On the Flexural Deflection of a Rectangular Beam with Moderately Large Depth

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Abstract

Equation for deflection of a beam of narrow rectangular cross-section with moderately large depth is derived from the equation for deflection of a thick plate proposed in the previous paper, considering that one of the lateral dimensions of a rectangular plate is much smaller than the other.

Assuming that the depth of beam is small compared with its length, the approximate equations for deflection of a beam can be obtained with any desired accuracy.

For a simply supported rectangular beam under uniform load, the exact solution for deflection of the beam and solutions for its approximate equations, are given in this paper.

The results obtained here are compared with those given by the usual beam theory.

§ 1. Notations and Fundamental Equations for Deflection of a Thick Plate

Notations

x_i: rectangular coordinates, $(i=1, 2, 3)$

ξ_i: components of displacement, $(i=1, 2, 3)$

$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i}+\frac{\partial \xi_i}{\partial x_j}\right)$: components of strain, $(i, j=1, 2, 3)$

$\varepsilon_{kk}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$,

$A_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}$: components of stress, $(i, j=1, 2, 3)$

with Lame's constants λ and μ,

$(l, m)=l\lambda+m\mu$, $\quad (l, m, n)=l\lambda^2+m\lambda\mu+n\mu^2$, etc.

h: depth of beam,

b: breadth of beam,

$E=\frac{(0, 1)(3, 2)}{(1, 1)}$: Young's modulus,

$I=\frac{bh^3}{12}$: moment of inertia of the cross-section of beam, and

W_0: deflection of beam.

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Fundamental Equations

The approximate equations for deflection of a moderately thick plate were obtained and solved in the previous papers[1,2]. The present author proposes here the equation for deflection of a rectangular beam with moderately large depth, with reference to the equation of a moderately thick plate.

At first the author summarizes the method to treat the bending problem of a moderately thick plate, then shows how the fundamental equations of a beam with large depth can be derived from the equation of the plate.

We shall take x_1- and x_2- axes on the middle plane of the plate, x_3- axis being directed downwards. The thickness of the plate shall be taken to be h.

When one treats the problem of a plate in a bending state, one expands the components of displacement as follows:

$$\left.\begin{aligned} \xi_1 &= \sum_{k=0}^{\infty} u_{2k+1}\, x_3^{2k+1}\,, \\ \xi_2 &= \sum_{k=0}^{\infty} v_{2k+1}\, x_3^{2k+1}\,, \\ \xi_3 &= \sum_{k=0}^{\infty} w_{2k}\, x_3^{2k}\,, \end{aligned}\right\} \tag{1-1}$$

where u_k, v_k, and w_k, are functions of x_1 and x_2.

Introducing (1-1) into equations of equilibrium of an elastic body:

$$0 = \frac{\partial A_{ij}}{\partial x_j} = (1,1)\frac{\partial}{\partial x_i}\varepsilon_{kk} + (0,1)\,\Delta_3 \xi_i\,, \quad (i = 1, 2, 3) \tag{1-2}$$

with

$$\Delta_3 = \frac{\partial^2}{\partial x_k^2} = \Delta + \frac{\partial^2}{\partial x_3^2}\,,$$

and comparing the coefficients of the same power in x_3, one finds that all the coefficients of power series in eqs. (1-1) can be expressed in terms of u_1, v_1, and w_0, as follows*:

$$\left.\begin{aligned} u_{2k+1} &= \frac{(-1)^k}{(2k+1)!}\left[\Delta^k u_1 + \frac{k(1,1)}{(1,2)}\Delta^{k-1}\frac{\partial}{\partial x_1}\{\Xi_1 - \Delta w_0\}\right], \\ &\qquad (k=0, 1, 2, \cdots) \\ v_{2k+1} &= \frac{(-1)^k}{(2k+1)!}\left[\Delta^k v_1 + \frac{k(1,1)}{(1,2)}\Delta^{k-1}\frac{\partial}{\partial x_2}\{\Xi_1 - \Delta w_0\}\right], \\ &\qquad (k=0, 1, 2, \cdots) \\ &\text{and} \\ w_{2k} &= \frac{(-1)^k}{(2k)!}\left[\Delta^k w_0 + \frac{k(1,1)}{(1,2)}\Delta^{k-1}\{\Xi_1 - \Delta w_0\}\right], \ (k=0, 1, 2, \cdots) \end{aligned}\right\} \tag{1-3}$$

with

* We understand that the zero-th power of the Laplacian operator is unity, *i. e.* $\Delta^0 = 1$.

$$\Xi_1=\frac{\partial u_1}{\partial x_1}+\frac{\partial v_1}{\partial x_2}\,. \tag{1-4}$$

Boundary conditions at the surfaces of the plate are given:

$$\left.\begin{aligned}\pm\frac{p}{2}&=A_{33}=(1,0)\,\varepsilon_{kk}+2(0,1)\,\varepsilon_{33}\,, &&\text{at}\quad x_3=\pm\frac{h}{2}\\ 0&=A_{3i}=2(0,1)\,\varepsilon_{3i}\,,\quad(i=1,2) &&\text{at}\quad x_3=\pm\frac{h}{2}\end{aligned}\right\} \tag{1-5}$$

where $p=p(x_1, x_2)$ is the distributed external pressure over the upper surface of the plate. Introducing (1-1) with (1-3) into (1-5), we obtain the following equations:

$$\frac{4}{3}(1,1)\frac{p}{D}=\sum_{k=0}^{\infty}\frac{(-1)^k(2k+2)}{(2k+3)\,!}\left(\frac{h}{2}\right)^{2k}\times$$
$$\times\Big[(2k+1,2k)\,\Delta^{k+2}w_0-(2k+3,2k+4)\,\Delta^{k+1}\Xi_1\Big]\,, \tag{1-6}$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)\,!}\left(\frac{h}{2}\right)^{2k}\Big[(2k+1,2k+2)\,\Delta^k\Xi_1-(2k-1,2k-2)\,\Delta^{k+1}w_0\Big]=0\,, \tag{1-7}$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)\,!}\left(\frac{h}{2}\right)^{2k}\Delta^k\Omega_1=0\,, \tag{1-8}$$

with flexural rigidity D of the plate of thickness h:

$$D=\frac{h^3(0,1)(1,1)}{3(1,2)}\,,$$

and

$$\Omega_1=\frac{\partial u_1}{\partial x_2}-\frac{\partial v_1}{\partial x_1}\,. \tag{1-9}$$

§ 2. Equation for Deflection of a Beam with Rectangular Cross-Section

When the breadth, say b, of a plate is small, the equation of a thick plate can be reduced to the equation of a narrow rectangular beam[3] of large depth.

Because of the narrow breadth of the beam, we can take:

$$0=A_{2i}=(1,0)\,\varepsilon_{kk}\delta_{2i}+2(0,1)\,\varepsilon_{2i}\,,\quad(i=1,2,3) \tag{2-1}$$

in the interior of the beam.

Introducing (1-1) into (2-1), equating the coefficients of the same power of x_3, and taking (1-3) into account, we obtain the following equations:

$$\frac{\partial v_{2k+1}}{\partial x_1}+\frac{\partial u_{2k+1}}{\partial x_2}=0\,,\quad(k=0,1,2,\cdots) \tag{2-2}$$

$$\frac{\partial w_{2k}}{\partial x_2}+(2k+1)\,v_{2k+1}=0\,,\quad(k=0,1,2,\cdots) \tag{2-3}$$

and

$$\frac{\partial v_{2k+1}}{\partial x_2}+\frac{(1,0)}{4(1,1)}\left\{\frac{\partial u_{2k+1}}{\partial x_1}-\frac{1}{2k+1}\frac{\partial^2 w_{2k}}{\partial x_1^2}\right\}=0\,. \quad (k=0,1,2,\cdots) \tag{2-4}$$

From (2-2)∼(2-4) with $k=0$, one can find:

$$\frac{\partial^{s-1} v_1}{\partial x_2^{s-1}}=0\,, \quad \text{and} \quad \frac{\partial^s u_1}{\partial x_2^s}=\frac{\partial^s w_0}{\partial x_2^s}=0\,. \quad (s\geqq 3) \tag{2-5}$$

By means of (2-2)∼(2-5), eqs. (1-3) read:

$$u_{2k+1}=\frac{(-1)^k}{4(1,1)(2k+1)!}\frac{\partial^{2k}}{\partial x_1^{2k}}\left[(3k+4,2k+4)\,u_1-k(3,2)\frac{\partial}{\partial x_1}w_0\right], \quad (k=0,1,2,\cdots) \tag{2-6}$$

$$v_{2k+1}=\frac{(-1)^k}{(2k+1)!}\frac{\partial^{2k}}{\partial x_1^{2k}}v_1\,, \quad (k=0,1,2,\cdots) \tag{2-7}$$

and

$$w_{2k}=\frac{(-1)^k}{4(1,1)(2k)!}\frac{\partial^{2k-1}}{\partial x_1^{2k-1}}\left[k(3,2)\,u_1-(3k-4,2k-4)\frac{\partial}{\partial x_1}w_0\right]. \quad (k=0,1,2,\cdots) \tag{2-8}$$

From eqs. (2-5) and (2-2)∼(2-4) with $k=0$, terms u_1, v_1, and w_0, are expressed as follows:

$$\left.\begin{aligned} u_1&=U_1+\frac{(1,0)}{8(1,1)}x_2^2\frac{d^2}{dx_1^2}\left[U_1-\frac{d}{dx_1}W_0\right],\\ v_1&=-\frac{(1,0)}{4(1,1)}x_2\frac{d}{dx_1}\left[U_1-\frac{d}{dx_1}W_0\right],\\ &\text{and}\\ w_0&=W_0+\frac{(1,0)}{8(1,1)}x_2^2\frac{d}{dx_1}\left[U_1-\frac{d}{dx_1}W_0\right], \end{aligned}\right\} \tag{2-9}$$

where U_1 and W_0 are functions of x_1 alone.

Introducing (2-9) into (1-6)∼(1-8) and retaining terms of $O(x_2^0)$, we obtain the following equations:

$$\frac{q}{EI}=\frac{3}{2(3,2)}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2k+2)}{(2k+3)!}\left(\frac{h}{2}\right)^{2k}\times \times\frac{d^{2k+3}}{dx_1^{2k+3}}\left[(3k+5,2k+4)\,U_1-(3k+1,2k)\frac{d}{dx_1}W_0\right], \tag{2-10}$$

and

$$0=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{h}{2}\right)^{2k}\frac{d^{2k}}{dx_1^{2k}}\left[(3k+2,2k+2)\,U_1-(3k-2,2k-2)\frac{d}{dx_1}W_0\right], \tag{2-11}$$

where $q=\int_{-b/2}^{b/2}p(x_1,x_2)\,dx_2$ is the intensity of external load, $E=(0,1)(3,2)/(1,1)$ is

Young's modulus, and $I=bh^3/12$ is moment of inertia of the cross-section of the beam.

When the depth h of beam is small compared with its length, U_1 in eq. (2-11) can be expressed as follows:

$$U_1=-\frac{d}{dx_1}\left[1+\frac{(3,2)}{2(1,1)}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}+\frac{(3,2)(13,10)}{24(1,1)^2}\left(\frac{h}{2}\right)^4\frac{d^4}{dx_1^4}+\right.$$
$$\left.+\frac{(3,2)(287,444,172)}{480(1,1)^3}\left(\frac{h}{2}\right)^6\frac{d^6}{dx_1^6}+\cdots\right]W_0. \qquad (2\text{-}12)$$

Introducing (2-12) into (2-10), we obtain:

$$q=EI\frac{d^4}{dx_1^4}\left[1+\frac{(21,16)}{20(1,1)}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}+\frac{(1957,3018,1166)}{1680(1,1)^2}\left(\frac{h}{2}\right)^4\frac{d^4}{dx_1^4}+\right.$$
$$\left.+\frac{(6941,15378,9852,2828)}{60480(1,1)^3}\left(\frac{h}{2}\right)^6\frac{d^6}{dx_1^6}+\cdots\right]W_0. \qquad (2\text{-}13)$$

We can also obtain (2-13) from eqs. (2-10) and (2-11) in a quite similar manner as was described in the previous paper[1] for a thick plate. Applying operators:

$$\sum_{k=0}^{\infty}(-1)^k\frac{(3k+2,2k+2)}{(2k)!}\left(\frac{h}{2}\right)^{2k}\frac{d^{2k}}{dx_1^{2k}}, \quad \text{and}$$
$$\sum_{k=0}^{\infty}(-1)^k\frac{(2k+2)(3k+5,2k+4)}{(2k+3)!}\left(\frac{h}{2}\right)^{2k}\frac{d^{2k+3}}{dx_1^{2k+3}},$$

respectively to (2-10) and (2-11), and eliminating U_1, we have:

$$\sum_{k=0}^{\infty}(-1)^k\frac{(3k+2,2k+2)}{(2k)!}\left(\frac{h}{2}\right)^{2k}\frac{d^{2k}}{dx_1^{2k}}q=$$
$$=6(1,1)EI\sum_{k=0}^{\infty}\sum_{i=0}^{k}(-1)^k\frac{(2i+2)(2i+1-k)}{(2i+3)!\{2(k-i)\}!}\left(\frac{h}{2}\right)^{2k}\frac{d^{2k+4}}{dx_1^{2k+4}}W_0. \qquad (2\text{-}14)$$

The inverse of the operator in the left-hand side of (2-14) can be written as:

$$\vartheta=\frac{1}{2(1,1)}\left[1+\frac{(5,4)}{4(1,1)}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}+\frac{(67,106,42)}{48(1,1)^2}\left(\frac{h}{2}\right)^4\frac{d^4}{dx_1^4}+\right.$$
$$\left.+\frac{(4447,10500,8274,2176)}{2880(1,1)^3}\left(\frac{h}{2}\right)^6\frac{d^6}{dx_1^6}+\cdots\right]. \qquad (2\text{-}15)$$

Applying the inverse operator ϑ to both sides of (2-14), we obtain eq. (2-13).

Truncating the series in h in the right-hand side of eq. (2-13), we obtain the approximate equation for deflection of a beam with large depth. For instance, if we retain the term of $O(h^0)$ in the right-hand side of (2-13), we have:

$$q=EI\frac{d^4}{dx_1^4}W_0. \qquad (2\text{-}16)$$

Eq. (2-16) is the equation for deflection of a beam in the zero-th order approximation in the present theory, and is nothing but the usual equation for a beam of small depth.

While, retaining terms of $O(h^2)$ in the right-hand side of (2-13), we obtain the equation:

$$q=EI\frac{d^4}{dx_1^4}\left[W_0+\frac{(21,16)}{20(1,1)}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}W_0\right], \tag{2-17}$$

for deflection of a beam in the first order approximation.

In a similar manner, we can obtain the approximate equations for deflection of a beam with any desired accuracy, after truncating the series in eq. (2-13) at the terms of higher order of h. Eq. (2-13) shows the terms to $O(h^6)$ explicitly.

§ 3. Solution of Equation for Deflection of a Simply Supported Beam under Uniform Load

We shall solve a set of equations (2-10) and (2-11) for a simply supported beam of moderately large depth under uniformly distributed external load q_0 per unit of longitudinal length.

Let a beam of breadth b and length l be simply supported at $x_1=\pm l/2$. The boundary conditions shall be taken to be:

$$\xi_3=0, \quad \text{and} \quad A_{11}=0. \quad \text{at} \quad x_1=\pm\frac{l}{2} \tag{3-1}$$

The solution of eqs. (2-10) and (2-11), satisfying the boundary conditions (3-1), can be written as:

$$W_0=\sum_{n=1}^{\infty}A_n\cos(\alpha_n x_1), \tag{3-2}$$

$$U_1=\sum_{n=1}^{\infty}B_n\sin(\alpha_n x_1), \tag{3-3}$$

where,

$$\alpha_n=\frac{(2n-1)}{l}\pi, \quad (n=1, 2, 3, \cdots)$$

with constants A_n and B_n. While, the uniform load $q=\text{const.}=q_0$ can be expressed as:

$$q_0=\sum_{n=1}^{\infty}q_n\cos(\alpha_n x_1), \tag{3-4}$$

with

$$q_n=\frac{4}{\pi}\frac{(-1)^{n-1}}{(2n-1)}q_0. \quad (n=1, 2, 3, \cdots) \tag{3-5}$$

Introducing (3-2)~(3-4) into eqs. (2-10) and (2-11), we obtain:

$$A_n=\frac{1}{6EI}\frac{q_n}{\alpha_n^4}(\alpha_n h)^3\frac{\cosh\left(\alpha_n\frac{h}{2}\right)+\frac{(3,2)}{4(1,1)}\sinh\left(\alpha_n\frac{h}{2}\right)}{\sinh(\alpha_n h)-\alpha_n h}, \quad (n=1, 2, 3, \cdots) \tag{3-6}$$

and

$$B_n=\frac{1}{6EI}\frac{q_n}{\alpha_n^4}(\alpha_n h)^3\frac{\cosh\left(\alpha_n\frac{h}{2}\right)-\frac{(3,2)}{4(1,1)}\sinh\left(\alpha_n\frac{h}{2}\right)}{\sinh(\alpha_n h)-\alpha_n h}. \qquad (n=1,2,3,\cdots) \tag{3-7}$$

Expressing the right-hand side of eq. (3-6) in the form:

$$A_n=\frac{1}{EI}\frac{q_n}{\alpha_n^4}\left[1+\frac{(21,16)}{20(1,1)}\left(\alpha_n\frac{h}{2}\right)^2-\frac{(262,227)}{4200(1,1)}\left(\alpha_n\frac{h}{2}\right)^4+ +\frac{(14407,11572)}{756000(1,1)}\left(\alpha_n\frac{h}{2}\right)^6-\cdots\right], \quad (n=1,2,3,\cdots) \tag{3-8}$$

and truncating the series at $O(h^{2n})$, we obtain the solution of equation in the n-th order approximation.

For example, retaining the term of $O(h^0)$ in the right-hand side of (3-8), we have:

$$W_0=\frac{q_0}{24EI}\left\{\left(\frac{l}{2}\right)^2-x_1^2\right\}\left\{5\left(\frac{l}{2}\right)^2-x_1^2\right\}. \tag{3-9}$$

Eq. (3-9) is the solution of equation for deflection of a beam in the zero-th order approximation and is nothing but the usual solution for a simply supported beam of small depth.

While, if we retain terms of $O(h^2)$ in the right-hand side of eq. (3-8), W_0 can be written as:

$$W_0=\frac{q_0}{24EI}\left\{\left(\frac{l}{2}\right)^2-x_1^2\right\}\left[5\left(\frac{l}{2}\right)^2\left\{1-\frac{3(21,16)}{25(1,1)}\left(\frac{h}{l}\right)^2\right\}-x_1^2\right]. \tag{3-10}$$

Eq. (3-10) is the solution of equation for deflection of a beam in the first order approximation.

In a similar manner, we can obtain the solution of equation in the n-th order approximation, after truncating the series in the right-hand side of eq. (3-8) at the terms of $O(h^{2n})$.

§ 4. Discussions

A) Approximate equations obtained from eq. (2-14)

From eq. (2-14), we can have approximate equations (2-16), (2-17), etc. It is also possible to obtain approximate equations from (2-14) in a somewhat different manner.

Retaining terms of $O(h^2)$ in the left-hand side of eq. (2-14), we get:

$$\left\{1-\frac{(5,4)}{4(1,1)}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}\right\}q=EI\frac{d^4}{dx_1^4}W_0. \tag{4-1}$$

While, if we retain terms of $O(h^2)$ in both sides of eq. (2-14), we arrive at the following equation:

$$\left\{1-\frac{(5,4)}{4(1,1)}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}\right\}q=EI\frac{d^4}{dx_1^4}\left\{1-\frac{1}{5}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}\right\}W_0, \tag{4-2}$$

which can be transformed into:

$$\left\{1-\frac{(21,16)}{20(1,1)}\left(\frac{h}{2}\right)^2\frac{d^2}{dx_1^2}\right\}q=EI\frac{d^4}{dx_1^4}W_0. \tag{4-3}$$

Eqs. (4-1) and (4-3) correspond to the equation given by Timoshenko[4]:

$$-\frac{1}{EI}\left\{M+\frac{(3,2)}{2(1,1)}\left(\frac{h}{2}\right)^2q\right\}=\frac{d^2}{dx_1^2}W_0, \tag{4-4}$$

where M is the bending moment of the beam ($q=-d^2M/dx_1^2$). Eq. (4-4) is obtained from the usual equation of deflection for a beam of small depth, taking into account the effect of shearing stress at the cross-section of the beam.

Eqs. (4-1) and (4-3) are reduced to eq. (2-16), when the external load q is uniformly distributed along the length of the beam, *i. e.* the effect of the depth of the beam disappears in the left-hand side of (4-1) and (4-3). The author showed here eqs. (4-1) and (4-3), for the sake of comparison with eq. (4-4).

B) Numerical Example

As for numerical examples, the maximum deflections $W_{\max}$ of a simply supported rectangular beam under uniform load are calculated from solutions obtained in this paper.

The maximum deflection corresponding to the exact solution is given by eq. (3-2) with $x_1=0$:

$$W_{\max}=\sum_{n=1}^{\infty}A_n, \tag{4-5}$$

where expression (3-6) is to be used. The maximum deflections in the zero-th and the first order approximations are derived from eqs. (3-9) and (3-10), and are written as:

$$W_{\max}=\frac{5}{24}\frac{q_0}{EI}\left(\frac{l}{2}\right)^4, \tag{4-6}$$

and

$$W_{\max}=\frac{5}{24}\frac{q_0}{EI}\left(\frac{l}{2}\right)^4\left\{1+\frac{3(21,16)}{25(1,1)}\left(\frac{h}{l}\right)^2\right\}. \tag{4-7}$$

While, from the solution of eq. (4-4), we can obtain the maximum deflection:

$$W_{\max}=\frac{5}{24}\frac{q_0}{EI}\left(\frac{l}{2}\right)^4\left\{1+\frac{6(3,2)}{5(1,1)}\left(\frac{h}{l}\right)^2\right\}. \tag{4-8}$$

Numerical results W_e for the exact solution are calculated after truncating the series (4-5) at appropriate terms ($n=11$). For several values of h/l, numerical results $W_{\max}/W_e$ calculated from eqs. (4-6)~(4-8) are listed in Table 1, in which the ratio λ/μ is taken to be 3/2, with Poisson's ratio $(\lambda/2)/(\lambda+\mu)=0.3$.

TABLE 1. Comparison of the maximum deflections of a simply supported rectangular beam under uniform load. ($\lambda/\mu=3/2$, and Poisson's ratio$=0.3$)

h/l	W_{max}/W_e (W_e: exact solution)		
	0-th order approx. (usual beam theory)	1st order approx.	Effect of shearing force[4] (Eq. (4-8))
0.05	0.994	1.000	1.002
0.10	0.978	1.000	1.008
0.15	0.951	1.000	1.018
0.20	0.917	1.000	1.031
0.25	0.876	1.001	1.047
0.30	0.831	1.002	1.065
0.35	0.784	1.003	1.084
0.40	0.737	1.005	1.104

In the zero-th order approximation (the usual beam theory), the relative error $|1-(W_{max}/W_e)|$ is comparatively small for a beam of small depth (*e.g.* less than 5% for $h/l=0.15$), while, the error increases with increasing depth of the beam (from *ca.* 8% for $h/l=0.2$ to *ca.* 26% for $h/l=0.4$).

In case of eq. (4-8), the error is comparatively small for a beam of small depth and also for a beam of relatively large depth, *e.g.* the error is less than 5% for $h/l<0.25$.

In concluding this paper, the author wishes to mention that the values of W_{max} in the first order approximation in our theory agree very well with the values from the exact solution W_e for any value of h/l, *e.g.* the error is less than 0.1% for $h/l<0.2$ and is 0.5% for $h/l=0.4$.

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