On the Flexural Deflection of a Moderately Thick Plate

Part II. Solution of Equation for Deflection of the Plate

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Résumé

The equation of deflection for a moderately thick plate presented in our previous paper, is solved for two cases, namely, a) a simply supported rectangular plate under distributed pressure, and b) a simply supported circular plate under uniformly distributed pressure.

Results obtained here are used to calculate the maximum deflection of the plate, and are compared with the results hitherto obtained.

§ 1. Notations and Equation of Flexural Deflection of a Moderately Thick Plate

Notations

x_i: rectangular coordinates, $(i=1, 2, 3)$

ξ_i: components of displacement, $(i=1, 2, 3)$

$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial \xi_j}{\partial x_i}+\frac{\partial \xi_i}{\partial x_j}\right)$: components of strain, $(i, j=1, 2, 3)$

$\varepsilon_{kk}=\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}$,

$A_{ij}=\lambda\varepsilon_{kk}\delta_{ij}+2\mu\varepsilon_{ij}$: components of stress, $(i, j=1, 2, 3)$

with Lamé's constants λ and μ,

$(l, m)=l\lambda+m\mu$, $\quad (l, m, n)=l\lambda^2+m\lambda\mu+n\mu^2$, etc.

h: thickness of plate,

$D=\frac{h^3(0, 1)(1, 1)}{3(1, 2)}$: flexural rigidity of plate, and

w_0: deflection of plate, *i. e.* vertical displacement of the middle plane of plate.

We shall take x_1- and x_2-axes on the middle plane of the plate, x_3-axis being directed downwards.

Equation of Flexural Deflection of a Moderately Thick Plate

In the previous paper[1], the fundamental equations for deflection of a moderately thick plate were presented, and it was shown that one can obtain approximate

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equations with any desired accuracy after considering that the thickness of plate is small. We shall summarize the results here.

Expanding components of displacement ξ_i $(i=1, 2, 3)$ of the plate into power series in x_3:

$$\left.\begin{aligned} \xi_1 &= \sum_{k=0}^{\infty} u_{2k+1}\, x_3^{2k+1}\,, \\ \xi_2 &= \sum_{k=0}^{\infty} v_{2k+1}\, x_3^{2k+1}\,, \\ \xi_3 &= \sum_{k=0}^{\infty} w_{2k}\, x_3^{2k}\,, \end{aligned}\right\} \tag{1-1}$$

and introducing (1-1) into equations of equilibrium of an elastic body:

$$0 = \frac{\partial A_{ij}}{\partial x_j} = (1,1)\frac{\partial}{\partial x_i}\varepsilon_{kk} + (0,1)\,\Delta_3 \xi_i\,, \quad (i=1,2,3) \tag{1-2}$$

with

$$\Delta_3 = \frac{\partial^2}{\partial x_k^2} = \Delta + \frac{\partial^2}{\partial x_3^2}\,,$$

we obtain the following relations* among the coefficients of power series in eqs. (1-1):

$$\left.\begin{aligned} u_{2k+1} &= \frac{(-1)^k}{(2k+1)!}\left[\Delta^k u_1 + \frac{k(1,1)}{(1,2)}\Delta^{k-1}\frac{\partial}{\partial x_1}\{\Xi_1 - \Delta w_0\}\right], \quad (k=0,1,2,\cdots) \\ v_{2k+1} &= \frac{(-1)^k}{(2k+1)!}\left[\Delta^k v_1 + \frac{k(1,1)}{(1,2)}\Delta^{k-1}\frac{\partial}{\partial x_2}\{\Xi_1 - \Delta w_0\}\right], \quad (k=0,1,2,\cdots) \\ &\text{and} \\ w_{2k} &= \frac{(-1)^k}{(2k)!}\left[\Delta^k w_0 + \frac{k(1,1)}{(1,2)}\Delta^{k-1}\{\Xi_1 - \Delta w_0\}\right], \quad (k=0,1,2,\cdots) \end{aligned}\right\} \tag{1-3}$$

with

$$\Xi_1 = \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\,. \tag{1-4}$$

Boundary conditions at the surfaces of the plate read:

$$\left.\begin{aligned} \pm\frac{p}{2} &= A_{33} = (1,0)\,\varepsilon_{kk} + 2(0,1)\varepsilon_{33}\,, \quad \text{at} \quad x_3 = \pm\frac{h}{2} \\ 0 &= A_{31} = 2(0,1)\,\varepsilon_{31}\,, \quad \text{at} \quad x_3 = \pm\frac{h}{2} \\ 0 &= A_{32} = 2(0,1)\,\varepsilon_{32}\,, \quad \text{at} \quad x_3 = \pm\frac{h}{2} \end{aligned}\right\} \tag{1-5}$$

where $p=p(x_1, x_2)$ is the distributed external pressure over the upper surface of the plate. Introducing (1-1) with (1-3) into (1-5), we have the following equations:

* We understand that the zero-th power of the Laplacian operator is equal to unity, *i. e.* $\Delta^0=1$.

$$\frac{4}{3}(1,1)\,p = D\sum_{k=0}^{\infty}\frac{(-1)^k(2k+2)}{(2k+3)!}\left(\frac{h}{2}\right)^{2k}\Big[(2k+1,2k)\,\varDelta^{k+2}w_0 - $$
$$-(2k+3,2k+4)\,\varDelta^{k+1}\varXi_1\Big], \qquad (1\text{-}6)$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{h}{2}\right)^{2k}\Big[(2k+1,2k+2)\,\varDelta^{k}\varXi_1-(2k-1,2k-2)\,\varDelta^{k+1}w_0\Big]=0\,, \qquad (1\text{-}7)$$

and

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{h}{2}\right)^{2k}\varDelta^{k}\varOmega_1=0\,, \qquad (1\text{-}8)$$

with flexural rigidity D of the plate of thickness h:

$$D=\frac{h^3(0,1)(1,1)}{3(1,2)}\,,$$

and

$$\varOmega_1=\frac{\partial u_1}{\partial x_2}-\frac{\partial v_1}{\partial x_1}\,. \qquad (1\text{-}9)$$

When the thickness of plate is small compared with its lateral dimensions, one can find:

$$\varOmega_1=\frac{\partial u_1}{\partial x_2}-\frac{\partial v_1}{\partial x_1}=0\,. \qquad (1\text{-}10)$$

Eq. (1-10) shows that there is a function ϕ such that:

$$u_1=\frac{\partial\phi}{\partial x_1}\,, \qquad \text{and} \qquad v_1=\frac{\partial\phi}{\partial x_2}\,, \qquad (1\text{-}11)$$

with

$$\varXi_1=\varDelta\phi\,. \qquad (1\text{-}12)$$

From (1-7)~(1-12), ϕ is expressed in the form:

$$\phi=-w_0-\frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2\varDelta w_0-\frac{2(1,1)(4,5)}{3(1,2)^2}\left(\frac{h}{2}\right)^4\varDelta\varDelta w_0-$$
$$-\frac{2(1,1)(27,68,43)}{15(1,2)^3}\left(\frac{h}{2}\right)^6\varDelta\varDelta\varDelta w_0-\cdots\,. \qquad (1\text{-}13)$$

Eliminating $\varXi_1$ from (1-6) by means of (1-12) and (1-13), we obtain:

$$p=D\varDelta\varDelta\Big\{w_0+\frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2\varDelta w_0+\frac{(1479,3704,2332)}{840(1,2)^2}\left(\frac{h}{2}\right)^4\varDelta\varDelta w_0+$$
$$+\frac{(35969,135768,171420,72400)}{15120(1,2)^3}\left(\frac{h}{2}\right)^6\varDelta\varDelta\varDelta w_0+\cdots\Big\}\,. \qquad (1\text{-}14)$$

Retaining terms of $O(h^{2n})$ in the right-hand side of eqs. (1-13) and (1-14), we obtain the equation for deflection of a moderately thick plate in the n-th order approximation in our theory. We shall cite here merely the equations in the zero-th, the first, the second, and the third order approximations. For the sake of simplicity, we shall omit the subscript of w_0 hereafter, writing w instead of w_0.

A) Zero-th order approximation

$$\phi = -w\,, \tag{1-15}$$

and

$$p = D\Delta\Delta w\,. \tag{1-16}$$

B) First order approximation

$$\phi = -w - \frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2 \Delta w\,, \tag{1-17}$$

and

$$p = D\Delta\Delta\left\{w + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta w\right\}. \tag{1-18}$$

C) Second order approximation

$$\phi = -w - \frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2 \Delta w - \frac{2(1,1)(4,5)}{3(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w\,, \tag{1-19}$$

and

$$p = D\Delta\Delta\left\{w + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta w + \frac{(1479,3704,2332)}{840(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w\right\}. \tag{1-20}$$

D) Third order approximation

$$\begin{aligned}\phi = -w &- \frac{2(1,1)}{(1,2)}\left(\frac{h}{2}\right)^2 \Delta w - \frac{2(1,1)(4,5)}{3(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w - \\ &- \frac{2(1,1)(27,68,43)}{15(1,2)^3}\left(\frac{h}{2}\right)^6 \Delta\Delta\Delta w\,,\end{aligned} \tag{1-21}$$

and

$$\begin{aligned}p = D\Delta\Delta\Bigg\{w &+ \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta w + \frac{(1479,3704,2332)}{840(1,2)^2}\left(\frac{h}{2}\right)^4 \Delta\Delta w + \\ &+ \frac{(35969,135768,171420,72400)}{15120(1,2)^3}\left(\frac{h}{2}\right)^6 \Delta\Delta\Delta w\Bigg\}.\end{aligned} \tag{1-22}$$

§ 2. Solution of Equation of Deflection for a Simply Supported Rectangular Thick Plate

Eqs. (1-6)∼(1-8) as well as equations in any order approximation can be solved for a simply supported rectangular plate under distributed pressure.

Let the rectangular plate occupy the region: $-a/2 \leqslant x_1 \leqslant a/2$ and $-b/2 \leqslant x_2 \leqslant b/2$, and be simply supported at $x_1 = \pm a/2$ and $x_2 = \pm b/2$. We shall take the boundary conditions in this case as follows:

$$\xi_2 = \xi_3 = 0\,, \qquad \text{and} \quad A_{11} = 0\,, \qquad \text{at} \quad x_1 = \pm a/2 \tag{2-1}$$

and

$$\xi_1 = \xi_3 = 0\,, \qquad \text{and} \quad A_{22} = 0\,. \qquad \text{at} \quad x_2 = \pm b/2 \tag{2-2}$$

Under the boundary conditions (2-1) and (2-2), solution of eqs. (1-6)~(1-8) can be expressed by:

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \cos \frac{(2m-1)\,\pi}{a} x_1 \cdot \cos \frac{(2n-1)\,\pi}{b} x_2\,, \tag{2-3}$$

$$\Omega_1 = 0\,, \tag{2-4}$$

and

$$\phi = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Phi_{mn} \cos \frac{(2m-1)\,\pi}{a} x_1 \cdot \cos \frac{(2n-1)\,\pi}{b} x_2\,, \tag{2-5}$$

with (1-11) and (1-12).

The distributed pressure $p = p(x_1, x_2)$ is expressed in a double Fourier series:

$$p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} \cos \frac{(2m-1)\,\pi}{a} x_1 \cdot \cos \frac{(2n-1)\,\pi}{b} x_2\,, \tag{2-6}$$

with

$$p_{mn} = \frac{4}{ab} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} p(x_1, x_2) \cos \frac{(2m-1)\,\pi}{a} x_1 \cdot \cos \frac{(2n-1)\,\pi}{b} x_2 dx_1 dx_2\,. \qquad (m, n=1, 2, 3, \cdots) \tag{2-7}$$

When the pressure is uniformly distributed, *i. e.* $p=\text{const}\ (=p_0)$, expression (2-7) turns simply to be:

$$p_{mn} = \frac{2^4}{\pi^2} \frac{(-1)^{m+n}}{(2m-1)(2n-1)} p_0\,. \qquad (m, n=1, 2, 3, \cdots) \tag{2-8}$$

Introducing (2-3)~(2-6) with (1-12) into (1-6)~(1-8), we obtain:

$$W_{mn} = \frac{1}{6D} \frac{p_{mn}}{\gamma_{mn}{}^4} (\gamma_{mn} h)^3 \frac{\frac{(1,1)}{(1,2)}\left(\gamma_{mn}\frac{h}{2}\right)\sinh\left(\gamma_{mn}\frac{h}{2}\right) + \cosh\left(\gamma_{mn}\frac{h}{2}\right)}{\sinh(\gamma_{mn} h) - \gamma_{mn} h}\,, \qquad (m, n=1, 2, 3, \cdots) \tag{2-9}$$

$$\Phi_{mn} = \frac{1}{6D} \frac{p_{mn}}{\gamma_{mn}{}^4} (\gamma_{mn} h)^3 \frac{\frac{(1,1)}{(1,2)}\left(\gamma_{mn}\frac{h}{2}\right)\sinh\left(\gamma_{mn}\frac{h}{2}\right) - \cosh\left(\gamma_{mn}\frac{h}{2}\right)}{\sinh(\gamma_{mn} h) - \gamma_{mn} h}\,, \qquad (m, n=1, 2, 3, \cdots) \tag{2-10}$$

with

$$\gamma_{mn}{}^2 = \left[\frac{(2m-1)\,\pi}{a}\right]^2 + \left[\frac{(2n-1)\,\pi}{b}\right]^2. \qquad (m, n=1, 2, 3, \cdots) \tag{2-11}$$

By means of (2-3) and (2-9), we have the expression for w, which is found to be identical with the result obtained by Iyenger et al.[2] They obtained the solution of the equation derived from Vlasov's method[3].

If we express the right-hand side of eq. (2-9) into power series in h:

$$W_{mn}=\frac{1}{D}\frac{p_{mn}}{\gamma_{mn}{}^4}\left[1+\frac{(13,16)}{10(1,2)}\left(\gamma_{mn}\frac{h}{2}\right)^2-\frac{(297,454)}{4200(1,2)}\left(\gamma_{mn}\frac{h}{2}\right)^4+\cdots\right],$$
$$(m, n=1, 2, 3, \cdots) \qquad (2\text{-}12)$$

and truncate the series at the terms of $O(h^{2n})$, we can obtain the solution of equation in the n-th order approximation.

§ 3. Solution of Equation of Deflection for a Simply Supported Circular Thick Plate under Uniform Pressure

Let us take cylindrical coordinates $r=\sqrt{x_1^2+x_2^2}$, $\theta=\mathrm{arctg}\,(x_2/x_1)$, and $z=x_3$. Then, Laplacian operator Δ reads:

$$\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$

which appears in eqs. (1-6)∼(1-8), (1-16), (1-18), (1-20), etc.

Components of displacement and components of stress are to be written also in cylindrical coordinates:

$$\xi_r=\xi_1\cos\theta+\xi_2\sin\theta, \qquad \xi_\theta=-\xi_1\sin\theta+\xi_2\cos\theta, \qquad \xi_z=\xi_3,$$
$$A_{rr}=\lambda\varepsilon_{kk}+2\mu(\partial\xi_r/\partial r), \quad \text{etc.}$$

with

$$\varepsilon_{kk}=\frac{1}{r}\frac{\partial(r\xi_r)}{\partial r}+\frac{1}{r}\frac{\partial\xi_\theta}{\partial\theta}+\frac{\partial\xi_z}{\partial z}.$$

We shall solve eqs. (1-16), (1-18), and (1-20) for a simply supported circular plate under *uniform* pressure p_0 over the surface of the plate. Accordingly we have:

$$\xi_\theta=0, \qquad \frac{\partial}{\partial\theta}=0, \qquad \text{and} \quad \Delta=\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right).$$

Let the circular plate occupy the region: $0\leqslant r\leqslant a$, and be simply supported at $r=a$. As for the boundary conditions, we shall take:

$$\xi_z=0, \qquad \text{and} \quad M_r\equiv\int_{-\frac{h}{2}}^{\frac{h}{2}}A_{rr}z\,dz=0, \qquad \text{at} \quad r=a \qquad (3\text{-}1)$$

and let us start from the zero-th order approximation.

A) Zero-th Order Approximation

The solution $w^{(0)}$ of equation for deflection in the zero-th order approximation (1-16), is expressed by a particular solution of (1-16), say $w_{\mathrm{I}}^{(0)}$, plus a biharmonic function $w_{\mathrm{II}}^{(0)}$, namely

$$w^{(0)} = w_{\mathrm{I}}^{(0)} + w_{\mathrm{II}}^{(0)} , \tag{3-2}$$

with

$$D\Delta\Delta w_{\mathrm{I}}^{(0)} = p_0 , \tag{3-3}$$

and

$$\Delta\Delta w_{\mathrm{II}}^{(0)} = 0 . \tag{3-4}$$

The solution of eq. (3-3) is given by:

$$w_{\mathrm{I}}^{(0)} = \frac{1}{64}\frac{p_0}{D} r^4 . \tag{3-5}$$

While, the solution of eq. (3-4), which is finite at $r=0$, reads

$$w_{\mathrm{II}}^{(0)} = C_1 r^2 + C_0 , \tag{3-6}$$

with constants C_1 and C_0. Determining C_1 and C_0, so as to satisfy the boundary conditions (3-1) for $w^{(0)}$, we obtain the solution of eq. (1-16):

$$w^{(0)} = w_{\mathrm{I}}^{(0)} + w_{\mathrm{II}}^{(0)} = \frac{1}{64}\frac{p_0}{D}(a^2 - r^2)\left\{\frac{(11,10)}{(3,2)} a^2 - r^2\right\} . \tag{3-7}$$

Expression (3-7) is nothing but the usual solution for a circular thin plate.

B) First Order Approximation

The solution $w^{(1)}$ of eq. (1-18) for deflection in the first order approximation is the sum of a particular solution of (1-18), say $w_1^{(1)}$, and the solution $w_2^{(1)}$ of the homogeneous equation for eq. (1-18), namely

$$w^{(1)} = w_1^{(1)} + w_2^{(1)} , \tag{3-8}$$

with

$$D\Delta\Delta\left\{w_1^{(1)} + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta w_1^{(1)}\right\} = p_0 , \tag{3-9}$$

and

$$\Delta\Delta\left\{1 + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta\right\} w_2^{(1)} = \left\{1 + \frac{(13,16)}{10(1,2)}\left(\frac{h}{2}\right)^2 \Delta\right\} \Delta\Delta w_2^{(1)} = 0 . \tag{3-10}$$

The solution $w_2^{(1)}$ of eq. (3-10) is decomposed into two parts, namely

$$w_2^{(1)} = w_{\mathrm{II}}^{(1)} + w_{\mathrm{III}}^{(1)} , \tag{3-11}$$

where

$$\Delta\Delta w_{\mathrm{II}}^{(1)} = 0 , \tag{3-12}$$

and

$$\Delta w_{\mathrm{III}}^{(1)} + \alpha^2 w_{\mathrm{III}}^{(1)} = 0 , \tag{3-13}$$

with

$$\alpha^2=\frac{10(1,2)}{(13,16)}\left(\frac{2}{h}\right)^2.$$

The solutions of eqs. (3-9) and (3-12) are:

$$w_{\mathrm{I}}^{(1)}=\frac{1}{64}\frac{p_0}{D}r^4, \tag{3-14}$$

and

$$w_{\mathrm{II}}^{(1)}=C_1r^2+C_0, \tag{3-15}$$

with constants C_1 and C_0. While, the solution of eq. (3-13) is expressed by a Bessel function of order zero and is written as:

$$w_{\mathrm{III}}^{(1)}=AJ_0(\alpha r), \tag{3-16}$$

with a constant A.

Accordingly, the solution of eq. (1-18) under the boundary conditions (3-1) can be written by:

$$w^{(1)}=w_{\mathrm{I}}^{(1)}+w_{\mathrm{II}}^{(1)}+w_{\mathrm{III}}^{(1)}=\frac{1}{64}\frac{p_0}{D}r^4+C_1r^2+C_0+AJ_0(\alpha r), \tag{3-17}$$

with constants A, C_1, and C_0, expressed as follows:

$$A=-\frac{(13,16)}{80(3,2)}a^4\frac{p_0}{D}\left(\frac{h}{2a}\right)^2\frac{1+\dfrac{4(35,66,32)}{5(1,2)^2}\left(\dfrac{h}{2a}\right)^2}{J_0(\alpha a)-\dfrac{4(1,0)}{5(3,2)}\alpha a\left(\dfrac{h}{2a}\right)^2J_1(\alpha a)}, \tag{3-18}$$

$$C_1=-\frac{1}{16}\frac{p_0}{D}a^2+\frac{1}{4}\alpha^2AJ_0(\alpha a), \tag{3-19}$$

and

$$C_0=\frac{3}{64}\frac{p_0}{D}a^4-\left(1+\frac{\alpha^2a^2}{4}\right)AJ_0(\alpha a). \tag{3-20}$$

Expression (3-17) obtained here has a somewhat different feature from a solution given by Love[4]:

$$w=\frac{1}{64}\frac{p_0}{D}(a^2-r^2)\left\{\frac{(11,10)}{(3,2)}a^2-r^2+\frac{8(35,66,32)}{5(1,2)(3,2)}\left(\frac{h}{2}\right)^2\right\}, \tag{3-21}$$

where he took another boundary condition. Eq. (3-21), however, has a similar expression to (3-7) and also contains a term of $O(h^2)$.

C) *Second Order Approximation*

In a similar manner, the solution $w^{(2)}$ of eq. (1-20) in the second order approximation can be decomposed into three terms, namely

$$w^{(2)}=w_{\mathrm{I}}^{(2)}+w_{\mathrm{II}}^{(2)}+w_{\mathrm{III}}^{(2)}, \tag{3-22}$$

with

$$D\Delta\Delta\left\{w_{\mathrm{I}}^{(2)}+\frac{2\alpha_2^2}{\beta_2^4}\Delta w_{\mathrm{I}}^{(2)}+\frac{1}{\beta_2^4}\Delta\Delta w_{\mathrm{I}}^{(2)}\right\}=p_0, \tag{3-23}$$

$$\Delta\Delta w_{\mathrm{II}}^{(2)}=0\,, \tag{3-24}$$

and

$$\Delta\Delta w_{\mathrm{III}}^{(2)}+2\alpha_2^2\Delta w_{\mathrm{III}}^{(2)}+\beta_2^4 w_{\mathrm{III}}^{(2)}=0\,, \tag{3-25}$$

with

$$\alpha_2{}^2=\frac{42(1,2)(13,16)}{(1479,3704,2332)}\left(\frac{2}{h}\right)^2, \qquad \text{and} \quad \beta_2{}^4=\frac{840(1,2)^2}{(1479,3704,2332)}\left(\frac{2}{h}\right)^4.$$

The solutions of eqs. (3-23) and (3-24) are given as:

$$w_{\mathrm{I}}^{(2)}=\frac{1}{64}\frac{p_0}{D}r^4\,, \tag{3-26}$$

and

$$w_{\mathrm{II}}^{(2)}=C_1 r^2+C_0\,, \tag{3-27}$$

with constants C_1 and C_0. While, the finite solution of (3-25) at $r=0$ reads:

$$w_{\mathrm{III}}^{(2)}=A_1\,\mathfrak{Re}\,I_0(\kappa r)+B_1\,\mathfrak{Im}\,I_0(\kappa r)\,, \tag{3-28}$$

where $I_0(\kappa r)$ is a modified Bessel function of order zero, with constants A_1, and B_1. We wrote one of the roots of the equation:

$$\kappa^4+2\alpha_2{}^2\kappa^2+\beta_2{}^4=0\,, \tag{3-29}$$

to be $\kappa=\zeta+i\eta$ $\quad(\zeta\geqslant 0$ and $\eta\geqslant 0)$,
with

$$\zeta=\sqrt{\frac{\beta_2{}^2-\alpha_2{}^2}{2}}\,, \qquad \text{and} \quad \eta=\sqrt{\frac{\beta_2{}^2+\alpha_2{}^2}{2}}\,. \tag{3-30}$$

The solution $w^{(2)}$ of eq. (1-20) under the boundary conditions (3-1) can be written as:

$$\begin{aligned} w^{(2)} &= w_{\mathrm{I}}^{(2)}+w_{\mathrm{II}}^{(2)}+w_{\mathrm{III}}^{(2)}= \\ &=\frac{1}{64}\frac{p_0}{D}r^4+C_1r^2+C_0+A_1\,\mathfrak{Re}\,I_0(\kappa r)+B_1\,\mathfrak{Im}\,I_0(\kappa r)\,, \end{aligned} \tag{3-31}$$

with

$$A_1=\frac{p_0}{D}a^4\frac{\mathfrak{Im}\,L(\kappa a)-\left\{\dfrac{1}{8}+\dfrac{(17,16)}{10(1,2)}\left(\dfrac{h}{2a}\right)^2\right\}\mathfrak{Im}\,\{(\kappa a)^4 I_0(\kappa a)\}}{\mathfrak{Re}\,L(\kappa a)\,\mathfrak{Im}\,\{(\kappa a)^4 I_0(\kappa a)\}-\mathfrak{Im}\,L(\kappa a)\,\mathfrak{Re}\,\{(\kappa a)^4 I_0(\kappa a)\}}\,, \tag{3-32}$$

$$B_1=-\frac{p_0}{D}a^4\frac{\mathfrak{Re}\,L(\kappa a)-\left\{\dfrac{1}{8}+\dfrac{(17,16)}{10(1,2)}\left(\dfrac{h}{2a}\right)^2\right\}\mathfrak{Re}\,\{(\kappa a)^4 I_0(\kappa a)\}}{\mathfrak{Re}\,L(\kappa a)\,\mathfrak{Im}\,\{(\kappa a)^4 I_0(\kappa a)\}-\mathfrak{Im}\,L(\kappa a)\,\mathfrak{Re}\,\{(\kappa a)^4 I_0(\kappa a)\}}\,, \tag{3-33}$$

$$C_1=-\frac{1}{16}\frac{p_0}{D}a^2-\frac{1}{4a^2}\Big[A_1\,\mathfrak{Re}\big\{(\kappa a)^2 I_0(\kappa a)\big\}+B_1\,\mathfrak{Im}\big\{(\kappa a)^2 I_0(\kappa a)\big\}\Big], \tag{3-34}$$

and

$$C_0 = -\frac{1}{64}\frac{p_0}{D}a^4 - C_1 a^2 - A_1 \Re e\, I_0(\kappa a) - B_1 \Im m\, I_0(\kappa a)\,, \tag{3-35}$$

where

$$L(\kappa a) = (\kappa a)^2 I_0(\kappa a) - 2(\kappa a)\, I_1(\kappa a) + \frac{(17,16)}{5(1,2)}\left(\frac{h}{2a}\right)^2 \left\{(\kappa a)^4 I_0(\kappa a) - (\kappa a)^3 I_1(\kappa a)\right\} +$$
$$+ \frac{(1479, 3704, 2332)}{420(1,2)^2}\left(\frac{h}{2a}\right)^4 \left\{\frac{2(1,1)}{(1,2)}(\kappa a)^6 I_0(\kappa a) - \right.$$
$$\left. - \frac{(1919, 4248, 2332)}{(1479, 3704, 2332)}(\kappa a)^5 I_1(\kappa a)\right\}. \tag{3-36}$$

§ 4. Numerical Example

As for numerical examples, we shall consider the maximum deflection of a plate under uniform pressure calculated from solutions of several approximate equations obtained in this paper.

A) Rectangular Plate

Uniform pressure $p = p_0$ ($=$const) is expressed in a double Fourier series in (2-7), giving

$$p_{mn} = \frac{2^4}{\pi^2}\frac{(-1)^{m+n}}{(2m-1)(2n-1)}p_0. \quad (m, n = 1, 2, 3, \cdots) \tag{4-1}$$

The maximum deflection $w_{\max}$ of a rectangular plate is obtained from (2-3) with $x_1 = x_2 = 0$, *i. e.*

$$w_{\max} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\,. \tag{4-2}$$

W_{mn} are given in eq. (2-9) for the exact solution of the plate and in eq. (2-12) for approximate solutions.

For a square plate, *i. e.* $a = b$, we calculate $w_{\max}$ for various values of h/a,

TABLE 1. Comparison of the maximum deflections of a simply supported square plate ($a=b$) under uniform pressure.
($\lambda/\mu = 3/2$, and Poisson's ratio $= 0.3$)

h/a	$w_{\max}/w_e$ (w_e : exact solution)		
	0-th order approx. (thin plate)	1st order approx.	Reissner[5)]
0.05	0.989	1.000	1.000
0.10	0.956	1.000	0.998
0.15	0.907	1.000	0.996
0.20	0.846	1.001	0.995
0.25	0.779	1.003	0.993
0.30	0.711	1.005	0.993

after truncating series (4-2) at appropriate terms ($m, n \simeq 5 \sim 6$). In Table 1, the values w_{max}/w_e are shown in the zero-th and the first order approximations in our theory and are compared with the values calculated from Reissner's theory[5]. w_e means the exact solution given by (4-2) with (2-9). The ratio λ/μ is taken to be 3/2, with Poisson's ratio $(\lambda/2)/(\lambda+\mu)=0.3$.

In the zero-th order approximation (*i.e.* thin plate), the relative error $|1-(w_{max}/w_e)|$ is comparatively small (*e.g.* less than 5% for $h/a=0.1$), while, for increasing thickness, the error increases from *ca.* 9% (for $h/a=0.15$) to *ca.* 30% (for $h/a=0.3$). In the first order approximation in our theory, the values of w_{max} agree very well with the exact solution, *e.g.* the error is 0.5% for $h/a=0.3$. In the second order approximation, the values of w_{max} coincides very well with the exact values (*e.g.* the error is less than 0.1% for $h/a \leqslant 0.4$).

B) Circular Plate

The maximum deflection w_{max} of a circular plate under uniform pressure is obtained from eqs. (3-7), (3-17), and (3-31), with $r=0$.

In the zero-th order approximation, we have:

$$w_{max}=\frac{(11,10)}{64(3,2)}\frac{p_0}{D}a^4, \tag{4-3}$$

while the first order approximation gives:

$$w_{max}=\frac{3}{64}\frac{p_0}{D}a^4-\left(1+\frac{\alpha^2 a^2}{4}\right)AJ_0(\alpha a)+A, \tag{4-4}$$

with (3-18). The maximum deflection in the second order approximation reads:

$$\begin{aligned} w_{max}=\frac{3}{64}\frac{p_0}{D}a^4+A_1+\left[\frac{1}{4}\mathfrak{Re}\left\{(\kappa a)^2 I_0(\kappa a)\right\}-\mathfrak{Re}\, I_0(\kappa a)\right]A_1+ \\ +\left[\frac{1}{4}\mathfrak{Im}\left\{(\kappa a)^2 I_0(\kappa a)\right\}-\mathfrak{Im}\, I_0(\kappa a)\right]B_1, \end{aligned} \tag{4-5}$$

with (3-32) and (3-33).

Numerical results calculated from (4-3)$\sim$(4-5) and from Love's solution are compared in Table 2. In this case, the exact solution can not be obtained, and numerical values of w_{max} are shown as a ratio of w_{max} to $w^{(0)}$, for values of $h/2a$. $w^{(0)}$ means the values in the zero-th order approximation (thin plate). The ratio λ/μ is taken to be 3/2 and Poison's ratio 0.3.

For the plate of small thickness, the values $w_{max}/w^{(0)}$ in the first and the second order approximations and also in the theory of Love, do not differ very much. For example, the values of the difference $|1-(w_{max}/w^{(0)})|$ are less than 10% for $h/2a \leqslant 0.1$. The value increases with increasing thickness of the plate.

Although the features of (3-31) in our second order approximation and (3-21) given by Love are quite different from each other, the numerical values given in our second order approximation (4-5) show a very good agreement with those obtained by (3-21) for all the values of $h/2a$. The maximum difference between them is merely less than 7%.

TABLE 2. Comparison of the maximum deflections of a simply supported circular plate under uniform pressure. ($\lambda/\mu=3/2$, and Poisson's ratio$=0.3$)

$h/2a$	$w_{\max}/w^{(0)}$ ($w^{(0)}$: 0-th order approximation)		
	1st order approx.	2nd order approx.	Love[4]
0.05	0.997	1.012	1.009
0.10	1.090	1.047	1.036
0.15	1.016	1.111	1.081
0.20	1.441	1.221	1.145
0.25	1.462	1.314	1.226
0.30	1.683	1.337	1.326

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