Correspondence of Arcs in Spanning Trees and its Application to Finding Shortest Tours

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Abstract

A correspondence of arcs is established for the shortest spanning tree and an arbitrary spanning tree. This correspondence gives some properties to the shortest tour. These properties unifies the ways of proving the shortest tour in a few special cases and gives a new approach to the shortest tour in general cases.

Introduction

Shortest or efficient spanning trees are easily found (Kruskal 1956, Corley 1985), but it is very difficult to obtain the shortest tour (Cook 1970, Karp 1972 and Papadimitrious 1977). No one has presented any geometrical method for finding the shortest tour in the general case from Dantzig et al. (1959) to Padberg (1985), but some researchers such as Held and Karp (1970) suppose that the shortest tour should have close relations with the shortest spanning tree. We are convinced that there are many common arcs for the two. If these common arcs are available, most algorithms for the traveling salesman problem could be improved.

Unfortunately these are unknown until the shortest tour is found, and we have searched for ways of applying the "existance of common arcs" to construct the shortest tour (Sakakibara 1980-1985).

A one-to-one correspondence of arcs is set up for the shortest spanning tree and an arbitrary spanning tree. This correspondence characterizes arcs corresponding to each arc of the shortest spanning tree. This characterization gives some properties of the shortest tour. The ways of proving the shortest tour in some special cases are unified by the properties. And the properties give a new approach to the shortest tour in the general case. In this approach the shortest tour should be obtained quicker when it has more common arcs with the shortest spanning tree.

1. A correspondence of arcs for spanning trees.

Spanning trees in the undirected complete graph G (V, A), a node set V, $|V| = n \geqq 4$ and the arc set $A = \{a\}$, are treated.

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The weight (or length) of each arc is denoted by $\tilde{a}$. An arbitrary spanning tree is represented by an arc set $\{x_k\}=X\subset A$, and the shortest by $\{b_k\}=B\subset A$, where k is an integer $\leqq n-1$. The arc sets C, X^*, and B^* are defined as

$$C=X\cap B,\ X\sim C=X^* \text{ and } B\sim C=B^*,$$

which give

$$x^*\cap B^*=\phi. \quad\cdots\cdots(1)$$

The arcs $c_i\in C$ are called the "common arcs" of the spanning tree and the shortest, where $i = 1, 2, ..., i_0$, and $0\leqq i_0\leqq n-1$. These definitions give

$$B\cup X^*=(C\cup B^*)\ \cup X^*=(C\cup X^*)\cup B^*=X\cup B^*. \quad\cdots\cdots(2)$$

This means that the shortest spanning tree becomes the spanning tree X when interchanging B^* into X^*.

Here a one-to-one correspondence of arcs for an arbitrary spanning tree and the shortest spanning tree is set up as follows:

Proposition 1. Each arc x_k of an arbitrary spanning tree can correspond to an arc in a path of the shortest spanning tree ;
the path connects the ends of the arc x_k.

Proof. This proof has five steps :

(1). If x_k is c_i, let it correspond to the arc equal to c_i in B. The arc c_i is the path itself of the shortest spanning tree which connects the ends of $x_k = c_i$.

(2). If x_k is not c_i for all i, that is, if x_k is $h_j\in X^*$, let it correspond to an are in $B^*=\{t_j\}$ according to j_0 as follows. Here $j=1, 2, ..., j_0$, and $i_0+j_0=n-1$.

(2)−1. $j_0=0$. This case is covered in (1).

(2)−2. $j_0=1$. Here $X^*=\{h_1\}$ and $B^*=\{t_1\}$. The graph for $B\cup\{h_1\}$ has a loop consisting of h_1 and the path of B (the graph of B) which connects the ends of h_1. From Equation (2) the graph becomes a spanning tree when removing t_1, and the graph cannot become a spanning tree without removing an arc from the loop. Therefore, from equation (1), t_1 must be contained in the path of B which connects the ends of h_1. Let the arc h_1 correspond to t_1.

(2)−3. $j_0=m$. In this case it is assumed that each $h_j\in X^*$ corresponds to an arc$\in B^*$ in the path of B which connects the ends of h_j.

(2)−4. $j_0=m+1$. Let h_{m+1} denote an arbitrary arc in X^*. The graph for $B\cup\{h_{m+1}\}$ has a loop consisting of h_{m+1} and the path connecting the ends of h_{m+1} (see (a), (b) and (c) in Fig. 1). From equation (2) the path contains one or more arcs in B^*. One of these arcs joins the two parts of the spanning tree X which are separated by removing h_{m+1} (see (c) and (d) of Fig. 1). The reason is shown in the next paragraph.

Both parts contain the respective ends of h_{m+1} (see solid circles in (d) of Fig. 1). An arc in the path of B which connects the ends of h_{m+1} showld join the two parts. Since the common arcs c_i are contained in the two parts, no common arcs can join the two parts. For these reasons, an arc in B^* and in the path of B showld join the two parts. The arc is defined as t_{m+1}.

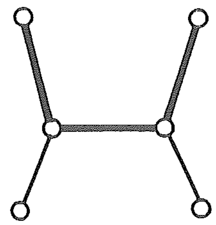
Let h_{m+1} correspond to t_{m+1}. It is clear that the arc t_{m+1} is in the path of the shortest spanning tree which connects the ends of h_{m+1}.

Next, we consider the graph for $(\{t_{m+1}\} \cup X \sim \{h_{m+1}\})$. This graph is a spanning tree (see (d) of Fig. 1), and

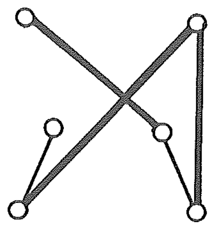
$$\{t_{m+1}\} \cup X \sim \{h_{m+1}\} = (C \cup \{t_{m+1}\}) \cup X^* \sim \{h_{m+1}\}.$$

All the arcs of $(C \cup \{t_{m+1}\})$ are common to the shortest spanning tree. The number of arcs of $(X^* \sim \{h_{m+1}\})$ is m. By the assumption in (2)−3, each arc $h_j \in (X^* \sim \{h_{m+1}\}$ for $j \leqq m$ corresponds to an arc of $(B^* \sim \{t_{m+1}\}$ in the path of B connecting the ends of h_j. The arc of $(B^* \sim \{t_{m+1}\})$ is denoted by t_j. Let h_j correspond to t_j.

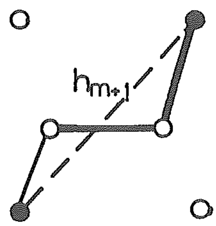
This proves Proposition I by mathematical induction.



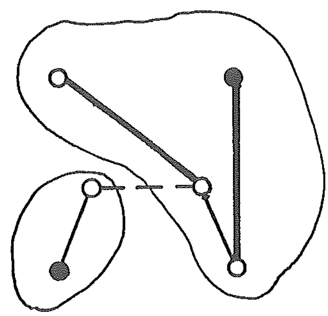
(a)



(b)



(c)



(d)

Fig. 1 An example of a set of nodes.
(a) The shortest spanning tree.
(b) A spanning tree ; common arcs drawn thin, and notcommon arcs drawn thick.
(c) The loop consisting of h_{m+1} and the path in the shortest spanning tree which connects the ends of h_{m+1}. The ends are drawn in black.
(d) The two parts of the spanning tree separated by removing h_{m+1}, and a not common arc in the shortest spanning tree which connects the two parts.

The following theorem is obtained from Proposition 1.

Theorem 1. Each arc of an arbitrary spanning tree corresponds to a not-longer arc in the shortest spanning tree.

Proof. Proposition 1 states that an arc x_k in an arbitrary spanning tree corresponds to b_k in the path of the shortest spanning tree which connects the ends of x_k. Every arc in this path is not longer than x_k, because if the path contains an arc longer than x_k, we can make the shortest spanning tree shorter by interchanging the arc into x_k. Therefore

$$\tilde{b}_k \leqq \tilde{x}_k \quad \text{for all } k \cdots\cdots (3)$$

and

$$\tilde{x}_k = \tilde{b}_k \quad \text{for} \quad x_k = c_l. \cdots\cdots (4)$$

Where the length of a graph, the sum of the lengths of its arcs, is shown by a tilde,

$$0 \leqq \tilde{X} - \tilde{B} = \sum_{k=1}^{n-1} (\tilde{x}_k - \tilde{b}_k) = \sum_{j=1}^{jo} (\tilde{h}_j - \tilde{t}_j). \cdots\cdots (5)$$

This means that the spanning tree is longer than the shortest by $\Sigma_j\ (\tilde{h}_j - \tilde{t}_j)$, and that in general a spanning tree with more common arcs is shorter.

2. Characterizing arcs which can correspond to each b_k

In all arcs (denoted by the arc set A), the above correspondence defines the arcs which can correspond to each arc in the shortest spanning tree. The arc b_k (which corresponds to x_k) is contained in the path of the shortest spanning tree which connects the ends of x_k. Therefore, in all spanning trees, only the arcs with the same ends as the paths of the shortest spanning tree which contain b_k can correspond to b_k.

Such arcs are the ones which can join the two parts of the shortest spanning tree separated by removing b_k. The set of these arcs is denoted by $A^*(k) = \{a_{k,q}\}$ for $q = 1, 2, \ldots, q(k)$ for each k, where $(n-1) \leqq q(k) = \alpha(n-\alpha) \leqq n^2/4$ when one of the two parts of the shortest spanning tree has α nodes. The suffix q is defined as

$$\tilde{a}_{k,q} \leqq \tilde{a}_{k,q'} \quad \text{for} \quad q < q', \cdots\cdots (6)$$

accordingly

$$\tilde{a}_{k,1} = \tilde{b}_k \quad \text{for all } k.$$

An arc $a \in A$ is contained in every $A^*(k)$ for k of all b_k of the path of the shortest spanning tree which connects the ends of the arc a, and such a path exists for each a. Therefore, if $a = b_k$, the arc is contained in only one $A^*(k)$ set for $k = k$. If the shortest spanning tree is a spanning path, the arc joining the ends of the spanning path is contained in every $A^*(k)$ set.

It is clear from the correspondence of arcs in Proposition 1 that the spanning tree has an arc in every $A^*(k)$. The shortest spanning tree consists of the first element in each $A^*(k)$.

Providing tables of $(\tilde{a}_{k,q} - \tilde{b}_k)$ for each k and all q, the set consisting of an arc in each $A^*(k)$ for all k can be constructed in the order of the sum of the lengths of the arcs. Every such set does not represent a spanning tree, but every spanning tree can be represented by such a set. Thus, all spanning trees can be obtained, ordered by length, and they contain all spanning paths in a similar order. The mettod of efficient construction of arc sets which represent spanning paths will be shown in Section 4.

3. Application to find the shortest tour

A tour (through every nodes exactly once) becomes a spanning path when removing an arc, thus the tour contains an arc in A^* (k) for each k. At the same time, since the set A^* (k) consists of all arcs which can join the parts of the shortest spanning tree separated by removing b_k, thus at least two arcs in the tour are contained in A^* (k) for each k. The reason is that if it does not, the tour cannot pass through all nodes (Bellmore and Nemhauser, 1968).

The first element of A^* (k) is contained in no other set, accordingly the tour must have at least one arc not shorter than the maximum of $\{\tilde{a}_{k,2}\}$, where $\{a_{k,2}\}$ consists of the second element of all the A^* $(k)=\{a_{k,q}\}$ sets. Let $a_{k,2}$ denote the arc with the maximum weight of $\{\tilde{a}_{k,2}\}$. The longest arc in A^* (k) of the tour is denoted by $a_{k,p}$, then $2 \leqq p \leqq q$ (k). All tours can be classified by $a_{k,p}$. When the shortest spanning path with the same ends as $a_{k,p}$ is denoted by

P $(k, p)=\{x\ (p)_k\}$ and $x(p)_k \in A^*$ (k) for all k, the shortest tour denoted by D (k, p) which has $a_{k,p}$ is

$$D\ (k,\ p)=\{a_{k,p}\} \cup P\ (k,\ p), \cdots\cdots(7)$$

The shortest of the tours D (k, p) for all p is the shortest of G (V, A). Equation (6), (7) and $2 \leqq p$ give

$$0 \leqq \tilde{D}\ (k,\ p)-(\tilde{a}_{k,2}+\tilde{B}). \cdots\cdots(8)$$

The graph of $(B \cup \{a_{k,2}\})$ is similar in shape to the 1−tree given by Held and Karp (1970).

Lemma 1. If the graph of $B \cup \{a_{k,2}\}$ is a tour, the tour is the shortest tour of G (V, A).

Proof. See Equation (8).

Lemma 1 unifies the ways of proving the shortest tour of some special cases. The graph for $B \cup$ $\{a_{k,2}\}$ can only be a tour in the case that the shortest spanning tree is a spanning path. In this case the arc joining the ends of the spanning path is contained in every A^* (k) and is not shorter than $a_{k,2}$ for all k. If this arc is contained in $\{a_{k,2}\}$, the arc is $a_{k,2}$, because the second element of A^* (k) which contains the arc (joining the ends of the spanning path) as the third or that following is not longer than the arc. Thus the following is obtained :

(i). The regular polygon on the Euclidean plane is the shortest tour for the set of nodes of the polygon, because, in this case, the shortest spanning tree is a spanning path, the arc jeoining its ends is an arc of the polygon and the arc is the second element of A^* (k) for every k.

(ii). For a set of nodes at every lattice crossing with the same length 1 on both sides, every spanning path having ends at a distance 1 forms the shortest tour together with the arc joining the ends. This is because the arc joining the ends can be the second element of every A^* (k) (see Figure 2).

(iii). Figure 3 presents a shortest spanning tree and a circle with center V_1 and radius v_1 v_{16}. This shortest spanning tree is a spanning path and the arc $\overline{v_1v_{16}}$ is

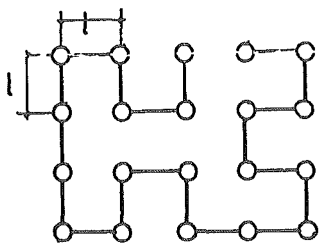


Fig. 2 A shortest spanning path for a set of nodes given at every lattice crossing.

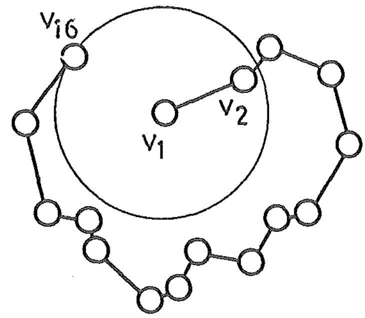


Fig. 3 A shortest spanning tree which is a spanning path, and the circle with center v_1 and radius $\overline{v_1 v_{16}}$.

the second element of A^* (k) for k of $b_k = v_1\ v_2$. Therefore the spanning path constructs the shortest tour together with $v_1\ v_{16}$.

4. Practical techniques

From Eq. (7), and if

$$0 \leqq \tilde{D}\ (k,\ p) - \tilde{D}\ (k,\ p') = (\tilde{a}_{k,p}') + \Sigma_k\ (\tilde{x}\ (p)_k - \tilde{x}\ (p')_k), \quad \cdots\cdots(9)$$

for all p, the tour D $(k,\ p')$ is the shortest tour of G (V, A).

The following is useful for obtaining P $(k,\ p)$ and comparing tours D $(k,\ p)$.

4-1. Shape of the shortest spanning tree

(i) Using the tables $(\tilde{a}_{k,q} - \tilde{b}_k)$ for each k and all q, every set of (n−1) arcs chosen from each A^* (k) over all k can be constructed in the order of the length. But every such set does not represent a spanning path. The shape (the degree of each node) of the shortest spanning tree is applied to efficient construction of sets representing spanning paths.

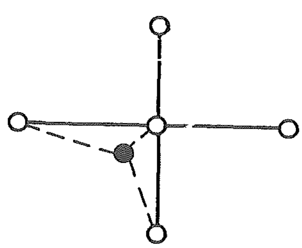


Fig. 4 A shotest spanning tree, a temporary node drawn in black, and new arcs.

Constructing P (k, p) is equivalent to changing the shortest spanning tree into P (k, p) by interchanging some b_k into $a_{k,q}$. The arcs b_k to be interchanged are defined by the shape of the shortest spanning tree. The degree of nodes in the shortest spanning tree can be assumed to be three or fewer without eliminating the generality, because the shortest spanning tree can be reduced to one having nodes with three or fewer degrees by placing a few temporary nodes near each node with four or more degrees (see Fig. 4).

For the shortest spanning tree of this type, it is necessary to change the shortest spanning tree into a spanning path by disconnecting at least one arc from each node with three degrees and adding at least one arc to each node with one degree except the two end nodes of the spanning path. This severely restricts the interchanges of b_k and $a_{k,q}$ that are needed to change B into P, and this restriction makes it simple to find P.

When the number of nodes with one degree of the shortest spanning tree is denoted by β,

$$2 \leqq \beta \leqq (n+2)/2.$$

The number of nodes with three degrees is $\beta-2$. An interchange of b_k with $a_{k,q}$ subtracts one degree from each of two nodes and adds one degree to each of two other, so the number of interchanges of arcs is not smaller than $(\beta-2)/2$. This number gives the maximum number of common arcs of the shortest spanning tree and a tour. This maximum number is not larger than $(n-\beta/2)$.

Generally, P is obtained quicker when β is smaller. This means that the shortest tour is found quicker when it has more arcs in common with the shortest spanning tree. In some special cases of $\beta=2$, the shortest tour can be geometrically obtained as shown above. An experimental example will be shown later.

(ii) The comparison in Eq. (9) starts with D(2) and D(3), the smaller of the two is compared with D(4), and so on. If both parts of the shortest spanning tree separated by removing $b_k = a_{k,1}$ contain two or more nodes, and if $a_{k,2}$ has a common node with $a_{k,1}$, the comparison can be started with a larger p.

This is because when both parts contain two or more nodes, the tour must have two arcs with no common nodes which join the twe parts (Bellmore and Nemhauser, 1968).

4-2. Equation (9)

The first term of the right hand side in Eq. (9) is not negative and increases according to p. Therefore, there is p' so every $p \geqq p'$ satisfies Eq. (9). Such that p is not so much larger than the starting p of the comparison in Eq. (9), because $\tilde{a}_{k,p}$ increases and $\Sigma_k\ \tilde{\bar{x}}\ (p)_k$ decreases step by step.

4-3. Experimental example

The above-mentioned techniques were applied to an experimental example shown in Fig. (4), which has 12 nodes including five extreme nodes. The longest arc of $\{a_{k,2}\}$ is $a_{10,2}$, the shortest arc in A* (10) which has no common node with b_{10} is $a_{10,4}$, and $a_{10,2}$ and $a_{10,3}$ cross each other.

The starting p was 4, the seventh of the $\{x(4)_k\}$ sets was the first set representing a spanning path, p' was 6, and this spanning path constructs the shortest tour together with $a_{k,p} = a_{10,4}$. The shortest tour is represented by

$a_{10,4}$ and $(a_{1,1}, a_{2,1}, a_{3,1}, a_{4,1}, a_{5,2}, a_{6,1}, a_{7,2}, a_{8,2}, a_{9,1}, a_{10,1}, a_{11,2})$.

This shortest tour is obtained by interchanging only four b_k into the second element of the respective A^* (k) sets. In this example, there is no path with the same ends as $a_{10,5}$ which is shorter than D (10, 4), and

$\tilde{a}_{k,6} + \tilde{B} \geqq \tilde{D}$ (10, 4) for all $p \geqq 6$.

Then q (k) = q (10) is 2 (12−2) = 20. The weight of the arcs is in Table 1.

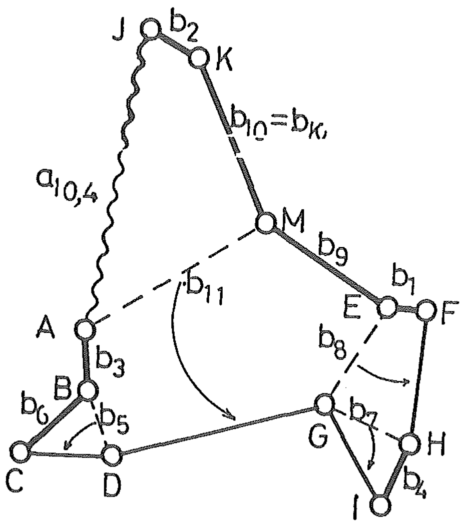


Fig. 5 An experimental example. The shortest spanning tree consists of thick lines and broken lines. The spanning path D (10, 4) is the thick and thin lines, the wave line is $a_{k,p}$. Arrows show the interchanges of b_k with $a_{k,q}$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
EF	KJ	AB	HI	BD	BC	CD	GH	EG	GI	AD	FH	EH	FG	AC	EM	FM
1.0	1.4	2.0	2.1	2.2	2.8	3.0	3.2	3.8	4.0	4.1	4.2	4.3	4.4	4.5	4.9	5.7
18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
GM	EI	FI	KM	AM	DG	BG	AG	JM	BM	HM	DI	DM	AK	IM	DH	AE
6.3	6.4	6.7	7.1	7.2	7.3	8.0	8.2	8.3	8.5	8.8	9.1	9.4	9.6	9.9	10.0	10.1
35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
CG	AJ	EK	DE	BE	BI	AI	BH	FK	AF	DF	CM	AH	BK	BF	EJ	GK
10.2	10.24	10.3	10.4	10.5	10.6	10.9	11.0	11.03	11.1	11.27	11.31	11.4	11.46	11.5	11.8	12.0
52	53	54	55	56	57	58	59	60	61	62	63	64	65	66		
CI	BJ	FJ	CH	CE	DK	GJ	CK	DJ	HK	CJ	CF	HJ	IK	IJ		
12.1	12.2	12.4	13.0	13.1	13.2	13.3	14.0	14.1	14.5	14.6	14.8	15.7	15.9	17.3		

Table 1 Weight of arcs in Fig. 5.

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REFERENCE

Bellmore, M. and G. L. Nemhauser (1968).
"The Traveling Salesman Problem : a survey".
O. P., 16, 538-558.
Cook, S. A. (1971).
"The Complexity of Theorem-proving Procedures".
Proc. ACM Symp, Theory of Computing, 3, 151-158.
Corley, H. W. (1985).
"Efficient Spanning Tree".
J. Optimization Theory and Applications, 45, no. 3, 481-485.
Dantzig, G. B., D. R. Fulkerson, and S. Johson, (1958).
"On a Linear Programming Combinatorial Approach to the Traveling Salesman Problem". O. R., 7, 58-66.
Held, M. and R. M. Karp, (1970).
"The Traveling Salesman Problem and Minimum Spanning Trees". Operations Res. 18, 1138-1162.
Hoffman, K. and M. Padbeg, (1985).
"LP-based Combinatorial Problem Solving" ann. Oper. Res. (Switzerland), vol. 4, no. 1-4, 145-194.
Karp, R. M. (1975)
"On the Combinatorial Problems". Networks, 5, 45-68.
Kruskal, J. B. Jr., (1956).
"On the Salesman Spanning Subtree of a Graph and the Traveling Salesman Problem". Proc. Amer. Math. Soc. 7, 45-50.
Papdimitrous, C. H., (1977).
"The Euclidean Traveling Salesman Problem is NP-Comlete". Ther. Comput. Sci. 4, 237-244.
Sakakibara, K. (1980)
"A few Geometric Properties on the Traveling Salesman Problem". Bulletine of the Faculty of Engineering of Hokkaido University, No. 101, 101-107.
(1982)
"A few Geometric Properties on the Traveling Salesman Problem, part-2". ibids. No. 108, 53-60.
(1984)
"A MYthod to obtain the Shortest Hamilton-line on a Traveling Salesman Problem". ibid. No. 122, 23-28.
(1984)
"Method to obtain the Shortest Hamilton-line of the Traveling Salesman Problem, part-2". ibid. No. 126, 69-80.
(1985)
"A Geometric Method to solve the Traveling Salesman Problem". ibid. No. 27, 61-70.