Collocation Method for Static and Dynamic Analysis of Shells of Revolution

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Abstract

The static and dynamic analysis of shells of revolution using the collocation method is described. The proposed method in which the collocation points are taken at the roots of orthogonal polynomial is used to discretize the space-variable. The versatility and accuracy are illustrated through several numerical examples. The method appears to be easy to formulate and simple to use.

1. Introduction

The system of equations for the analysis of shells of revolution are, in general, too complicated to be solved exactly, except for a particular shell such as cylindrical tank. It is, therefore, natural that various methods have been developed to obtain approximate solutions.

The procedure adopted here is the collocation method, which is one method of the weighted residual techniques. This method has been modified and improved in recent years, and successfully used in chemical engineering[1]. However, the application of the method to structural mechanics, especially shell structures, is comparatively limited.

The subdomain (partition) method, analogous to the collocation method, has been used to analyze axisymmetric shell problems. This method was used by Langhaar, et al[2]. for a bending problem of shells of revolution. Subsequently, a similar method was used by several authors[3,4] to analyze a stability and static problems of axisymmetric shells. However, to the lest of the writer's knowledge applications to dynamic response problems are not available.

The main objective of this paper is to show an application of the collocation method to the static and dynamic analysis of shells of revolution, and to present some important features of this method. In the present analysis, the collocation method is used to replace the field equations by a set of ordinary differential equations in time. The resulting equations are solved by a direct time integration.

To demonstrate the applicability of the method, several example problems are presented, and the results are compared with other proposed solution techniques. This comparison shows that the method yields very good results with relatively coarse discretization patterns.

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2. Basic Equations of Shell

Let the shell be divided into N elements of arbitrary meridional length [Fig. 1(b)]. Denote a point on the meridional coordinate by i, where i varies from 1 to $N+1$. The top of the shell is at point 1 and the base is point $N+1$. The local coordinates (x, y, z) are shown in Fig, 1 (a), and x is measured from the top of the i th element as shown in Fig. 1 (b).

Basic equations of the shell will be based on the linear theory given by Novozhilov[5].

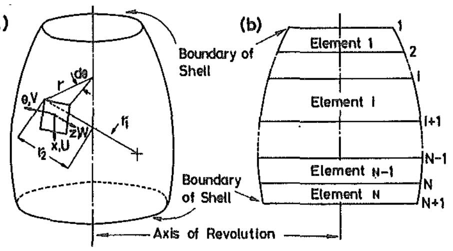

Fig. 1 (a) Typical shell of revolution ; (b) Divition of shell into N elements.

Equations of Motion. —— The present analysis consists of a system of fourth second-order linear differential equations. Therefore, the three displacement components (U, V, W) and the meridional moment M_x are used as dependent variables.

Neglecting the rotary inertia terms, the governing equations of arbitary shells of revolution are,

$$
\begin{aligned}
&\frac{\partial(rN_x)}{\partial x}+\frac{\partial N_{x\theta}}{\partial\theta}-N_\theta\frac{dr}{dx}-\frac{1}{r_1}\left[\frac{\partial(rM_x)}{\partial x}+\frac{\partial M_{x\theta}}{\partial\theta}-M_\theta\frac{dr}{dx}\right]+rq_x\\
&=r\rho h\frac{\partial^2 U}{\partial t^2}\,;\ \frac{\partial(rN_{x\theta})}{\partial x}+\frac{\partial N_\theta}{\partial\theta}+N_{\theta x}\frac{dr}{dx}-\frac{1}{r_2}\left[\frac{\partial M_\theta}{\partial\theta}+\frac{\partial(rM_{x\theta})}{\partial\theta}\right.\\
&\left.+M_{\theta x}\frac{dr}{dx}\right]+rq_\theta=r\rho h\frac{\partial^2 V}{\partial t^2}\,;\ r\left(\frac{N_x}{r_1}+\frac{N_\theta}{r_2}\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{\partial M_\theta}{\partial\theta}\right.\\
&\left.+\frac{\partial(rM_{x\theta})}{\partial x}+M_{\theta x}\frac{dr}{dx}\right]+\frac{\partial}{\partial x}\left[\frac{\partial(rM_x)}{\partial x}+\frac{\partial M_{\theta x}}{\partial\theta}-M_\theta\frac{dr}{dx}\right]\\
&+rq_z=r\rho h\frac{\partial^2 W}{\partial t^2} \qquad\cdots\cdots (1)
\end{aligned}
$$

in which h=shell thickness ; ρ=mass density ; t=time ; N_x, N_θ, $N_{x\theta}$ $N_{\theta x}$=membrane stress resultants ; M_x, M_θ, $M_{x\theta}$, $M_{\theta x}$=moment stress resultants ; q_x, q_θ, q_z=external loads ; r=distance of point on the middle surface of the shell from the axis of revolution [Fig. 1 (a)] ; and r_1, r_2=principal radii of the curvature of the middle surface of the shell [Fig. 1 (a)] .

The stress resultants can be expressed as

$$
\begin{aligned}
&N_x=K(\varepsilon_x+\nu\varepsilon_\theta)\,;\ N_\theta=K(\varepsilon_\theta+\nu\varepsilon_x)\,;\ N_{x\theta}=N_{\theta x}=Gh\gamma_{x\theta}\,;\\
&M_x=-D(\kappa_x+\nu\kappa_\theta)\,;M_\theta=-D(\kappa_\theta+\nu\kappa_x)\,;M_{x\theta}=M_{\theta x}=-D(1-\nu)\kappa_{x\theta}\ \cdots\cdots\ (2)
\end{aligned}
$$

in which ν=Poisson's ratio ; G=shear modulus ; K=extensional rigidity ; D=bending rigidity ; ε_x, ε_θ, $\gamma_{x\theta}$ =strains ; and κ_x, κ_θ, $\kappa_{x\theta}$=curvatures.

The strains and curvatures are given by

$$\varepsilon_x=\frac{\partial U}{\partial x}-\frac{W}{r_1};\ \varepsilon_\theta=\frac{1}{r}\frac{\partial V}{\partial\theta}+\frac{U}{r}\frac{dr}{dx}-\frac{W}{r_2};\ \gamma_{x\theta}=\frac{1}{r}\frac{\partial U}{\partial\theta}+\frac{\partial V}{\partial x}$$
$$-\frac{V}{r}\frac{dr}{dx};\ \kappa_x=\frac{\partial}{\partial x}\left(\frac{U}{r_1}+\frac{\partial W}{\partial x}\right);\ \kappa_\theta=\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{V}{r_2}-\frac{\partial W}{\partial\theta}\right)+\frac{1}{r}\left(\frac{U}{r_1}\right.$$
$$\left.+\frac{\partial W}{\partial x}\right)\frac{dr}{dx};\ 2\kappa_{x\theta}=\frac{2}{r}\left(\frac{\partial^2 W}{\partial x\partial\theta}+\frac{1}{r_1}\frac{\partial U}{\partial\theta}-\frac{1}{r}\frac{dr}{dx}\frac{\partial W}{\partial\theta}+\frac{1}{r}\frac{\partial W}{\partial x}\right)$$
$$-V\left(\frac{1}{r_2^2}\frac{dr_2}{dx}+\frac{1}{rr_2}\frac{dr}{dx}+\frac{1}{rr_1}\frac{dr}{dx}\right)\quad\cdots\cdots(3)$$

The displacements, stress resultants, and loads will be expanded in a Fourier series in the circumferential direction. Let a and σ be a reference of length and stress, respectively. Then we have

$$(U,V,W)=\frac{\sigma a^2}{Eh}\sum_{n=0}^{\infty}(u\cos n\theta,\ v\sin n\theta,\ w\cos n\theta);$$

$$(N_x,N_\theta,N_{x\theta})=\sigma a\sum_{n=0}^{\infty}(n_x\cos n\theta,\ n_\theta\sin n\theta,\ n_{x\theta}\cos n\theta);$$

$$(M_x,M_\theta,M_{x\theta})=\sigma a^2\sum_{n=0}^{\infty}(m_x\cos n\theta,\ m_\theta\sin n\theta,\ m_{x\theta}\cos n\theta);$$

$$(q_x,q_\theta,q_z)=\sigma\sum_{n=0}^{\infty}(p_x\cos n\theta,\ p_\theta\sin n\theta,\ p_z\cos n\theta)\quad\cdots\cdots(4)$$

in which E=Young's modulus ; and n=harmonic number. In Eq. 4 the superscript (n) on the Fourier coefficients has been omitted for brevity.

If the stress resultants are expressed in terms of the chosen dependent variables, Eq. 1 becomes

$$b_1u''+b_2u'+b_3u+b_4u'+b_5v+b_6w'+b_7w+b_8m_x'+b_9m_x$$
$$=-p_x+\ddot{u}\quad\cdots\cdots(5a)$$
$$b_{10}u'+b_{11}u+b_{12}v''+b_{13}v'+b_{14}v+b_{15}w''+b_{16}w'+b_{17}w$$
$$+b_{18}m_x=-p_\theta+\ddot{v}\quad\cdots\cdots(5b)$$
$$b_{19}v'+b_{20}v+b_{21}u''+b_{22}u'+b_{23}v+b_{24}w''+b_{25}w'+b_{26}w$$
$$+b_{27}m_x''+b_{28}m_x'+b_{29}m_x=-p_z+\ddot{w}\quad\cdots\cdots(5c)$$

in which $(\ )'=\partial(\ )/\partial\xi$; $(\dot{\ })=\partial(\ )/\partial\tau$; $\tau=(E/\rho)^{1/2}t/a$ =the nondimensional time variable ; and $\xi=x/l$=the nondimensional space variable. Furthermore, l=the meridional length of the element, and ξ takes the values 0 to 1 in the element.

The meridional moment-displacement relation is taken as the fourth equation. Then this relation can be expressed as

$$b_{30}u'+b_{31}u+b_{32}v+b_{33}w''+b_{34}w'+b_{35}w+b_{36}m_x=0\quad\cdots\cdots(5d)$$

Detailed expressions[6] for the coefficients b_1, ..., b_{36} in Eqs. 5 have been omitted here for brevity.

Boundary Conditions and Continuity Conditions.——In general, the quantities which appear in the boundary conditions and the continuity conditions are the generalized displacements U, V, W, and Φ_x and the generalized forces N_x, M_x, S_x, and T_x. The quantities S_x, T_x, and Φ_x are the effective shear resultants and the rotation defined as

$$T_x = Q_x + \frac{1}{r}\frac{\partial M_{x\theta}}{\partial \theta}\ ;\ \ S_x = N_{x\theta} - \frac{M_{x\theta}}{r_2}\ ;\ \ \Phi_x = \frac{U}{r_1} + \frac{\partial W}{\partial x} \qquad \cdots\cdots(6)$$

in which $Q_x=(1/r)[\partial(rM_x)/\partial x+\partial M_{\theta x}/\partial\theta-M_\theta\, dr/dx]$=the shear stress resultant. Let us express T_x, S_x and Φ_x as

$$(T_x, S_x) = \sigma a \sum_{n=0}^{\infty} (t_x \cos n\theta,\ s_x \sin n\theta)\ ;\ \ \Phi_x = \frac{\sigma a}{Eh}\sum_{n=0}^{\infty} \phi_x \cos n\theta \qquad \cdots\cdots(7)$$

As done before, the Fourier coefficents for the forces N_x, S_x, T_x and the rotation Φ_x can be expressed in terms of u, v, w, and m_x as follows :

$$n_x=c_1 u'+c_2 u+c_3 v+c_4 w;\ \ s_x=c_5 u+c_6 v'+c_7 v+c_8 w'$$

$$+c_9 w;\ \ t_x=c_{10} u+c_{11} v'+c_{12} v+c_{13} w'+c_{14} w+c_{15} m_x'$$

$$+c_{16} m_x;\ \ \phi_x=c_{17} u+c_{18} w' \qquad \cdots\cdots(8)$$

in which c_1, ..., c_{18}=the coefficients ; and detailed expressions[6] for these coefficients have been omitted here for brevity.

For the boundary conditions at the points 1 and $N+1$ [Fig. 1 (b)] , we will prescribe the appropriate four of the quantities U, V, W, Φ_x, N_x, M_x, S_x, and T_x. Thus, the following quantities are to be prescribed at each boundary.

$$U \text{ or } N_x\ ;\ V \text{ or } S_x\ ;\ W \text{ or } T_x\ ;\ \Phi_x \text{ or } M_x \qquad \cdots\cdots(9)$$

In the point between adjacent elements of the shell, the continuity conditions are

$$U^{(i-1)} = U^{(i)};\ \ V^{(i-1)} = V^{(i)};\ \ W^{(i-1)} = W^{(i)};\ \ \Phi_x^{(i-1)} = \Phi_x^{(i)};$$

$$N_x^{(i-1)} = N_x^{(i)};\ \ M_x^{(i-1)} = M_x^{(i)};\ \ S_x^{(i-1)} = S_x^{(i)};\ \ T_x^{(i-1)} = T_x^{(i)} \qquad \cdots\cdots(10)$$

in which i=2, 3, ..., N ; and the $i-1$ and i superscripts denote values for the $i-1$ and i elements, respectively.

3. Collocation Method

The basic procedure is as follows. If we assume that there is a problem described by second differential equation with respect to time τ and space ξ we have

$$Lu\ (\xi, \tau)+\ddot{u}\ (\xi, \tau)=0\ ;\ 0<\xi<1 \qquad \cdots\cdots(11)$$

with two boundary conditions. In above equation, L is a linear operator, and $(\dot{\ })$ denotes $\partial(\ )/\partial\tau$. We, for example, seek an approximate solution of the form

$$u(\xi,\ \tau) = \sum_{i=1}^{M+2} d_{i-1}(\tau)\cdot\xi^{i-1} \qquad \cdots\cdots(12)$$

in which d_{i-1} are unknown functions of time. Substituting Eq. 12 into Eq. 11 the residual $R\ (\xi, \tau)$ becomes

$$R(\xi,\ \tau) = d_{i-1}(\tau) \sum_{i=1}^{M+2} L(\xi^{i-1}) + \ddot{d}_{i-1}(\tau) \sum_{i=1}^{M+2} \xi^{i-1} \quad \cdots\cdots(13)$$

The residual is eliminated by setting $R\ (\xi,\ \tau)=0$ at M specified collocation points, ξ_j ($j=1, 2, ..., M$) and a set of M ordinary differential equations are obtained. Then, with two boundary conditions, $M+2$ equations are obtained for the $M+2$ unknowns.

The roots of the orthogonal polynomial are frequently used as collocation points with considerable success. In this paper, the collocation points, ξ_j ($j=1, 2, ..., M$), are selected to be zeros of the Mth shifted Legendre polynomial $P^*_M\ (\xi)$ defined on $0 \leq \xi \leq 1$. These M points, ξ_j, are called the interior collocation points.

It seems convenient to write the ordinary differential equations in terms of $u\ (\xi_j,\ \tau)$, $j=0, 1, ..., M+1$, i. e., the solution of the interior collocation points (ξ_j, $j=1, ..., M$) and two end points ($\xi_0=0$, $\xi_{M+1}=1$) rather than the τ-dependent unknowns $d_{i-1}\ (\tau)$. To this end, two matrices $[A]$ and $[B]$ given by Eq. 20 are used to approximate the first and second derivatives with respect to ξ, respectively. From Eq. 12 it follows that

$$u(\xi_j)_\tau = u(\xi_j,\ \tau) = \sum_{i=1}^{M+2} d_{i-1}(\tau)\, \xi_j^{i-1}\ ; \quad \left.\frac{\partial u(\xi,\ \tau)}{\partial \xi}\right|_{\xi_j} = \sum_{i=1}^{M+2} \left.\frac{\partial \xi^{i-1}}{\partial \xi}\right|_{\xi_j} d_{i-1}(\tau)\ ;$$

$$\left.\frac{\partial^2 u(\xi,\ \tau)}{\partial \xi^2}\right|_{\xi_j} = \sum_{i=1}^{M+2} \left.\frac{\partial^2 \xi^{i-1}}{\partial \xi^2}\right|_{\xi_j} d_{i-1}(\tau)\ ; \quad j = 0, 1, \cdots, M+1 \quad \cdots\cdots(14)$$

These can be rewritten in the matrix notation as follows :

$$\{u\}_\tau = [Q]\{d\}\ ; \quad \{u'\}_\tau = [C]\{d\}\ ; \quad \{u''\}_\tau = [D]\{d\} \quad \cdots\cdots(15)$$

in which the derivative with respect to ξ is denoted by primes. In above equations, the $(M+2)$ - dimensional vectors $\{d\}$, $\{u\}_\tau, \{u'\}_\tau$, and $\{u''\}_\tau$ are

$$\{d\} = \{d_0(\tau),\ d_1(\tau), \cdots,\ d_{M+1}(\tau)\}^T \quad \cdots\cdots(16)$$

and $\{u\}_\tau = \{u(\xi_0)_\tau, \cdots, u(\xi_{M+1})_\tau\}^T\ ; \quad \{u'\}_\tau = \{u'(\xi_0)_\tau, \cdots, u'(\xi_{M+1})_\tau\}^T\ ;$

$$\{u''\}_\tau = \{u''(\xi_0)_\tau, \cdots, u''(\xi_{M+1})_\tau\}^T \quad \cdots\cdots(17)$$

The $(M+2)\times(M+2)$ matrices $[Q]$, $[C]$, and $[D]$ have the following components :

$$Q_{ji} = \xi_{j-1}^{i-1}\ ; \quad C_{ji} = (i-1)\, \xi_{j-1}^{i-2}\ ; \quad D_{ji} = (i-1)(i-2)\, \xi_{j-1}^{i-3} \quad \cdots\cdots(18)$$

in which $j,\ i=1, 2, ..., M+2$.

Solving for of $\{d\}$ in Eq. 15 the resulting expressions are

$$\{u'\}_\tau = [A]\{u\}_\tau\ ; \quad \{u''\}_\tau = [B]\{u\}_\tau \quad \cdots\cdots(19)$$

in which the $(M+2)\times(M+2)$ matrices $[A]$ and $[B]$, respectively, are given by

$$[A] = [C][Q]^{-1}\ ; \quad [B] = [D][Q]^{-1} \quad \cdots\cdots(20)$$

In the next section, using the matrices $[A]$ and $[B]$ the formulation for the static and dynamic problems of shells of revolution is presented.

4. Formulation for Shell

If we assume that the shell is divided into N elements [Fig. 1 (b)] , the solutions for the kth element are taken in the following forms :

$$[u^{(k)},\ v^{(k)},\ w^{(k)},\ m_x^{(k)}] = \sum_{i=1}^{M+2} [d_{i-1}^{(k)}(\tau),\ e_{i-1}^{(k)}(\tau),\ f_{i-1}^{(k)}(\tau),$$

$$g_{i-1}^{(k)}(\tau)]\ \xi_j^{i-1};\ j=0, 1, \cdots, M+1 \quad \cdots\cdots(21)$$

in which d, e, f, and g are unknown functions of time. Following the procedure as described previously, the expressions representing the first and second derivatives of Eq. 21, say $w^{(k)}$, are given by the same formulas [Eq. 19] .

$$\{w'^{(k)}\} = [A]\{w^{(k)}\};\ \{w''^{(k)}\} = [B]\{w^{(k)}\} \quad \cdots\cdots(22)$$

in which the matrices $[A]$ and $[B]$ are given by Eq. 20, and the vectors $\{w^{(k)}\}$, etc. are given by an expressions similar to Eq. 17. For brevity, in Eq. 22 the subscripts with respect to time, τ, are dropped.

There are 4 $(M+2)$ unknowns in Eq. 21. That is, the total number of unknowns is 4 $(M+2)$ N. However, we obtain $4MN$ equations from Eqs. 5. In addition to this, we have eight boundary conditions and 8 $(N-1)$ continuity conditions. Accordingly, we have 4 $(M+2)$ N equations for 4 $(M+2)$ N unknowns.

4MN Equations. —— For the kth element, Eq. 21 has to satisfy the partial differential equations (Eqs. 5) at the interior collocation points. Then, with the help of Eq. 20 this set of partial differential equations is reduced to the following set of ordinary differential equations.

For Eq. 5a, we have

$$\sum_{j=1}^{M+2} [b_1^{(k)}(\xi_i)\, B_{i+1,j} + b_2^{(k)}(\xi_i)\, A_{i+1,j}]\, u^{(k)}(\xi_{j-1}) + b_3^{(k)}(\xi_i)\, u^{(k)}(\xi_i)$$

$$+ \sum_{j=1}^{M+2} b_4^{(k)}(\xi_i)\, A_{i+1,j}\, v^{(k)}(\xi_{j-1}) + b_5^{(k)}(\xi_i)\, v^{(k)}(\xi_i)$$

$$+ \sum_{j=1}^{M+2} b_6^{(k)}(\xi_i)\, A_{i+1,j}\, w^{(k)}(\xi_{j-1}) + b_7^{(k)}(\xi_i)\, w^{(k)}(\xi_i)$$

$$+ \sum_{j=1}^{M+2} b_8^{(k)}(\xi_i)\, A_{i+1,j}\, m_x^{(k)}(\xi_{j-1}) + b_9^{(k)}(\xi_i)\, m_x^{(k)}(\xi_i)$$

$$= -\, p_x^{(k)}(\xi_i) + \dot{u}^{(k)}(\xi_i);\ i = 1, 2, \cdots, M \quad \cdots\cdots(23)$$

in which $A_{i+1,j}$ and $B_{i+1,j}$=components of the matrices $[A]$ and $[B]$, respectively.

A similar expression can be formed from the remaining equations, and the complete set of equations, for the kth element, may be written as

$$[\alpha_c^{(k)}]\{\delta_c^{(k)}\} + [\alpha_e^{(k)}]\{\delta_e^{(k)}\} = -\{p_c^{(k)}\} + [\beta_c^{(k)}]\{\dot{\delta}_c^{(k)}\} \quad \cdots\cdots(24)$$

in which the subscripts c and e are used to represent the interior collocation points and end points, respectively. In Eq. 24, the $4M$-dimensional vectors

$\{\delta_c^{(k)}\}$, $\{p_c^{(k)}\}$, and $\{\dot{\delta}_c^{(k)}\}$ are

$$\{\delta_c^{(k)}\}=\{u^{(k)}(\xi_1),\cdots,u^{(k)}(\xi_M),\ v^{(k)}(\xi_1),\cdots,v^{(k)}(\xi_M),\ w^{(k)}(\xi_1),\cdots,w^{(k)}(\xi_M),\ m_x^{(k)}(\xi_1),$$
$$\cdots,m_x^{(k)}(\xi_M)\}^T;\ \{p_c^{(k)}\}=\{p_x^{(k)}(\xi_1),\cdots,p_x^{(k)}(\xi_M),\ p_\theta^{(k)}(\xi_1),\cdots,p_\theta^{(k)}(\xi_M),\ p_z^{(k)}(\xi_1),\cdots,$$
$$p_z^{(k)}(\xi_M),0,\cdots,0\}^T;\ \{\dot{\delta}_c^{(k)}\}=\{\dot{u}^{(k)}(\xi_1),\cdots,\dot{u}^{(k)}(\xi_M),\ \dot{v}^{(k)}(\xi_1),\cdots,\dot{v}^{(k)}(\xi_M),\ \dot{w}^{(k)}(\xi_1),$$
$$\cdots,\dot{w}^{(k)}(\xi_M),\dot{m}_x^{(k)}(\xi_1),\cdots,\dot{m}_x^{(k)}(\xi_M)\}^T \quad \cdots\cdots(25)$$

and the 8-dimensional vector $\{\delta_e^{(k)}\}$ is

$$\{\delta_e^{(k)}\}=\{u^{(k)}(\xi_0),\ u^{(k)}(\xi_{M+1}),\ v^{(k)}(\xi_0),\ v^{(k)}(\xi_{M+1}),\ w^{(k)}(\xi_0),$$
$$w^{(k)}(\xi_{M+1}),\ m_x^{(k)}(\xi_0),\ m_x^{(k)}(\xi_{M+1})]^T \quad \cdots\cdots(26)$$

The $4M\times4M$ matrix $[\alpha_c^{(k)}]$ and $4M\times8$ matrix $[\alpha_e^{(k)}]$ are composed of the components of the matrices $[A]$ and $[B]$, and the $4M\times4M$ diagonal matrix $[\beta_c^{(k)}]$ may be expressed as

$$[\beta_c^{(k)}]=\mathrm{diag}[\underbrace{1,1,\cdots,1}_{3M},\underbrace{0,\cdots,0}_{M}] \quad \cdots\cdots(27)$$

For the overall shell, we finally arrived at $4MN$ equations

$$[\alpha_c]\{\delta_c\}+[\alpha_e]\{\delta_e\}=-\{p_c\}+[\beta_c]\{\dot{\delta}_c\} \quad \cdots\cdots(28)$$

in which $[\alpha_c]$ and $[\beta_c]$=the $4MN\times4MN$ matrices, respectively ; $[\alpha_e]$=the $4MN\times8N$ matrix ; $\{\delta_c\}$, $\{p_c\}$, and $\{\dot{\delta}_c\}$=the $4MN$-dimensional vectors, respectively ; and $\{\delta_e\}$= the $8N$-dimensional vector. A typical vector and matrix, say $\{\delta_c\}$ and $[\alpha_c]$, are made up from appropriate subvectors and submatrices as follows :

$$\{\delta_c\}=\begin{Bmatrix}\{\delta_c^{(1)}\}\\ \{\delta_c^{(2)}\}\\ \cdot\\ \cdot\\ \cdot\\ \{\delta_c^{(N)}\}\end{Bmatrix};\ [\alpha_c]=\begin{bmatrix}[\alpha_c^{(1)}] & & & [0]\\ & [\alpha_c^{(2)}] & & \\ & & \ddots & \\ [0] & & & [\alpha_c^{(N)}]\end{bmatrix} \quad \cdots\cdots(29)$$

Eight Equations. —— From the boundary conditions at the points 1 and $N+1$ [Fig. 1 (b)], we obtain eight equations. Let us consider the case of a free point 1 that is clamped at point $N+1$.

The boundary conditions at point 1 may be expressed as

$$M_x=0\ ;\ N_x=0\ ;\ S_x=0\ ;\ T_x=0 \quad \cdots\cdots(30)$$

With the help of Eqs. 20 and 8, for example, the first and second equations of Eq. 30 can be written as

$$m_x^{(1)}(\xi_0)=0\ ;\ c_1^{(1)}(\xi_0)\sum_{j=1}^{M+2}A_{1j}\,u^{(1)}(\xi_{j-1})+c_2^{(1)}(\xi_0)\,u^{(1)}(\xi_0)$$

$$+\,c_3^{(1)}(\xi_0)\,v^{(1)}(\xi_0)+c_4^{(1)}(\xi_0)\,w^{(1)}(\xi_0) \quad\cdots\cdots(31)$$

The remaining conditions may be written as an expressions similar to Eq. 31.

The boundary conditions at point $N+1$ are

$$U=0\ ;\ V=0\ ;\ W=0\ ;\ \Phi_x=0 \quad\cdots\cdots(32)$$

Following the same procedure Eq. 32 becomes

$$u^{(N)}(\xi_{M+1})=0\ ;\ v^{(N)}(\xi_{M+1})=0\ ;\ w^{(N)}(\xi_{M+1})=0\ ;$$

$$c_{17}^{(N)}(\xi_{M+1})\,u^{(N)}(\xi_{M+1})+c_{18}^{(N)}(\xi_{M+1})\sum_{j=1}^{M+2}A_{M+2,j}\,w^{(N)}(\xi_{j-1})=0 \quad\cdots\cdots(33)$$

Finally, eight equations can be written in the matrix form as follows :

$$[\gamma_c^{(1)}]_1\{\delta_c^{(1)}\}+[\gamma_e^{(1)}]_1\{\delta_e^{(1)}\}=\{0\}\ \text{at point } 1 \quad\cdots\cdots(34a)$$

$$[\gamma_c^{(N)}]_{N+1}\{\delta_c^{(N)}\}+[\gamma_e^{(N)}]_{N+1}\{\delta_e^{(N)}\}=\{0\}\ \text{at point } N+1 \quad\cdots\cdots(34b)$$

in which $[\gamma_c^{(1)}]_1$ and $[\gamma_c^{(N)}]_{N+1}$=the $4\times 4M$ matrices, respectively ; $[\gamma_e^{(1)}]_1$ and $[\gamma_e^{(N)}]_{N+1}$= the 4×8 matrices, respectively ; and $\{\delta_c^{(\ )}\}$ and $\{\delta_e^{(\ )}\}$ =the $4M$ and 8-dimensional vectors, respectively, with components by Eqs. 25 and 26. In Eqs. 34, the subscripts c and e refer to the interior collocation points and end points, respectively, and the superscript (i), $i=1$, N, is used to represent the ith element. Furthermore, $[\ \ ]_i$ ($i=1, N+1$) denotes the point i on the meridional coordinate.

$8(N-1)$ *Equations.* —— From the continuity conditions at the points i ($=2, \ldots, N$), we have $8(N-1)$ equations. Here consider one of the eight conditions (Eq. 10), say $\Phi_x^{(i-1)}=\Phi_x^{(i)}$. With Eqs. 20 and 8 this condition yields

$$c_{17}^{(i-1)}(\xi_{M+1})\,u^{(i-1)}(\xi_{M+1})+c_{18}^{(i-1)}(\xi_{M+1})\sum_{j=1}^{M+2}A_{M+2,j}\,w^{(i-1)}(\xi_{j-1})$$

$$=c_{17}^{(i)}(\xi_0)\,u^{(i)}(\xi_0)+c_{18}^{(i)}(\xi_0)\sum_{j=1}^{M+2}A_{1,j}\,w^{(i)}(\xi_{j-1}) \quad\cdots\cdots(35)$$

The remaining conditions may be similary formed, and the complete set of equations may be written as an expression similar to Eqs. 34.

$$[\gamma_c^{(i-1)}]_i\{\delta_c^{(i-1)}\}+[\gamma_e^{(i-1)}]_i\{\delta_e^{(i-1)}\}+[\gamma_c^{(i)}]_i\{\delta_c^{(i)}\}$$

$$+\,[\gamma_e^{(i)}]_i\{\delta_e^{(i)}\}=\{0\} \quad\cdots\cdots(36)$$

in which $i=2, 3, \ldots, N$; $[\gamma_c^{(i-1)}]_i$ and $[\gamma_c^{(i)}]_i$=the $8\times 4M$ matrices, respectively ; $[\gamma_e^{(i-1)}]_i$ and $[\gamma_e^{(i)}]_i$=the 8×8 matrices, respectively ; $\{\delta_c^{(i-1)}\}$ and $\{\delta_c^{(i)}\}$ =the $4M$-dimensional vectors, respectively ; and $\{\delta_e^{(i-1)}\}$ and $\{\delta_e^{(i)}\}$ =the 8-dimensional vectors, respectively.

$4(M+2)N$ *Equations.* —— Writing Eqs. 34 and 36 together, as a single matrix equation, we have

$$[\gamma_c]\{\delta_c\} + [\gamma_e]\{\delta_e\} = \{0\} \quad \cdots\cdots(37)$$

in which $[\gamma_c]$ and $[\gamma_e]$ = the $8N \times 4MN$ and $8N \times 8N$ matrices, respectively ; and $\{\delta_c\}$ and $\{\delta_e\}$ = the $4MN$ and $8N$-dimensional vectors, respectively. A typical matrix, say $[\gamma_c]$, is made up from appropriate submatrices as follows :

$$[\gamma_c] = \begin{bmatrix} [\gamma_c^{(1)}]_1 & & & [0] \\ [\gamma_c^{(1)}]_2 & [\gamma_c^{(2)}]_2 & & \\ & \ddots & \ddots & \\ & & [\gamma_c^{(N-1)}]_N & [\gamma_c^{(N)}]_N \\ [0] & & & [\gamma_c^{(N)}]_{N+1} \end{bmatrix} \quad \cdots\cdots(38)$$

Finally, combining Eqs. 28 and 37, we obtain the following matrix equation for any time τ :

$$\begin{bmatrix} [\alpha_c] & [\alpha_e] \\ [\gamma_c] & [\gamma_e] \end{bmatrix} \begin{Bmatrix} \{\delta_c\} \\ \{\delta_e\} \end{Bmatrix}_{,\tau} = -\begin{Bmatrix} \{p_c\} \\ \{0\} \end{Bmatrix}_{,\tau} + \begin{bmatrix} [\beta] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\dot{\delta}_c\} \\ \{0\} \end{Bmatrix}_{,\tau} \quad \cdots\cdots(39)$$

If the matrix $[\gamma_e]$ is nonsigular, the lower half of Eq. 39 can be rearranged

$$\{\delta_e\} = -[\gamma_e]^{-1}[\gamma_c]\{\delta_c\} \quad \cdots\cdots(40)$$

Substituting Eq. 40 into the upper half of Eq. 39, we arrive at the modified equation

$$([\alpha_c] - [\alpha_e][\gamma_e]^{-1}[\gamma_c])_{,\tau}\{\delta_c\} = -\{p_c\}_{,\tau} + [\beta_c]\{\dot{\delta}_c\}_{,\tau} \quad \cdots\cdots(41)$$

For dynamic problems, a time integration scheme can be used to establish an algebraic system of pseudstatic equations from Eqs. 39 and/or 41. On the other hand, Eqs. 39 and/or 41 are reduced to static problems when intertia effects are not considered.

5. Numerical Studies

The versatility and accuracy of the present method are demonstrated by several numerical examples.

(1) Static Analysis

Cylindrical Shell under Liquid Pressure.——A cylindrical shell under liquid pressure is analyzed. Details of the shell are given in Fig. 2. The shell was modeled as one element with five different values of the interior collocation points, M. Fig. 2 shows the results for the normal displacement and the meridional moment. Also shown are numerical results from a closed-form solutions[7].

It is observed that the present solutions are in excellent agreement with the closed-form solutions. Also, it is found that high accuracy can be achieved with a relatively small number of collocation points. Based on the results in this analysis and previous results[8], we used in all following analyses 11 collocation points (i. e. $M=11$) per element.

Deep Spherical Cap.——To examine the applicability of the present method to the

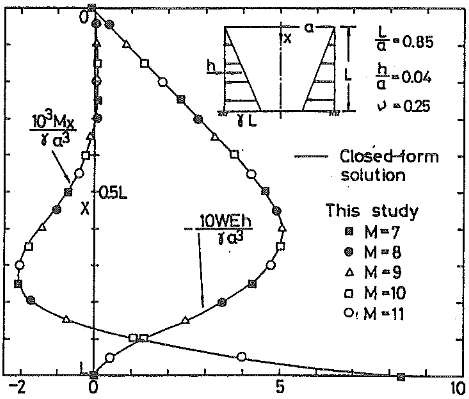

Fig. 2 Normal displacement and meridional bending moment for cylindrical shell under liquid pressure.

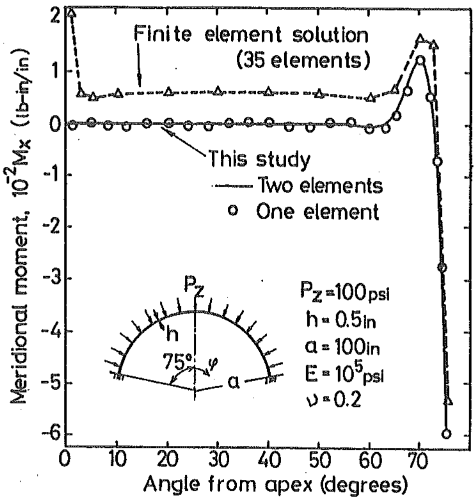

Fig. 3 Meridional bending moment for spherical cap.

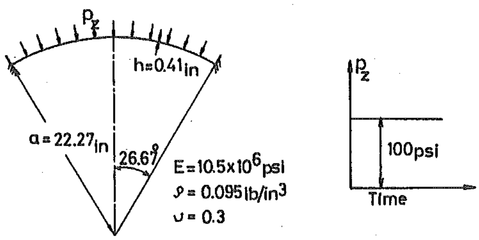

Fig. 4 Spherical cap under pulse loading.

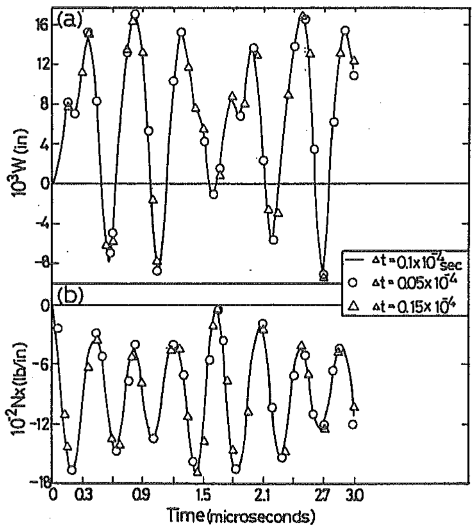

Fig. 5 Dynamic responses of spherical cap ; (a) normal displacement at apex ; (b) meridional stress resultant at clamped end.

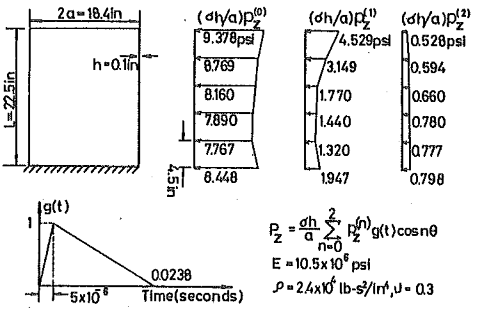

Fig. 6 Cylindrical shell under blast loading.

differential equations with variable coefficients, and to verify the method sensitivity to complicated stress variation, a spherical cap under uniform loading is considered. The dimensions and material properties used are given in Fig. 3.

The same cap has also been discussed by several authors who used conical and curved shell elements, and it was pointed out that the application of the conical shell element to a deep cap yields residual bending moments in an area where membrance forces are predominant.

For the present study, the cap was modeled by two different subdivisions ; one element and two elements with equal meridional length. Fig. 3 shows the distribution of meridional moment. The results using the conical shell element is also shown in Fig. 3 for comparison. As can be seen the one elememt solution agrees reasonably well with the curved element solution whereas the two element solution almost coincides with the curved element solution[9], (which is not shown in Fig. 3).

(2) **Dynamic Analysis**

Shallow Spherical Cap. —— A shallow spherical cap subjected to a pulse loading, as shown in Fig. 4, is considered. The same cap was analyzed by Klein and Sylvester[10] who used a conical shell element, together with an integration scheme of Chan, Cox and Benifield[11].

In the present study, the cap was divided into 2 elements of equal meridional length. The time integration was carried out with the method of Chan et al[11]. and the time step was taken as $\Delta t=10^{-5}$ sec (about $T_0/54$) to correspond timewise to the Ref. 10), where T_0 is the fundamental period of the cap.

The normal displacement at the pole and meridional force at the clamped end are shown in Fig. 5 for a duration 3×10^{-3} sec. The results obtained are identical to the ones presented in Ref. 10). Additional solutions obtained with time steps of $\Delta t=0.15\times10^{-4}$ sec and $\Delta t=0.05\times10^{-4}$ sec showed no appreciable differences in the results.

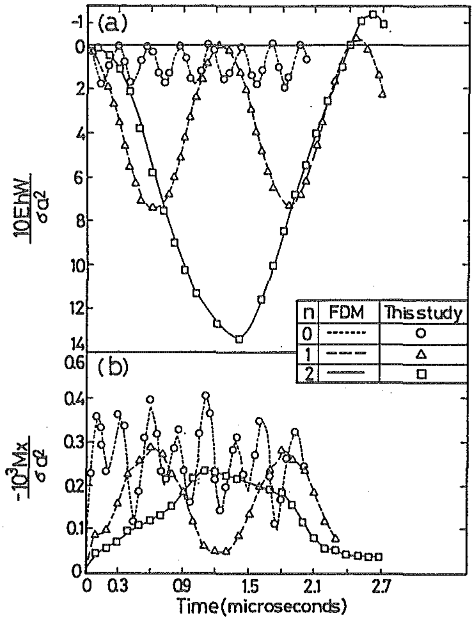

Fig. 7 Dynamic responses of cylindrical shell ; (a) normal displacement at free end ; (b) meridional bending moment at clamped end.

Cylindrical Shell. —— A cylindrical shell shown in Fig. 6 is considered. The shell is subjected to a blast loading which varies in the meridional direction and is expressed by using the Fourier harmonics for $n=0$ through $n=2$.

For the present analysis, the shell was divided into three equal-length elements. Houbolt's solution procedure[12] with the time step of 5×10^{-4} sec (about $T_0/88$) was used, where T_0 equals the fundamental period of the shell.

The normal displacement at the free end and the meridional moment at the clamped end are plotted against time in Fig. 7. The results are compared with those obtained by Johnson and Greif[13], using a finite difference method for the space and employing Houbolt's method for the time. It is noted that the present results using 11 collocation points per element agree reasonably well with those obtained using 90 finite difference stations.

6. Summary and Conclusions

A simple yet an efficient solution scheme utilizing the collocation method was presented for static and dynamic response analysis of shells of revolution. From the numerical examples studied, it is observed that the results obtained are in excellent agreement with those obtained from other analytical and numerical studies. Also, it is found that the proposed method will yield relatively high accuracy using coarse discretitations in space and large increment of time variables.

The method can handle, without special treatment, a variety of edge support conditions. Furthermore, the method may be extended, with the aid of suitable continuity equations, to systems consisting of several shells of revolution of different shapes.

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