$$

$$\frac{\partial p_2}{\partial y} = \rho_1 g \frac{\partial h_1}{\partial y} + \rho_2 g \frac{\partial}{\partial y}(h_b + h_2) = 0 \tag{12}$$

Thus, Eqs. (1) and (2) are transformable, with Eqs. (8), (11) and (12), into the following forms.

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} = -\frac{\rho_2 - \rho_1}{\rho_2} g \frac{\partial h_1}{\partial x} \tag{13}$$

$$\frac{\partial v_1}{\partial t} + u_1 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_1}{\partial y} = -\frac{\rho_2 - \rho_1}{\rho_2} g \frac{\partial h_1}{\partial y} \tag{14}$$

Those are final forms which govern the motion of the fresh water. If the flow is assumed to be steady, they are arranged, with Eq. (7), omitting the subscript 1, into the following.

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \varepsilon g \frac{\partial h}{\partial x} = 0 \tag{15}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \varepsilon g \frac{\partial h}{\partial y} = 0 \tag{16}$$

$$\frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0 \tag{17}$$

Now, assuming that the flow is irrotational, the velocity potential ϕ exists, and the following relationships are obtainable.

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y} \tag{18}$$

Transformation of Eq. (17) leads to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{h}\left(u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y}\right) = 0 \tag{19}$$

On the other hand, dividing Eqs. (15) and (16) by $\varepsilon g h$, we obtain

$$\frac{1}{\varepsilon g h}\left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}\right) + \frac{1}{h}\frac{\partial h}{\partial x} = 0 \tag{20}$$

$$\frac{1}{\varepsilon g h}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)+\frac{1}{h}\frac{\partial h}{\partial y}=0 \tag{21}$$

Multiplying Eq. (20) by u, and Eq. (21) by v, and adding both, while taking Eq. (19) into account, we get

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{1}{\varepsilon g h}\left\{u\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+v\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)\right\} \tag{22}$$

In addition, using the relationships in Eq. (18), this is transformable into the form of

$$\left(1-\frac{u^2}{\varepsilon g h}\right)\frac{\partial^2\phi}{\partial x^2}-2\frac{uv}{\varepsilon g h}\frac{\partial^2\phi}{\partial x\partial y}+\left(1-\frac{v^2}{\varepsilon g h}\right)\frac{\partial^2\phi}{\partial y^2}=0 \tag{23}$$

Eq. (23) is of the highest importance for a discussion of the flow characteristics. If the term $\varepsilon g h$ is replaced by the square of the sound velocity a, the equation coincides exactly with the two-dimensional equation of a compressible fluid flow. In other words, various features already discovered in the transonic flow can be expected also in the field of the two-layered flow. It may be easily understood that the interfacial Froude number F_i plays a dominant role on the entire flow field, in place of the Mach number M which is equal to q/a.

The discriminant D can be derived from Eq. (23), as follows.

$$D=\left(\frac{uv}{\varepsilon g h}\right)^2-\left(1-\frac{u^2}{\varepsilon g h}\right)\left(1-\frac{v^2}{\varepsilon g h}\right)=\frac{q^2}{\varepsilon g h}-1=F_i^2-1 \tag{24}$$

This implies that the mathematical situation of Eq. (23) is divided into the following three categories.

$D<0$ for $F_i<1$. . . . elliptic, corresponding to a subsonic flow

$D=0$ for $F_i=1$. . . . parabolic, sonic flow

$D>0$ for $F_i>1$. . . . hyperbolic, supersonic flow

The above conditions suggest that the flow of the fresh water gradually changes its property downwards, from an elliptic type, through parabolic at the outlet or the river mouth, and finally into the hyperbolic outside.

3. Discussions on singularities appearing in the flow at the outlet

It has been believed that the interfacial Froude number F_i must be unity at the outlet or the river mouth, if a heavier fluid or the sea water intrudes into a channel or a river.[1),2),3)] This can be understood as a kind of hydraulic jump. Namely, the lighter fluid decreases its depth gradually outwards with some transient acceleration of its velocity at the outlet, where the stagnant internal jump is formed, at the same time the numerical value of F_i becomes unity somewhere near the outlet.

The condition that $F_i=1$ plays an important part on dynamics of the salt wedge has long been accepted. Studies on the salt wedge are abundant and numerous data are available today, but in contrast, regarding detailed behavior of fresh water shows little or no advancement at or outside the river mouth, in spite of the fact of its importance in environmental problems.

The present author has previously published a paper[4], in which field data at the Ishikaria River show that F_i increases along the stream, from a numerical value smaller than unity inside the riven mouth, through unity at the mouth, finally to a considerably larger value than unity outside. These were measured along the center line of the fresh-water flow. However, it was difficult to determine the exact place where the value was unity. For example, numerical values of F_i were scattered even on the same cross-section, particularly in the vicinity of the mouth.

It is well known that the surface flow is fast around both corners at the river mouth, where F_i exceeds unity earlier than at any other place. This kind of acceleration in velocity seems to arise from the potential flow. The author has pointed out that there is another place of acceleration along the centerline at a short distance outward from the mouth[5]. It cannot be understood from a common potential flow in a homogeneous fluid, but it is peculiar to the density current, namely, the two-layered flow. This singular zone may be a place where many researchers consider that the interfacial Froude number F_i is unity. And beyond this zone, numerical values of F_i grow further towards the open sea.

Another important matter must be added, namely flow pattern. There are several kinds of flow patterns off the outlet or the river mouth, from one extreme type in which the flow spreads out in radial directions when the discharge is small, to the other extreme type which is a turbulent jet whose boundary extends initially parabolically and finally straight towards the open sea. There are some transitional patterns between those two extreme types[5]. Corresponding to a change of the flow pattern, the acceleration zone changes its position and area.

Recently, the thermal discharge from nuclear or thermoelectric power plants has been attracting public attention as one of environmental problems. This kind of discharge has, in general, a high velocity at the outlet and attains a high value of the interfacial Froude number, which gives rise to an intense mixing with the sorrounding sea water and a rapid decrease in temperature. According to the latest information, discharge of high velocity from deep layers seems more effective. A more rapid decay of temperature through a process in which the thermal plume buoys up to the sea surface may be expected. This requires a very high value of F_i at the outlet, more than 10. This kind of the flow is a forced discharge, and it differs from the outflow of a natural river. In a case where the discharge is released horizontally from the outlet situated at sea level, the motion follows Eq. (23) as well as the river discharge, and then the equation is hyperbolic as in the case of $F_i \gg 1$, and it corresponds to the supersonic flow.

On the other hand, as stated previously, a natural river gives a mixed flow, in which F_i varies from $F_i < 1$ on the inside, to $F_i > 1$ on the outside. This corresponds to the transonic flow, which can be seen around a slender body which is placed in a subsonic flow of a comparatively high speed. The interfacial Froude number at the upstream end of the salt wedge, F_{i0}, is a dominant factor, with which the flow pattern and the acceleration zone seem to change their figure and surface area. F_{i0} corresponds to M_∞, which is the Mach number giving a ratio of the speed of the body to the sonic speed at a distance sufficiently apart from the body.

In a flood season, the flow type sometimes becomes hyperbolic over the entire area, whether inside or outside, since the river discharge exceeds a critical value, beyond which the salt wedge cannot intrude into the river. This type is the same as that of horizontal thermal discharge as described before, and it is equivalent to the air flow which is supersonic everywhere over the entire field.

No attention has been paid as yet to what value the interfacial Froude number must attain at the outlet, when the discharge is extremely small. It is doubtful to consider that $F_i=1$ at the outlet even in this case, because Eq. (23) approaches $\nabla^2\phi=0$, namely it is elliptic everwhere, if the following conditions $u^2/\varepsilon gh \ll 1$, and $v^2/\varepsilon gh \ll 1$ are established. One of the two extreme patterns, previously mentioned, in which the flow expands horizontally in all directions under the condition of a very small discharge, may perhaps be caused by such circumstances. The flow pattern seems to approach the two-dimensional ideal fluid flow. In this case, the depth of the lighter fluid h, must be nearly constant over the entire area.

Numerous research work has been made to solve Eq. (23), in which $\sqrt{\varepsilon gh}$ is replaced by the sound velocity a, in aerodynamics. In almost all cases, flying bodies were assumed to be slender, namely, stream-lined in shape. Therefore, a few approximation methods were applied to fit a slender body. They are, numerical computations, series expansions, difference methods with relaxation technique, etc.[6] To obtain an exact solution, there is a hodograph method. But this method has some difficulties in taking boundary conditions into account. The present author has already derived a hodograph equation concerning the two-layered flow, from a set of fundamental equations of motion and of continuity[4]. However, this is incomplete yet because of diffculties to treat with the boundary conditions, particularly, boundary shapes with rectangular corners at the outlet which are not slender.

As the present paper has revealed the two-layered flow at the outlet is dynamically in the same category with the transonic flow in aerodynamics, hereafter, the study must be directed first, towards, investigation of previous work in aerodynamics, in more detail, particularly concerning a body, not slender but bluff, and next towards further developments.

4. Conclusion

This paper has revealed that the two-layered flow, of a lighter fluid superposing on a heavier fluid, which is observable at the outlet of an open channel, has the same characteristics as that of a transonic flow in aerodynamics. This means that there is a dynamically singular point between the inside and the outside of the outlet. Such examples are found at a river mouth with both the fresh water and the sea water superposed, and also in the case of a thermal discharge released horizontally from thermal power plants.

The problem has some difficulties which arise from the mixed type of the partial differential equation, from elliptic to hyperbolic. Thus, further studies must overcome such difficulties and attempts should be made to obtain a firm knowledge on the horizontal distribution of the interfacial Froude number around the outlet.

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