Free Vibration Analysis of Shells of Revolution Considering the Fluid-Structure Interaction

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Abstract

A simple and effective method is developed in this paper for free vibration analysis of shells of revolution with either internal or external fluids. The fluid region is treated analytically by the utilizing the eigenfunction expansions, and the collocation method using the roots of the orthogonal polynominal as collocation points is used to solve the integro-differential equations which govern the motion of the shell. The proposed approach is formulated in some detail. The versatility and accuracy are illustrated through several numerical examples. The method appears to be relatively easy to formulate and gives satisfactory results.

1. Introduction

The determination of the dynamic characteristics of shells of revolution in contact with fluid is probably the first item of interest in the dynamic analysis. Although extensive work has been directed towards the study of free vibration characteristics of circular cylindirical shells, little work has been done on the free vibration analysis of general shells of revolution.

Although the free vibration problems of shells of revolution, especially for cylindrical shells, have been solved using various numerical methods, no attempt has been made to date to analyze these problems by the collocation method. This method has been modified and improved over the recent years, and successfully used in chemical engineering.[1] The reason for employing the method in this paper lies firstly, in the simplicity of the theory and the brevity of the associated computer code. In addition, the method yields very good results even with a reasonably small number of collocation points, if the roots of the orthogonal polynomial are used as collocation points.[2]

In this paper, the fluid motion is treated analytically by the use of eigenfunction expansions, and the equations of motion of the shells are reduced to the integro-differential equations in terms of the displacements of the shell. The resulting equations are then solved by using the collocation method. The objectives of this paper: (1) To present a simple and effective solution procedure, based on the collocation method, for the vibration problem of shells of revolution with either internal or external fluids; and (2) to demonstrate the high accuracy of the method through several numerical examples.

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2. Shell-Fluid System and Coordinate

Fig. 1 shows two typical shells investigated: (1) the fluid is contained within the shell (this type will be referred to herein as an internal problem), such as in the case of storage tanks; and (2) when the shell is submerged in fluid (this type will be referred to herein as an external problem), such as in the case of offshore structures. The shell is of uniform thickness h, and height L, made of homogeneous, isotropic material with elasticity modulus E, Poisson's ratio ν and mass density ρ_s. The shell is in contact with the fluid of mass density ρ_f, to a height H and consists of the dry and wet portions.

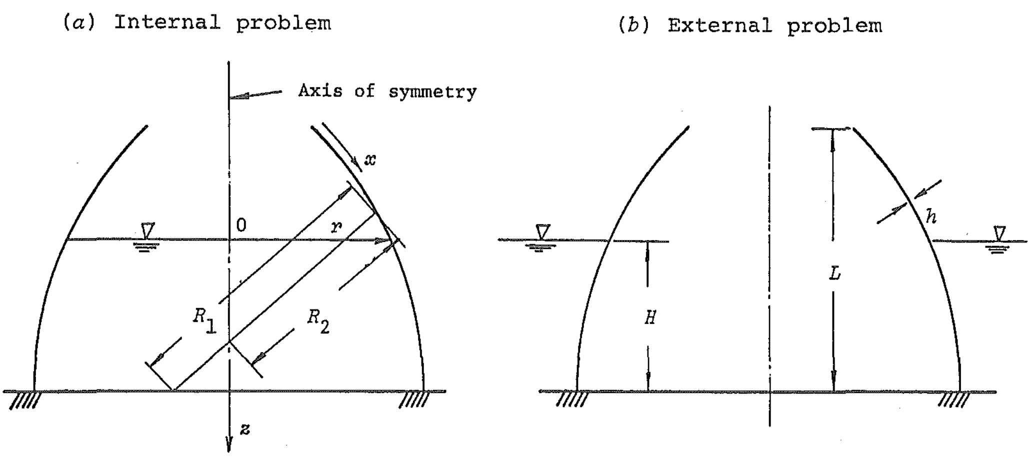

Fig. 1 Shells investigated and coordinate system

The locations of points in the shell are given by the orthogonal coordinates (x, θ, ς), where x is the distance measured from an arbitrary origin along meridian, θ is the circumferential angle, and ς is the normal, outward distance from the reference surface. The shape of the shell is determined by specifying the two principal radii of curvatures R_1, R_2. The locations of points in the fluid are specified by the cylindrical coordinates (r, θ, z), where z is the distance measured from the still-fluid level, and coincides with the axis of symmetry, and r is the distance from the z axis.

3. Equations Governing Fluid Motion

The fluid is assumed to be incompressible and inviscid and the fluid motion irrotational so that the flow can be described by a velocity potential, Φ, which satisfies the following Laplace equation:

$$\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{\partial^2\Phi}{\partial z^2}=0 \tag{1}$$

The shell-fluid boundary conditions for the velocity potential are as follows:

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=H}=0,\ \left.\frac{\partial\Phi}{\partial t}\right|_{z=0}=0,\ \Phi|_{r\to\infty}=0 \tag{2}$$

$$\left.\frac{\partial \Phi}{\partial n}\right|_{r=R}=\left.\frac{\partial W}{\partial t}\right|_{r=R} \tag{3}$$

where n is the outward normal to the surface of the shell, W is the normal displacement of the shell, which will appear in the next section, and R is the radius at any level. The third expression of Eq.(2) is applicable only to the external problem. The hydrodynamic pressure, p, acting on the surface of the shell can be determined from the linearized Bernoulli equation and is given by

$$p=-\rho_f \frac{\partial \Phi}{\partial t} \tag{4}$$

4. Equations Governing Shell Motion

The analytical formulation is based on an improved shell theory with the effects of transverse shear deformation and rotary inertia. This results in the same equations as those given in Ref.(3), except for the term representing the hydrodynamic pressure due to the fluid. Therefore, the material presented in this section will be discussed briefly.

The generalized displacement field consists of the displacement componets (U, V, W) in the (x, θ, ς) directions and the rotation components (β_x, β_θ). All dependent variables are expanded in Fourier series in the circumferential variable θ. Assuming that the shell-fluid system is undergoing free vibration with a frequency ω, then the displacements, the velocity potential Φ and the hydrodynamic pressure p are described as

$$(U,\ V,\ W)=\frac{\sigma a^2}{Eh}\sum_n (u \cos n\theta,\ v \sin n\theta,\ w \cos n\theta) e^{i\omega t} \tag{5}$$

$$(\beta_x,\ \beta_\theta)=\frac{\sigma a}{Eh}\sum_n (\bar{\beta}_x \cos n\theta,\ \bar{\beta}_\theta \sin n\theta) e^{i\omega t} \tag{6}$$

$$\Phi=\frac{i\omega\sigma a^3}{Eh}\sum_n \phi \cos n\theta e^{i\omega t},\ p=\rho_f\omega^2\frac{\sigma a^3}{Eh}\sum_n \bar{p} \cos n\theta e^{i\omega t} \tag{7}$$

where $i=\sqrt{-1}$, a is a reference length, σ is a reference stress, and n is the number of circumferential waves.

Hydrodynamic pressure

An eigenvalue problem such as described by Eqs.(1) and (2) is a Sturm-Liouville problem. Using the method of separation of variables, the solution ϕ can be expressed as

$$\phi_{in}=\sum_{i=1}^{\infty} A_i I_n(\lambda_i\rho) f_i(\eta) \tag{8. a}$$

and

$$\phi_{ex}=\sum_{i=1}^{\infty} A_i K_n(\lambda_i\rho) f_i(\eta) \tag{8. b}$$

where $\rho=r/a$, $\eta=z/H\,(0\leq\eta\leq1)$, A_i are unknown coefficients to be determined from the boundary condition (Eq.(3)) at the shell-fluid interface, the subscripts *in* and *ex* hold for the internal and external problems, respectively, and $I_n(\lambda_i\rho)$ and $K_n(\lambda_i\rho)$ are the modified Bessel functions of the order n of the first and second kind, respectively. In Eq.(8) the eigenvalues, λ_i, and corresponding eigenfunctions, f_i, are given by

$$\lambda_i=\frac{(2i-1)\pi}{2}\frac{a}{H} \quad f_i(\eta)=\sin\left(\lambda_i\frac{H}{a}\eta\right) \tag{9}$$

Substituting Eq.(8) and the third expression of Eq.(5) into Eq.(3) gives

$$\sum_{i=1}^{\infty}A_i g_i(\eta)=w \tag{10}$$

For the internal problem, $g_i(\eta)$ in the above equation becomes

$$\begin{aligned} g_i(\eta)=&\left\{-\frac{n}{\rho}I_n(\lambda_i\rho)+\lambda_i I_{n-1}(\lambda_i\rho)\right\}\sin\left(\lambda_i\frac{H}{a}\eta\right)n_r \\ &+\lambda_i I_n(\lambda_i\rho)\cos\left(\lambda_i\frac{H}{a}\eta\right)n_z \end{aligned} \tag{11. a}$$

and for the external problem, $g_i(\eta)$ is

$$\begin{aligned} g_i(\eta)=&\left\{-\frac{n}{\rho}K_n(\lambda_i\rho)+\lambda_i K_{n-1}(\lambda_i\rho)\right\}\sin\left(\lambda_i\frac{H}{a}\eta\right)n_r \\ &+\lambda_i K_n(\lambda_i\rho)\cos\left(\lambda_i\frac{H}{a}\eta\right)n_z \end{aligned} \tag{11. b}$$

where n_r, n_z are the direction cosines of the outward normal n to the surface, and n_r, n_z, and ρ are functions that depend on the variable η.

The orthogonality properties of the eigenfunctions f_i with respect to η can now be utilized to determine the unknown coefficients A_i. Both sides of Eq.(10) are multiplied by $f_l(\lambda_l\eta)$ for $l=1, 2, \cdots$ in turn and integrated with respect to η over (0, 1):

$$\sum_{i=1}^{\infty}A_i\int_0^1 g_i(\eta)f_l(\eta)d\eta=\int_0^1 wf_l(\eta)d\eta \quad (l=1, 2,\cdots) \tag{12}$$

which is an infinite linear system of algebraic equation. In the matrix form it becomes

$$[GF]\{A\}=\{WF\} \tag{13}$$

with

$$GF(l,\ i)=\int_0^1 g_i(\eta)f_l(\eta)d\eta=\int_0^1 g_i(\eta)\sin\left(\lambda_l\frac{H}{a}\eta\right)d\eta \tag{14. a}$$

and

$$WF(l)=\int_0^1 wf_l(\eta)d\eta=\int_0^1 w\sin\left(\lambda_l\frac{H}{a}\eta\right)d\eta \tag{14. b}$$

where l, $i=1, 2, \cdots, \infty$.

In the case of a cylindrical shell, the integral in expression (14. a) can be evaluated exactly, and the resulting matrix $[GF]$ is a diagonal matrix. In the case of other shells, the integral should be evaluated numerically, choosing an appropriate quadrature rule, and the matrix $[GF]$ is a full matrix. Eq.(13) can not, in general, be solved. An approximate solution is obtained by truncating the series appearing in Eq.(8) to a finite number of terms, I, and by solving the resulting linear system of I equations with I unknowns.

Since the matrix $[GF]$ possesses an inverse, the solution for the unknowns $\{A\}$ can be obtained from the matrix multiplication of $[GF]^{-1}\{WF\}$. Once the potential is known, the hydrodynamic pressure acting on the surface of the shell can be evaluated by using the second expression in Eq.(7).

Derivation of fundamental set of equations of shell

Considering the hydrodynamic pressure exerted on the surface of the shell, the governing equations are described by a system of integro-differential equations for harmonic amplitudes of the displacement variables. These equations can be written in matrix form:

$$[C]\{X''\}+[D]\{X'\}+[E]\{X\}=\Omega^2([F]\{X\}+\{p\}) \tag{15}$$

where the primes indicate differentiation with respect to a nondimensional meridional variable s, which takes values 0 to 1, and $[G]$, $[D]$, $[E]$, and $[F]$ are the 5×5 matrices whose elements have been given in Ref.(5). In Eq.(15), a frequency parameter Ω^2 is defined as

$$\Omega^2=\rho_s(1-\nu^2)a^2\omega^2/E \tag{16}$$

and $\{X\}$ and $\{P\}$ are the displacement and hydrodynamic pressure vectors given by

$$\{X\}^T=(u,\ v,\ w,\ \bar{\beta}_x,\ \bar{\beta}_\theta),\quad \{p\}^T=(0,\ 0,\ p_w,\ 0,\ 0) \tag{17}$$

where p_w is identical to an expression of added mass of shell-fluid system, and using Eq.(8) it can be written as

$$p_w=-\frac{\rho_f}{\rho_s}\frac{a}{h}\phi_{in} \text{ for internal problems} \tag{18. a}$$

and

$$p_w=-\frac{\rho_f}{\rho_s}\frac{a}{h}\phi_{ex} \text{ for internal problems} \tag{18. b}$$

The evaluation of the potential appearing in Eq.(18) involves the integral in expression (14. b), therefore, Eq.(15) is the so-called integro-differential equations which govern the motion of shell.

The stress resultants that appear in the statement of the boundary conditions are N_x, $N_{x\theta}$, Q_x, M_x and $M_{x\theta}$. As before, these resultants for each Fourier harmonic are taken as

$$(N_x,\ N_{x\theta},\ Q_x)=\sigma a^2\sum_n[n_x\cos n\theta,\ n_{x\theta}\sin n\theta,\ q_x\cos n\theta]e^{i\omega t} \tag{19. a}$$

$$(M_x,\ M_{x\theta})=\sigma a^2\sum_n[m_x\cos n\theta,\ m_{x\theta}\sin n\theta]e^{i\omega t} \tag{19. b}$$

The Fourier coefficients in Eq.(19) can be expressed in terms of the displacements, i. e.,

$$\{T\}^T=[G]\{X'\}+[H]\{X\} \tag{20}$$

where $[G]$ and $[H]$ are the 5×5 coefficient matrices whose elements can be found in Ref. 5), and $\{T\}$ is the stress resultant vector given by

$$\{T\}^T=(n_x,\ n_{x\theta},\ q_x,\ m_x,\ m_{x\theta}) \tag{21}$$

Finally, the boundary conditions at each edge of the shell are specified as a set of five conditions, one from each of the following five pairs:

$$(u,\ n_x),\ (v,\ n_{x\theta}),\ (w,\ q_x),\ (\bar{\beta}_x,\ m_x),\ (\bar{\beta}_\theta,\ m_{x\theta}) \tag{22}$$

5. Method of Solution

For the present study the shell is assumed to consist of dry and wet portions. As shown in Fig. 2, the dry and wet portions are divided into N_d and N_w elements, respectively; i. e., the total number of elements is $N=N_d+N_w$. A local nondimensional independent variable is denoted by ξ, where ξ takes the values 0 to 1 in each element. Denote the point along the meridional coordinate by i, where i varies from 0 to N. Of these points, the points from 1 to $N-1$ are identified at the common boundaries of

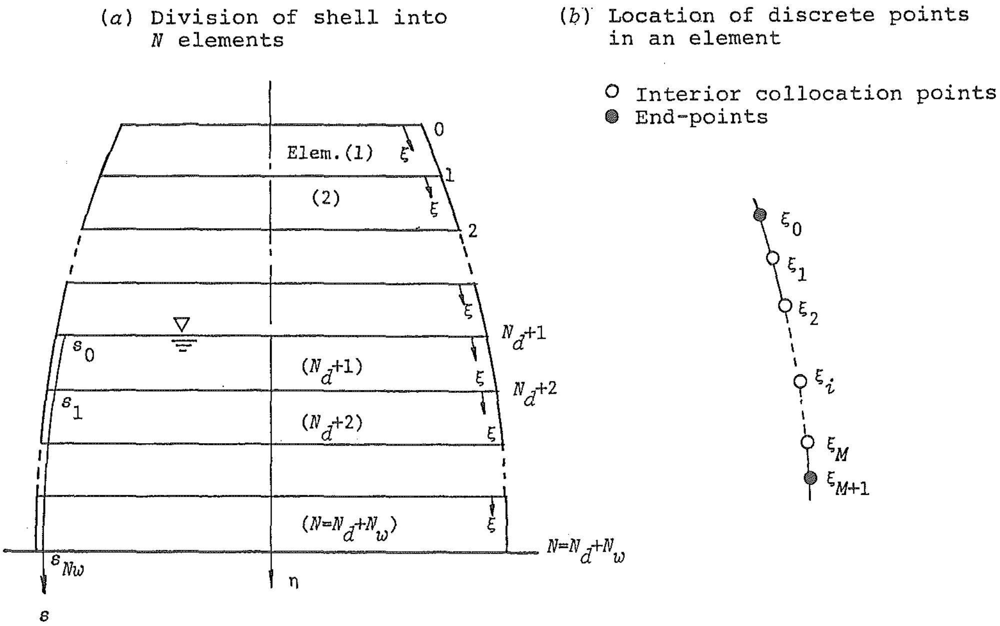

Fig. 2 Division of shell into N elements and location of discrete points in an element

different elements, and these points will be called "dividing points". The remaining points 0 and N of the ends of the shell will be called "boundary points". Let us construct a set of N_w+1 points $0=s_0<s_1<\cdots<s_{N_w}=1$ in the range (0, 1) of the wet part of the shell, so that the location of these points coincides with that of the dividing points.

The proposed method is to approximate a derivative and an integral as a linear sum of the displacement values at discrete points so that the integro-differential equations can be reduced to a set of algebraic equations. To this end, over each element we place a set of $M+2$ discrete points in Fig. 2 which are composed of the end-points $\xi_0=0$, $\xi_{M+1}=1$ and the interior collocation points $\xi_j (j=1\sim M)$, such that $0=\xi_0<\xi_1<\cdots<\xi_{M+1}=1$. In this paper, the interior collocation points are selected to be zeros of the Mth shifted Legendre polynomial[4] $P^*_M(\xi)$ since these zeros are distributed near two end-points and are therefore optimal for the boundary value problems.

The displacement functions for the kth element are interpolated by

$$X_j^{(k)}=\sum_{i=1}^{M+2} N_i(\xi) X_{j,i}^{(k)} \quad (j=1,\ 5) \tag{23}$$

where the notation $(\ )^{(k)}$ will designate quantities associated with the kth element, $X_1\sim X_5$ correspond to u, v, w, $\bar{\beta}_x$, $\bar{\beta}_\theta$, respectively, $N_i(\xi)$ are the $(M+2)$th interpolation functions, and $X_{j,i}$ are the values of the displacements X_j at the ith discrete points.

Before describing the details of the proposed method, the following comments seem to be in order:

(1) To decrease the computational effort required, the following two matrices $[A]$ and $[B]$ are used to approximate the first and second derivatives that appear in Eqs.(15) and (20):

$$\{X_j^{(k)\prime}\}=[A]\{X_j^{(k)}\},\ \{X_j^{(k)\prime\prime}\}=[B]\{X_j^{(k)}\} \quad (j=1\sim 5) \tag{24}$$

where $[A]$ and $[B]$ are the $(M+2)\times(M+2)$ matrices and are obtained by differentiation of the interpolation function, and the displacement vector $\{X_j^{(k)}\}$, etc. are $\{X_j^{(k)}\}^T=(X_j^{(k)}(\xi_0),\ X_j^{(k)}(\xi_1),\cdots,\ X_j^{(k)}(\xi_{M+1}))$, etc.

(2) The complete set of displacement vector $\{X_j^{(\ )}\}$ is partitioned into two groups $\{X_{j,c}^{(\ )}\}$ and $\{X_{j,e}^{(\ )}\}$, and the first one is associated with the interior collocation points while the second one is associated with the end-points; thus

$$\begin{aligned}\{X_{j,c}^{(k)}\}^T&=(x_j^{(k)}(\xi_1),\ X_j^{(k)}(\xi_2),\cdots,\ X_j^{(k)}(\xi_M))\\ \{X_{j,e}^{(k)}\}^T&=(x_j^{(k)}(\xi_0),\ X_j^{(k)}(\xi_{M+1}))\end{aligned}\quad (j=1\sim5) \tag{25}$$

Henceforth, the subscripts c and e appearing in Eq.(25) are used to designate quantities associated with the interior collocation points and end-points, respectively.

(3) The integral involved in the evaluation of Eq.(14. b) is carried out by an appropriate numerical integration rule.[6] Let us recall that we select the interior collocation points as the zeros of the shifted Legendre polynomial. It is natural, therefore, to choose the Gauss-Legendre quadrature formula, with this set of points as the sampling points. Before the application of the quadrature formula, the integral variable η is related to the local meridional coordinate ξ of each element in the wet portion by the following equations:

$$d\eta=L_w\sin\varphi ds \tag{26. a}$$

$$s^{(i)}(\xi)=s_{i-1}+\Delta s\xi,\ (i=1\sim N_w) \tag{26. b}$$

$$\eta=\psi(s) \tag{26. c}$$

where $\Delta s=s_i-s_{i-1}$, L_w is the meridional length of the wet part of the shell, φ is the meridional angle, and Eq.(26. c) represents the one-to-one relationship between the coordinates η and s.

With the aid of Eq.(26), the integral in Eq.(14. b) is approximated by

$$\int_0^1 w\sin\left(\lambda_l\frac{H}{a}\eta\right)d\eta=\sum_{i=1}^{N_w}[Y^{(i)}]\{w_c^{(N_d+i)}\} \tag{27}$$

where $\{w_c^{(N_d+i)}\}$ is the vector composed of the normal displacements at the interior collocation points (i. e., the sampling points), and $\{Y_l^{(i)}\}$ is the $1\times M$ row matrix whose elements are defined as

$$Y_l^{(i)}(j)=\frac{L_w}{H}\Delta sW_j\sin\left[\lambda_l\frac{H}{a}\psi(s^{(i)}(\xi_j))\right]\sin\varphi|_{s^{(i)}(\xi_j)} \tag{28}$$

where the subscript l represents the number of terms in the series expansion of the velocity potential and W_j are the weights in the interval (0, 1).

From Eq.(23), the number of unknowns per element is $5(M+2)$. That is, the total number of unknowns for the shell having N elements is $5(M+2)N$. The application of the present method to Eq.(15) yields $5MN$ linear algebraic equations. In addition to this, there are 10 boundary conditions at the boundary points and $10(N-1)$ continuity conditions of the displacements and stress resultants at the dividing points. Since $5MN+10+10(N-1)=5(M+2)N$ we have the same number of equations as unknowns. These equations will be explained further in the following subsections.

5MN equations

By using Eq.(24) to approximate the derivatives and by using Eq.(27) to compute the integral, Eq.(15) for the kth element leads to $5M$ linear equations. After dividing all of the unknowns into two groups as discussed previously, these equations can be expressed in the matrix form as

$$[\alpha_c]\{\delta_c^{(k)}\}+[\alpha_e^{(k)}\delta_e^{(k)}]=\Omega^2([MS_c^{(k)}]\{\delta_c^{(k)}\}+\sum_{j=1}^{N_w}[MF^{(j)}]\{w_c^{(N_d+1)}\}) \quad (29)$$

where $k=1, \cdots, N$. For the element in the dry portion, the second term on the right hand-side of Eq.(29) vanishes. $[\alpha_c^{(k)}]$ and $[\alpha_e^{(k)}]$ are the $5M\times5M$ and $5M\times10$ matrices, which depend on the elements of $[A]$, $[B]$ (given by Eq.(24)) and $[C]$, $[D]$, $[E]$ (appearing in Eq.(15)). $[MS_c^{(k)}]$ is the $5M\times5M$ matrix, which is dependent only on the elements of $[F]$ in Eq.(15). Moreover, by making use of the expression Eq.(25) of the displacement vector, the vectors $\{\delta_c^{(k)}\}$ and $\{\delta_e^{(k)}\}$ are as follows:

$$\begin{aligned}\{\delta_c^{(k)}\}^T&=(\{u_c^{(k)}\}^T, \{v_c^{(k)}\}^T, \{w_c^{(k)}\}^T, \{\bar{\beta}_{xc}^{(k)}\}^T, \{\bar{\beta}_{\theta e}^{(k)}\}^T)\\ \{\delta_e^{(k)}\}^T&=(\{u_e^{(k)}\}^T, \{v_e^{(k)}\}^T, \{w_e^{(k)}\}^T, \{\bar{\beta}_{xe}^{(k)}\}^T, \{\bar{\beta}_{\theta c}^{(k)}\}^T)\end{aligned} \quad (30)$$

In Eq.(29), $[MF^{(j)}]$ is the $M\times M$ matrix computed as

$$[MF^{(j)}]=\sum_{l=1}^{\infty}[F_l^{(j)}][G_l^{(j)}] \quad (31)$$

where $[F_l^{(j)}]$ is the $M\times1$ column matrix whose elements are defind as

$$F_l^{(j)}(j)=I_n(\lambda_l\rho|_{\varepsilon_l})\sin\left[\lambda_l\frac{H}{a}\psi(s^{(i)}(\xi_j))\right] \text{ for internal problem} \quad (32.\ a)$$

$$F_l^{(j)}(j)=K_n(\lambda_l\rho|_{\varepsilon_l})\sin\left[\lambda_l\frac{H}{a}\psi(s^{(i)}(\xi_j))\right] \text{ for external problem} \quad (32.\ b)$$

and using the inverse of the matrix $[GF]$ in Eq.(13), the matrix $[G_l^{(j)}]$ can be written in the following form:

$$[G_l^{(j)}]=[GF]^{-1}[YY^{(j)}] \quad (33)$$

where the matrix $[YY^{(j)}]$ can be obtained by using the matrix $[Y_l^{(j)}]$ in Eq.(27) as follows:

$$[YY^{(j)}]=\begin{bmatrix}[Y_1^{(j)}]\\ [Y_2^{(j)}]\\ \vdots\\ [Y_l^{(j)}]\end{bmatrix} \quad (34)$$

Eq.(29) can be determined for each element separately, and for the overall shell, these equations yield a system of $5MN$ algebraic equations and can be expressed in the matrix form as

$$[\alpha_c]\{\delta_c\}+[\alpha_e]\{\delta_e\}=\Omega^2([MS_c]+[MF_c])\{\delta_c\} \quad (35)$$

where $[\alpha_c]$, $[\alpha_e]$ and $[MS_c]$ are the global matrices with submatrices only on the diagonal position; i. e.,

$$[\alpha_i]=\ulcorner[\alpha_i^{(1)}], [\alpha_i^{(2)}], \cdots, [\alpha_i^{(N)}]\lrcorner, (i=c, e) \quad (36.\ a)$$

$$[MS_c]=\ulcorner[MS_c^{(1)}], [MS_c^{(2)}], \cdots, [MS_c^{(N)}]\lrcorner \quad (36.\ b)$$

and $\{\delta_c\}$ and $\{\delta_e\}$ are the global vectors denoted by

$$\{\delta_i\}^T=(\{\delta_i^{(1)}\}^T, \{\delta_i^{(2)}\}^T, \cdots, \{\delta_i^{(N)}\}^T), (i=c, e) \quad (37)$$

In Eq.(35), $[MS_c]$ and $[MF_c]$ are the structural mass matrix and the added mass matrix, respectively. $[MF_c]$ is obtained by adding the submatrix $[MF_c^{(j)}]$ (Eq.(29)) into the appropriate positions related to the normal displacement vectors $\{w_c^{(N_d+j)}\}(j=1 \sim N_w)$ in the global vector $\{\delta_c\}$.

10 *and* 10$(N-1)$ *equations*

From any given set of boundary conditions at the boundary points 0 and N, we have 10 equation. Using Eqs.(20), (22) and (24), and repeating the similar procedure which is used to obtain Eq.(29), these equations can be written as

$$\begin{aligned}&[r_{c,0}^{(1)}]\{\delta_c^{(1)}\}+[r_{e,0}^{(1)}]\{\delta_e^{(1)}\}=0\\&[r_{c,N}^{(N)}]\{\delta_c^{(N)}\}+[r_{e,N}^{(N)}]\{\delta_e^{(N)}\}=0\end{aligned} \tag{38}$$

where the subscripts $i(=0,\ N)$ following a comma represent the boundary points, $[r_{c,0}^{(1)}]$ and $[r_{c,N}^{(N)}]$ are the $5\times5M$ matrices, and $[r_{e,0}^{(1)}]$ and $[r_{e,N}^{(N)}]$ are the 5×5 matrices.

The remaining $10(N-1)$ equations are obtained from the compatibility and equilibrium conditions at the dividing points. These conditions can be expressed as

$$\{X^{(i)}\}_1=\{X^{(i+1)}\}_0,\ \{T^{(i)}\}_1=\{T^{(i+1)}\}_0,\ (i=1\sim N-1) \tag{39}$$

where $\{X\}$ and $\{T\}$=the displacement and stress resultant vectors given by Eqs.(17. a) and (21), respectively, and the subscripts 0 and 1 which appear outside the braces, refer to the values at $\xi=0$ and $\xi=1$, respectively. Utilizing Eqs.(20) and (24) for Eq.(39), we obtain an expression similar to Eq. (38) as follows:

$$[r_{c,i}^{(i)}]\{\delta_c^{(i)}\}+[r_{e,i}^{(i)}]\{\delta_e^{(i)}\}+[r_{c,i}^{(i+1)}]\{\delta_c^{(i+1)}\}+[r_{e,i}^{(i+1)}]\{\delta_e^{(i+1)}\}=0\ \ (i=1\sim N-1) \tag{40}$$

where the subscript i refers to the dividing points, and $[r_{c,i}^{(\ )}]$ and $[r_{e,i}^{(\ )}]$ are the $10\times5M$ and 10×10 matrices, respectively.

Eqs.(38) and (40) can be combined into a single matrix equation of the form

$$[r_c]\{\delta_c\}+[r_e]\{\delta_e\}=\{0\} \tag{41}$$

where $[r_c]$ and $[r_e]$ are the $10N\times5MN$ and $10N\times10N$ matrices, respectively.

Eigenvalue problem

When Eq.(41) is solved for $\{\delta_e\}$ and the result is substituted into Eq.(35), we obtain

$$([\alpha_c]-[\alpha_e][r_e]^{-1}[r_c])\{\delta_c\}=\Omega^2([MS_c]+[MF_c])\{\delta_c\} \tag{42}$$

Eq.(42) represents the generalized eigenvalue problem, and is the condensed form that contains only the unknowns associated with the interior collocation points. The solution of Eq.(42) yields the estimate for the $5MN$ eigenvalues and the corresponding eigenvectors.

6. Numerical Examples

In order to test the validity of the present method, three types of shells are employed as illustrative examples; i. e., (1) a cylindrical shell, (2) a spherical shell, and (3) a hyperboloidal shell. Based on the past work[5] of the author, the dry portion of the shell was modelled by one element and the number of collocation points was taken as $M=11$. In all the computations, unless otherwise stated, the following shell and fluid properties were used: $E=206$GPa, $\rho_s=7.84\times10^3$kg/m^3, $\rho_f=10^3$kg/m^3, and $\nu=0.3$. Also in all the following tables and figures, m denotes the number of half waves in the meridional direction, n the number of circumferential waves. The numerical computations were carried out a HITAC—M682H computer.

Cylindrical shells

For a fixed M, the convergence of the proposed method depends on the number of elements in the wet portion of the shell N_w, as well as, the number of terms I in the series expansion of the velocity potential. To examine the convergence characteristics of the method, two shells which are clamped at the base and free at the top, were considered. For convenience these are referred to as follows: (a) shell(A), its dimensions are $L=H=21.96$ m, $a=7.32$ m, and $h=1{,}09$ cm and (b) shell(B), its dimensions are $L=H=12.2$ m, $a=18.3$ m, and $h=2.54$ cm. calculations were performed by using $N_w=1$ and $N_w=2$, and using various terms (i. e., $I=18$, 10, and 12).

For the external problem, the frequencies for the first three modes are presented in Table 1 for $n=1$ and $n=5$. The convergence of the solutions is reasonable even with $N_w=1$, and is insensitive to choices of I. The convergence characteristics of the internal problem are the same as those of the external problem, although they are not shown here.

In order to check the accuracy of the frequencies obtained, some comparative studies were performed by using $N_w=1$ and $I=12$. Firstly comparisons were made with finite element solutions(FEM) of Ref. 7), for the internal problem of shell (A) mentioned previously. The results for various modes are given in Table 2 together with those of Ref. 7). There are no appreciable differences between both the results. A second set of comparisons was made with the Rayleigh-Ritz and matrix progression solutions of Ref. 8), for the external problem. The shell considered was: $L=80$ m, $a=40$ m, $H=64$ m, and $h=0.4$ m. The present solutions are in good agreement with those of Ref. 8), as summarized in Table 3.

Table 1 Convergence ot natural frequencies (Hz) of cylindrical shells (external problem)

(a) shell (A)

I	n	$N_w=1$			$N_w=2$		
		$m=1$	$m=2$	$m=3$	$m=1$	$m=2$	$m=3$
8	1	5.96	17.56	26.30	5.96	17.55	26.20
10	1	5.95	17.54	26.25	5.95	17.53	26.26
12	1	5.95	17.54	26.25	5.95	17.53	26.25
8	5	1.25	3.76	8.85	1.25	3.76	8.80
10	5	1.24	3.74	8.80	1.24	3.74	8.78
12	5	1.24	3.74	8.79	1.24	3.73	8.78

(b) shell (B)

I	n	$N_w=1$			$N_w=2$		
		$m=1$	$m=2$	$m=3$	$m=1$	$m=2$	$m=3$
8	1	7.18	11.99	15.59	7.18	11.99	15.60
10	1	7.17	11.98	15.58	7.18	11.99	15.59
12	1	7.17	11.98	15.58	7.18	11.98	15.59
8	5	2.74	8.47	13.21	2.74	8.48	13.21
10	5	2.74	8.46	13.20	2.74	8.47	13.20
12	5	2.74	8.45	13.20	2.74	8.46	13.20

Table 2 Natural frequencies (Hz) of a cylindrical shell (internal problem)

Method	m	n					
		1	2	3	4	5	6
Present (N_w=1)	1	3.548	1.638	0.934	0.632	0.531	0.584
Present (N_w=2)	1	3.545	1.636	0.933	0.632	0.531	0.584
FEM	1	3.559	1.650	0.950	0.650	0.550	0.600
Present (N_w=1)	2	10.338	6.550	4.401	3.162	2.397	1.923
Present (N_w=2)	2	10.334	6.579	4.429	3.188	2.421	1.944
FEM	2	10.450	6.660	4.520	3.280	2.520	2.050

FEM : Finite element method

Table 3 Natural frequencies (Hz) of a cylindrical shell (external problem)

Method	m	n			
		0	1	2	3
Present (N_w=1)	1	6.633	3.596	1.902	1.173
Present (N_w=2)	1	6.634	3.595	1.902	1.173
Present (N_w=1)	2	9.662	7.865	5.614	3.935
Present (N_w=2)	2	9.996	7.872	5.624	3.942
MPM	2				3.900
RRM	2				4.100

MPM : Matrix progression method
RRM : Rayleigh-Ritz method

Hemispherical shells

When closed shells of revolution are encountered, the equations of motion of the shell become singular at the pole. Therefore, the conventional analysis using FEM[9] and FDM[10] (finite difference method) requires special treatments, such as the use of a cap element and the use of a small hole with a free edge condition. The present method does not require these treatments, and the necessary conditions are imposed to ensure the existence of a finite solution at the pole.

A fixed hemispherical shell was considered. Its geomertric properties were : h/a (thickness-to-radius ratio)=0.01, and H/a (fluid height-to-radius ratio)=0.5. The number of elements (N_w) in the wet portion was taken to be 2.

For the internal and external problems, the convergence of the solutions is illustrated in Table 4 by computing the natural frequencies using various terms (i. e., $I=8$, 10, and 12) in the series expansion of the velocity potential. The results are expressed in terms of dimensionless frequency Ω (Eq.(16)). From this table, we can see that the convergence of the solution is insensitive to choices of I. The convergence of the external problem is the same as that of the internal problem, although they are not shown here.

For the internal and external problems, Table 5 presents the fundamental natural frequencies, Ω, for the range of n from 1 to 5. In the table comparisons are made with the RRM results obtained using the first five modes in air. Generally, the two methods

Table 4 Convergence of natural frequencies (Ω) of a hemispherical shell

(a) Internal problem

I	n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
8	1	0.366	0.633	0.732	0.896	0.939
10	1	0.365	0.632	0.731	0.895	0.938
12	1	0.365	0.632	0.731	0.895	0.937
8	5	0.596	0.750	0.946	0.985	1.048
10	5	0.596	0.759	0.945	0.983	1.047
12	5	0.596	0.748	0.944	0.983	1.047

(b) External problem

I	n	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$
8	1	0.438	0.663	0.763	0.898	0.942
10	1	0.438	0.663	0.763	0.898	0.942
12	1	0.438	0.663	0.763	0.898	0.942
8	5	0.643	0.779	0.947	0.988	1.053
10	5	0.643	0.779	0.947	0.988	1.053
12	5	0.643	0.779	0.947	0.988	1.053

Table 5 Comparison of fundamental frequencies (Ω) of a hemipherical shell

n	Internal problem		External problem	
	Present	RRM	Present	RRM
1	0.365	0.367	0.438	0.439
2	0.484	0.494	0.576	0.584
3	0.535	0.546	0.609	0.619
4	0.569	0.575	0.628	0.633
5	0.596	0.599	0.643	0.646

RRM: Rayleigh-Ritz method

give similar results. The mode shapes and associated hydrodynamic pressure distributions are shown in Fig. 3.

Hyperboloidal shells

As shown in Fig. 4, the five hyperboloidal shells with different throat radii, R_t, were considered. In the extreme case where $R_t/a=1$, the shell becomes a cylindrical one. The Poisson's ratio of the material was assumed to be 0.15. Calculations were carried out using $N_w=2$ and $I=12$.

For the internal and external problems, the fundamental natural frequencies, Ω, are given in Table 6 for shells having H/a of 2 and 2.5. Also included in the table are the results obtained for the shell without fluid. As expected, the frequency of the hyperboloidal shell approaches that of the cylindrical shell as the value of R_t/a increases. Owing to the added mass of the fluid, the frequencies for the shell with fluid are lower

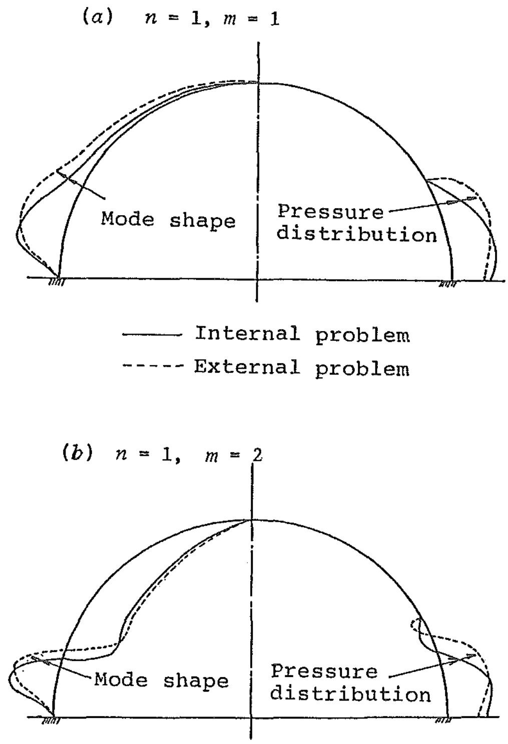

Fig. 3 Mode shapes and associated pressure distributions of hemispherical shell

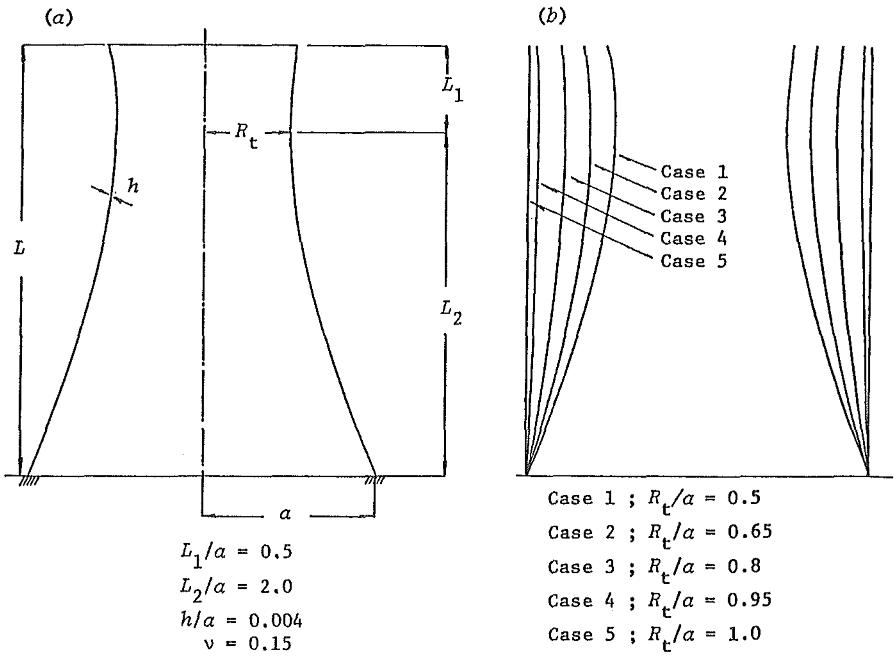

Fig. 4 Hyperboloidal shells used for numerical studies

than the corresponding ones for the shell without fluid. It can be seen that the frequencies of the shell with $H/a=2.5$ are lower than those of the same shell with $H/a=2.0$. This is the obvious result since the added mass of the fluid increases with H/a, which the structure stiffness properties remain unchanged.

To illustrate the frequency characteristics of the internal and external problems, the relationship between the fundamental frequency Ω and the circumferential wave number n are presented in Fig. 5, for shells having R_t/a of 0.5 and 0.8 and the same H/a of 2.5. Some points are worthy of note in these results. First, for the cases considered the minimum frequency occurs when $n=4$. Furthermore, we can see that when $n=0$ and 1, the frequencies of the external problem are larger than the corresponding ones of the internal problem. It should be pointed out, however, that for higher values of n (say $n \geq 2$), the frequencies of both internal and external problems are nearly equal.

Table 6 Variation of fundamental frequencies (Ω) of hyperboloidal shells with R_t/a ratio (n=1)

R_t/a	Internal proble		External problem		Shell without fluid
	$H/a=2$	$H/a=2.5$	$H/a=2$	$H/a=2.5$	
0.50	0.078	0.065	0.090	0.070	0.2860
0.65	0.069	0.055	0.079	0.065	0.2711
0.80	0.061	0.047	0.070	0.053	0.2532
0.95	0.053	0.041	0.061	0.046	0.2327
1.00	0.051	0.038	0.058	0.044	0.2255

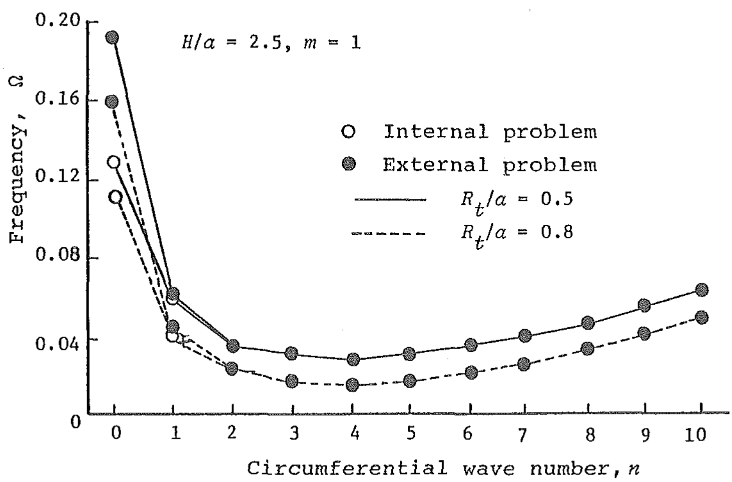

Fig. 5 Relationsship between frequency and circumferential wave number (hyperboloidal shell)

7. Concluding Remarks

A reliable and computationally effective method is presented in this paper for the free vibration analysis of shells of revolution with either internal or external fluids. A linear potential flow theory is used, and an improved shell theory including the effects of transverse shear and rotary inertia is used to describe the motion of shell. The fluid motion is treated analytically using eigenfunction expansions, and the collocation method using the Gaussian points as collocation points is used to solve the integro-differential equations which govern the shell motion.

Numerical results are presented for three types of shells of revolution. These examples show that the method yields relatively high accuracy even with a resonably small number of collocation points. Therefore, the proposed method is usefulnot only for a better understanding of the vibration characteristics of the shell but is also available for a check on other numerical methods such as FEM and BEM.

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