Note on Helmholtz Instability of Vortex Sheet

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Abstract

The effect of a background vorticity on the stability of a vortex sheet which lies near a plane wall is analytically studied. The background vorticity can make the vortex sheet stable to small disturbances in a particular range of wavenumbers. The background vorticity increases the growth rate of the deformed vortex sheet in which the vorticity has the same sign as the background vorticity. This tendency is consistent with experimental observations.

1. Introduction

It is well established that a vortex sheet is unstable with respect to small disturbances of all the wavenumbers, i. e. the Helmholtz instability [1]. However, the presence of background vorticity can make the vortex sheet stable within a particular range of wavenumbers as will be shown in the present note. A theoretical sanalysi of the rolling up of a vortex sheet in a shear flow was originally considered by Hama and Burke [2] in connection with the flow patterns associated with the laminar boundary-layer transition. They assumed a shear flow of constant vorticity, i. e. a uniform shear flow in which the vortex sheet was embedded. It is a matter of great interest whether or not such a vortex sheet is unstable to small disturbances in a similar manner to conventional Helmholtz instability without the background vorticity. Hama and Burke [2] omitted the analysis of stability of the vortex sheet by stating that the interaction of the background vorticity with the vortex sheet was of a nonlinear nature. If an inviscid fluid is assumed, however, no diffiusion of vorticity occurs and thus Kelvin's law of conservation of vorticity assures that the stability of the vortex sheet in the uniform shear flow can be analyzed by the well-established method of small disturbances. Such an analysis is the subject of the present note.

2. Analysis and Discussion of Results

The vortex sheet which lies in the vicinity of a plane wall will be considered. The distance between the initial position of the vortex sheet and the plane wall is written as h, as shown in Fig. 1. The intial position of the vortex sheet coincides with the x axis and the y axis is taken as normal to the x axis. The vortex sheet divides the flow field into two regions, i. e. $y>0$ and $-h<y<0$ which will be

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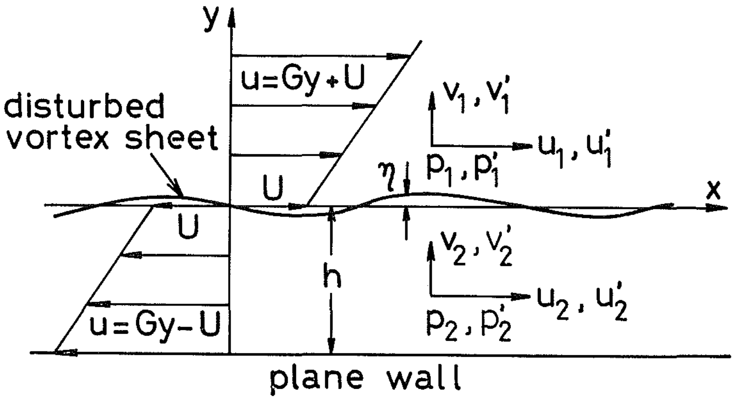

Fig. 1. Definition sketch and coordinate system.

designated as regions 1 and 2 respectively. The corresponding velocity components and pressure in region $j\,(=1$ and $2)$ are denoted by u_j, v_j and p_j. In the absence of disturbances the flow in the region j is specified by the velocity in the x direction $Gy+U_j$, in which $-G$ is the background vorticity and $U_j=(-1)^{j+1}U$, U being constant, and by a constant pressure p_∞. Since one considers the instability of the vortex sheet surrounded by the background vorticity of the original boundary layer along the wall based on the assumption of an inviscid fluid, non-zero slip velocity at the wall will be justified, at least during the time interval relevant to small disturbances of the vortex sheet.

On writing the disturbance velocity components and pressure associated with distortion of the vortex sheet as u'_j, v'_j and p'_j, one has

$$u_j=Gy+U_j+u'_j\,,\qquad v_j=v'_j\,,\qquad p_j=p_\infty+p'_j\,. \tag{1}$$

Substitution of Eq. (1) into Euler's equations of motion together with the equation of continuity and omission of quadratic terms in the disturbance velocity components yield three simultaneous linear partial differential equations of the first order from which the three quantities u'_j, v'_j and p'_j are to be determined:

$$\frac{\partial u'_j}{\partial t}+(Gy+U_j)\frac{\partial u'_j}{\partial x}+\frac{1}{\rho}\frac{\partial p'_j}{\partial x}=0\,, \tag{2 a}$$

$$\frac{\partial v'_j}{\partial t}+(Gy+U_j)\frac{\partial v'_j}{\partial x}+\frac{1}{\rho}\frac{\partial p'_j}{\partial y}=0\,, \tag{2 b}$$

$$\frac{\partial u'_j}{\partial x}+\frac{\partial v'_j}{\partial y}=0\,, \tag{2 c}$$

where ρ is the density of fluid.

Provided that the Fourier transform or Fourier-series expansion of the disturbances exists, the subsequent linearized history of the system following an arbitrary disturbance can be determined from the response to a disturbance of a particular wavenumber α. Accordingly, one defines u'_j, v'_j and p'_j as the real parts of the complex quantities $\hat{u}_j$, $\hat{v}_j$ and $\hat{p}_j$ of the from

$$(\hat{u}_j, \hat{v}_j, \hat{p}_j) = \left\{\tilde{u}_j(y), \tilde{v}_j(y), \tilde{p}_j(y)\right\} \exp\left\{i\alpha(x-ct)\right\}, \tag{3}$$

where $\tilde{u}_j$, $\tilde{v}_j$ and $\tilde{p}_j$ are functions of y alone, c is a complex phase velocity, t is time and $i=(-1)^{1/2}$. The imaginary part of c, say c_i, determines the degree of amplification or damping of the disturbance according to $c_i>0$ or $c_i<0$. Substitution of Eq. (3) into Eqs. (2) yields three ordinary differential equations for $\tilde{u}_j$, $\tilde{v}_j$ and $\tilde{p}_j$:

$$i\alpha(Gy+U_j-c)\,\tilde{u}_j+G\tilde{v}_j+i\alpha\rho^{-1}\tilde{p}_j=0\,, \tag{4 a}$$

$$i\alpha(Gy+U_j-c)\,\tilde{v}_j+\rho^{-1}(d\tilde{p}_j/dy)=0\,, \tag{4 b}$$

$$i\alpha\tilde{u}_j+(d\tilde{v}_j/dy)=0\,, \tag{4 c}$$

whose solutions are easily found to be

$$\tilde{u}_j=i(A_je^{\alpha y}-B_je^{-\alpha y})\,, \tag{5 a}$$

$$\tilde{v}_j=A_je^{\alpha y}+B_je^{-\alpha y}\,, \tag{5 b}$$

$$\begin{aligned}\tilde{p}_j/\rho = &-iA_je^{\alpha y}(Gy+U_j-c-\alpha^{-1}G)\\ &+iB_je^{-\alpha y}(Gy+U_j-c+\alpha^{-1}G)\,,\end{aligned} \tag{5 c}$$

where A_j and B_j are constants.

With the restriction that the disturbance motion vanishes at a position far from the vortex sheet, one has $A_1=0$. The boundary condition that the normal velocity component should be zero along the plane wall yields a relation between A_2 and B_2, namely $B_2=-A_2\exp(-2\alpha h)$. Moreover, since the pressure must be continuous across the vortex sheet, one obtains the relation

$$B_1/A_2=2e^{-\alpha h}\left\{(U+c)\cosh\alpha h+\alpha^{-1}G\sinh\alpha h\right\}\Big/(U-c+\alpha^{-1}G)\,. \tag{6}$$

The vertical displacement η of each element of the vortex sheet must satisfy the following kinematical equation because the vortex sheet is a material surface:

$$\frac{\partial\eta}{\partial t}+U_j\frac{\partial\eta}{\partial x}=(v_j)_{y=0}\,. \tag{7}$$

One assumes that η is the real part of the complex quantity $\hat{\eta}$ of the form

$$\hat{\eta}=\tilde{\eta}(y)\exp\left\{i\alpha(x-ct)\right\}. \tag{8}$$

Substitution of Eqs. (3) and (8), together with Eq. (5 b), into Eq. (7) yields

$$B_1/A_2=-\left\{(U-c)/(U+c)\right\}2e^{-\alpha h}\sinh\alpha h\,. \tag{9}$$

Equating the right-hand sides of Eqs. (6) and (9), one obtains a quadratic equation for the complex phase velocity c:

$$c^2(1+\coth\alpha h)-2cU(1-\coth\alpha h)+U^2(1+\coth\alpha h)+2U\alpha^{-1}G=0\,. \tag{10}$$

The discriminant D of this equation is

$$D = -2U^2 \coth \alpha h \left\{ 2 + (\alpha U)^{-1} G (1 + \tanh \alpha h) \right\} . \tag{11}$$

Since $\alpha > 0$ and U can be taken as positive without any loss of generality, the vortex sheet is unstable for all the wave numbers when $G > 0$, the nondimensional growth rate of the disturbance of a particular wavenumber α being

$$\alpha c_i h / U = \alpha h (-D/U^2)^{1/2} / (1 + \coth \alpha h) . \tag{12}$$

It is clear from Eqs. (11) and (12) that the growth rate of the disturbance increases with the increase of the background vorticity when its sign is the same as that of the vortex sheet. However, if $G < 0$, the vortex sheet becomes stable with respect to any small disturbance of the wavenumber which satisfies the inequality :

$$(1 + \tanh \alpha h)/\alpha < -2U/G . \tag{13}$$

The main features of the Helmholtz instability of the vortex sheet in the presence of the background vorticity have thus been clarified on the basis of the linear stability analysis. In accordance with Hama and Burke's findings based on the numerical calculation of the rolling up of the vortex sheet, the present results also suggest that the background vorticity promotes the rolling up of the vortex sheet of the same sign whereas it decelerates that of the opposite sign. The vortex sheet will not roll up with a sufficiently large background vorticity of the opposite sign.

Figure 2 shows an experimental vortex pattern in the wake of a circular cylinder placed in the laminar boundary layer along a plane wall. As was predicted by the forgoing theory, the upper shear layer separated from the surface of the cylinder, whose vorticity has the same sign as the background vorticity of the boundary layer, shows a much more significant rolling up than the lower shear layer of the opposite sign.

A final remark is that Eqs. (11) and (13) are reduced to

$$D = -4U^2 \left\{ 1 + (\alpha U)^{-1} G \right\} , \qquad \alpha^{-1} < -U/G \tag{14}$$

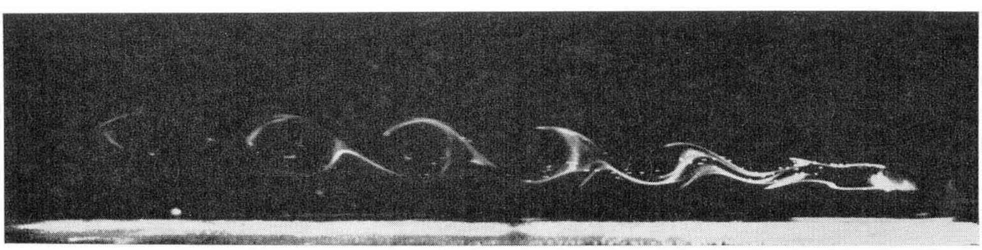

Fig. 2. Vortex pattern behind a circular cylinder in a laminar boundary layer along a flat plate (flow from right to left). Flow visualization by electrolytic dye production. $Gd/U_0 = 0.49$, $U_0 d/\nu = 52$; d = diameter of cylinder (3.2 mm), U_0 = undisturbed velocity at the center of cylinder (2.5 cm/s), ν = kinematic viscosity ; boundary layer thickness $\delta = 13.4$ mm, normal distance between the center of cylinder and the plane wall $= 0.4\,\delta$.

as $h \to \infty$, which is the case of a vortex sheet very far from the plane wall. The vortex sheet is neutrally stable in the range of wavenumber described by Eq. (14).

References

1) Batchelor, G. K.: An Introduction to Fluid Dynamics, Cambridge University Press, 1967.
2) Hama, F. R. and Burke, E. R.: On the Rolling-Up of the Vortex Sheet, University of Maryland, Tech. Note, No. BN-220, 1960.