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Time Operators of a Hamiltonian with Purely Discrete Spectrum

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Abstract

We develop a mathematical theory of time operators of a Hamiltonian with purely discrete spectrum. The main results include boundedness, unboundedness and spectral properties of them. In addition, possible connections of a time operator of H with regular perturbation theory are discussed.

Keywords. canonical commutation relation, Hamiltonian, time operator, time-energy uncertainty relation, phase operator, spectrum, regular perturbation theory.

Mathematics Subject Classification (2000). 81Q10, 47N50.

1 Introduction

This paper is concerned with mathematical theory of time operators in quantum mechanics [2, 3, 4, 6, 12]. There are some types of time operators which are not necessarily equivalent each other. For the reader's convenience, we first recall the definitions of them with comments.

Let \mathcal{H} be a complex Hilbert space. We denote the inner product and the norm of \mathcal{H} by $\langle \cdot, \cdot \rangle$ (antilinear in the first variable) and $\| \cdot \|$ respectively. For a linear operator A on a Hilbert space, $D(A)$ denotes the domain of A .

Let H be a self-adjoint operator on \mathcal{H} and T be a symmetric operator on \mathcal{H} .

The operator T is called a *time operator* of H if there is a (not necessarily dense) subspace $\mathcal{D} \neq \{0\}$ of \mathcal{H} such that $\mathcal{D} \subset D(TH) \cap D(HT)$ and the canonical commutation relation (CCR)

$$[T, H] := (TH - HT) = i \tag{1.1}$$

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holds on \mathcal{D} (i.e., $[T, H]\psi = i\psi$, $\forall \psi \in \mathcal{D}$), where i is the imaginary unit. In this case, T is called a *canonical conjugate* to H too.

The name “time operator” for the operator T comes from the quantum mechanical context where H is taken to be the Hamiltonian of a quantum system and the heuristic classical-quantum correspondence based on the structure that, in the classical relativistic mechanics, time is a canonical conjugate variable to energy in each Lorentz frame of coordinates. Note also that the dimension of T is that of time if the dimension of H is that of energy in the original unit system where the right hand side of (1.1) takes the form $i\hbar$ with \hbar being the Planck constant h divided by 2π . We remark, however, that this name is somewhat misleading, because, in the framework of the standard quantum mechanics, time is *not an observable*, but just a parameter assigning the time when a quantum event is observed. But we follow the convention in this respect. By the same reason as just remarked, T is not necessarily (essentially) self-adjoint. But this does not mean that it is “unphysical” [2, 12]. Note also that we do not require the denseness of the subspace \mathcal{D} in the definition stated above. This is more general. In fact, there is an example of the pair (T, H) satisfying (1.1) on a non-dense subspace \mathcal{D} [8, 10].

From a representation theoretic point of view, the pair (T, H) is a symmetric representation of the CCR with one degree of freedom. But one should remember that, as for this original form of representation of the CCR, the von Neumann uniqueness theorem ([13], [14, Theorem VIII.14]) does not necessarily hold. In other words, (T, H) is not necessarily unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom. Indeed, for example, it is obvious that, if T or H is bounded below or bounded above, then (T, H) cannot be unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

A classification of pairs (T, H) with T being a bounded self-adjoint operator has been done by G. Dorfmeister and J. Dorfmeister [7]. We remark, however, that the class discussed in [7] does not cover the pairs (T, H) considered in this paper, because the paper [7] treats only the case where T is bounded and absolutely continuous.

A weak form of time operator is defined as follows. We say that a symmetric operator T is a *weak time operator* of H if there is a subspace $\mathcal{D}_w \neq \{0\}$ of \mathcal{H} such that $\mathcal{D}_w \subset D(T) \cap D(H)$ and

$$\langle T\psi, H\phi \rangle - \langle H\psi, T\phi \rangle = \langle \psi, i\phi \rangle, \quad \psi, \phi \in \mathcal{D}_w,$$

i.e., (T, H) satisfies the CCR in the sense of sesquilinear form on \mathcal{D}_w . Obviously a time operator T of H is a weak time operator of H . But the converse is not true (it is easy to see, however, that, if T is a weak time operator of H and $\mathcal{D}_w \subset D(TH) \cap D(HT)$, then T is a time operator). An important aspect of a weak time operator T of H is that a *time-energy uncertainty relation* is naturally derived [2, Proposition 4.1]: for all unit vectors ψ in $\mathcal{D}_w \subset D(T) \cap D(H)$,

$$(\Delta T)_\psi (\Delta H)_\psi \geq \frac{1}{2},$$

where, for a linear operator A on \mathcal{H} and $\phi \in D(A)$ with $\|\phi\| = 1$,

$$(\Delta A)_\phi := \|(A - \langle \phi, A\phi \rangle)\phi\|,$$

called the *uncertainty of A in the vector ϕ* .

In contrast to the weak form of time operator, there is a strong form. We say that T is a *strong time operator* of H if, for all $t \in \mathbb{R}$, $e^{-itH}D(T) \subset D(T)$ and

$$Te^{-itH}\psi = e^{-itH}(T+t)\psi, \quad \psi \in D(T). \quad (1.2)$$

We call (1.2) the *weak Weyl relation* [2]. From a representation theoretic point of view, we call a pair (T, H) obeying the weak Weyl relation a *weak Weyl representation* of the CCR. This type of representation of the CCR was extensively studied by Schmüdgen [17, 18]. It is shown that a strong time operator of H is a time operator of H [12]. But the converse is not true. In fact, the time operators considered in the present paper are not strong ones.

There is a generalized version of strong time operator [2]. We say that T is a *generalized time operator* of H if, for each $t \in \mathbb{R}$, there is a *bounded self-adjoint* operator $K(t)$ on \mathcal{H} with $D(K(t)) = \mathcal{H}$, $e^{-itH}D(T) \subset D(T)$ and a *generalized weak Weyl relation* (GWWR)

$$Te^{-itH}\psi = e^{-itH}(T + K(t))\psi \quad (\forall \psi \in D(T)) \quad (1.3)$$

holds. In this case, the bounded operator-valued function $K(t)$ of $t \in \mathbb{R}$ is called the *commutation factor* of the GWWR under consideration.

We now come to the subject of the present paper. In his interesting paper [9], Galapon showed by an explicit construction that, for every self-adjoint operator H (a Hamiltonian) on an abstract Hilbert space \mathcal{H} which is *bounded below* and has purely discrete spectrum with some growth condition, there is a time operator T_1 on \mathcal{H} , which is a bounded self-adjoint operator under an additional condition (for the definition of T_1 , see (2.12) below). To be definite, we call the operator T_1 introduced in [9] the *Galapon time operator*.

An important point of Galapon's work [9] is in that it disproved the long-standing belief or folklore among physicists that there is no self-adjoint operator canonically conjugate to a Hamiltonian which is bounded below (for a historical survey, see Introduction of [9]).

Motivated by work of Galapon [9], we investigate, in this paper, properties of time operators of a self-adjoint operator H with purely discrete spectrum. In Section 2, we introduce a densely defined linear operator T whose restriction to a subspace yields the Galapon time operator T_1 and prove basic properties of T and T_1 , in particular the closedness of T . It follows that, if T is bounded, then T is self-adjoint with $D(T) = \mathcal{H}$ and a time operator of H . We denote by $T^\#$ one of T_1 , T and T^* (the adjoint of T). In Section 3, we discuss some general properties of $T^\#$. Moreover the reflection symmetry of the spectrum of $T^\#$ with respect to the imaginary axis is proved. Sections 4–6 are the main parts of this paper. In Section 4, we present a general criterion for T to be bounded with $D(T) = \mathcal{H}$, while, in Section 5, we give a sufficient condition for T to be unbounded. In Section 6, we present a necessary and sufficient condition for T to be Hilbert-Schmidt. In Section 7, we show that, under some condition, the Galapon time operator is a generalized time operator of H , too. We also discuss non-differentiability of the commutation factor K in the GWWR for (T_1, H) . In the last section, we consider a perturbation of H by a symmetric operator and try to draw out physical meanings of T_1 and K in the context of regular perturbation theory.

2 Time Operators

In this section, we recapitulate some basic aspects of the Galapon time operator in more apparent manner than in [9].

Let \mathcal{H} be a complex Hilbert space and

H be a self-adjoint operator on \mathcal{H} which has the following properties (H.1) and (H.2):

(H.1) The spectrum of H , denoted $\sigma(H)$, is purely discrete with $\sigma(H) = \{E_n\}_{n=1}^{\infty}$, where each eigenvalue E_n of H is simple and

$$0 < E_n < E_{n+1}$$

for all $n \in \mathbb{N}$ (the set of positive integers).

(H.2) $\sum_{n=1}^{\infty} \frac{1}{E_n^2} < \infty$.

Throughout the present paper we assume (H.1) and (H.2).

Remark 2.1 The positivity condition $E_n > 0$ for the eigenvalues of H does not lose generality, because, if H is bounded below, but not strictly positive, then one needs only to consider, instead of H , $\tilde{H} := H + c$ with a constant $c > -\inf \sigma(H)$, which is a strictly positive self-adjoint operator.

Property (H.2) implies that

$$E_n \rightarrow \infty \quad (n \rightarrow \infty). \tag{2.1}$$

Let e_n be a normalized eigenvector of H belonging to eigenvalue E_n :

$$He_n = E_n e_n, \quad n \in \mathbb{N}. \tag{2.2}$$

Then, by property (H.1), the set $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal system (C.O.N.S.) of \mathcal{H} .

Lemma 2.1

(i) For all $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{(E_n - E_m)^2} < \infty. \tag{2.3}$$

In particular, for each $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{E_n - E_m} e_n$$

converges in \mathcal{H} .

(ii) For all $n \in \mathbb{N}$ and vectors ψ in \mathcal{H} , the infinite series

$$\sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \quad (2.4)$$

absolutely converges.

Proof. (i) By (2.1), we have

$$C_m := \sup_{n \neq m} \frac{1}{\left(1 - \frac{E_m}{E_n}\right)^2} < \infty. \quad (2.5)$$

Hence we have

$$\sum_{n \neq m}^{\infty} \frac{1}{|E_n - E_m|^2} \leq C_m \sum_{n \neq m}^{\infty} \frac{1}{E_n^2} < \infty.$$

(ii) By the Cauchy-Schwarz inequality, the Parseval equality and part (i), we have

$$\begin{aligned} \sum_{m \neq n}^{\infty} \left| \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right| &\leq \left(\sum_{m \neq n}^{\infty} |\langle e_m, \psi \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{m \neq n}^{\infty} \left| \frac{1}{E_n - E_m} \right|^2 \right)^{\frac{1}{2}} \\ &\leq \|\psi\| \left(\sum_{m \neq n}^{\infty} \left| \frac{1}{E_n - E_m} \right|^2 \right)^{\frac{1}{2}} < \infty. \end{aligned} \quad (2.6)$$

□

By Lemma 2.1-(ii), one can define a linear operator T on \mathcal{H} as follows:

$$D(T) := \left\{ \psi \in \mathcal{H} \mid \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2 < \infty \right\}, \quad (2.7)$$

$$T\psi := i \sum_{n=1}^{\infty} \left(\sum_{m \neq n} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right) e_n, \quad \psi \in D(T). \quad (2.8)$$

Note that

$$\|T\psi\|^2 = \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2. \quad (2.9)$$

For a subset $\mathcal{D} \subset \mathcal{H}$, we denote by $\text{l.i.h.}(\mathcal{D})$ the subspace algebraically spanned by the vectors of \mathcal{D} .

The subspace

$$\mathcal{D}_0 := \text{l.i.h.}(\{e_n\}_{n=1}^{\infty}) \quad (2.10)$$

is dense in \mathcal{H} .

Lemma 2.2 *The operator T is densely defined with $\mathcal{D}_0 \subset D(T)$ and*

$$Te_k = i \sum_{n \neq k}^{\infty} \frac{1}{E_n - E_k} e_n, \quad k \in \mathbb{N}. \quad (2.11)$$

Proof. To prove $\mathcal{D}_0 \subset D(T)$, it is sufficient to show that $e_k \in D(T)$, $k \in \mathbb{N}$. Putting

$$c_n(k) := \sum_{m \neq n}^{\infty} \frac{\langle e_m, e_k \rangle}{E_n - E_m},$$

we have $c_k(k) = 0$ and $c_n(k) = 1/(E_n - E_k)$ for $n \neq k$. Hence, by Lemma 2.1-(i), we have

$$\sum_{n=1}^{\infty} |c_n(k)|^2 = \sum_{n \neq k}^{\infty} \frac{1}{(E_n - E_k)^2} < \infty.$$

Hence $e_k \in D(T)$ and (2.11) holds. \square

In general, it is not clear whether or not T is a symmetric operator. But a restriction of T to a smaller subspace gives a symmetric operator. Indeed, we have the following fact:

Lemma 2.3 ([9]) *The operator*

$$T_1 := T|_{\mathcal{D}_0} \quad (2.12)$$

(the restriction of T to \mathcal{D}_0) is symmetric.

Proof. It is enough to show that, for all $\psi \in \mathcal{D}_0$, $\langle \psi, T\psi \rangle$ is real. For a complex number $z \in \mathbb{C}$ (the set of complex numbers), we denote its complex conjugate by z^* . We have

$$\langle \psi, T\psi \rangle = i \sum_{n=1}^{\infty} \langle \psi, e_n \rangle \sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m}.$$

Hence

$$\langle \psi, T\psi \rangle^* = i \sum_{n=1}^{\infty} \langle e_n, \psi \rangle \sum_{m \neq n}^{\infty} \frac{\langle \psi, e_m \rangle}{E_m - E_n}.$$

Since ψ is in \mathcal{D}_0 , the double sum on m, n with $m \neq n$ is a sum consisting of a finite term. Hence we can exchange the sum on n and that on m to obtain

$$\langle \psi, T\psi \rangle^* = i \sum_{m=1}^{\infty} \langle \psi, e_m \rangle \sum_{n \neq m}^{\infty} \frac{\langle e_n, \psi \rangle}{E_m - E_n} = \langle \psi, T\psi \rangle.$$

Hence $\langle \psi, T\psi \rangle$ is real. \square

The operator T_1 defined by (2.12) is the time operator introduced by Galapon in [9]. Obviously we have

$$T_1 \subset T. \quad (2.13)$$

Hence

$$T^* \subset T_1^*. \quad (2.14)$$

Remark 2.2 It is asserted in [9] that T_1 is essentially self-adjoint without additional conditions. But, unfortunately, we find that this is not conclusive, because the proof of it given in [9] (pp.2678–2679) has some gap: the interchange of the double sum in Equation (2.30) on p.2678 in [9] may not be justified, at least, by the reasoning given there. The assertion is true in the case where T_1 becomes a bounded operator under an additional condition for $\{E_n\}_{n=1}^\infty$, as we show below in the present paper. But, in the case where T_1 is unbounded, it seems to be very difficult to prove or disprove the essential self-adjointness of T_1 . We leave this problem for future study.

Lemma 2.4 *The subspace*

$$\mathcal{D}_c := \text{l.i.h.}(\{e_n - e_m \in \mathcal{H} | n, m \geq 1\}). \quad (2.15)$$

is dense in the Hilbert space \mathcal{H} . Moreover

$$\mathcal{D}_c \subset \mathcal{D}_0 \subset D(T). \quad (2.16)$$

Proof. Let $\psi \in \mathcal{D}_c^\perp$ (the orthogonal complement of \mathcal{D}_c). Then, for all $m, n \geq 1$, $|\langle e_n, \psi \rangle|^2 = |\langle e_m, \psi \rangle|^2$. By the Parseval equality, $\|\psi\|^2 = \lim_{N \rightarrow \infty} N |\langle e_m, \psi \rangle|^2$. This implies that $|\langle e_m, \psi \rangle|^2 = 0$ for all $m \geq 1$ and $\|\psi\| = 0$. Hence $\psi = 0$. Thus \mathcal{D}_c is dense in \mathcal{H} . Inclusion relation (2.16) is obvious. \square

Theorem 2.5 ([9]) *It holds that*

$$\mathcal{D}_c \subset D(T_1 H) \cap D(H T_1) \quad (2.17)$$

and

$$[T_1, H]\psi = i\psi, \quad \psi \in \mathcal{D}_c. \quad (2.18)$$

Theorem 2.5 shows that T_1 is a time operator of H .

Remark 2.3 It is easy to see that, for all $k \in \mathbb{N}$, $T_1 e_k \notin D(H)$. Hence $\mathcal{D}_0 \not\subset D(H T_1)$. Therefore one can not consider the commutation relation $[T_1, H]$ on \mathcal{D}_0 . Moreover, by direct computation, we have

$$\langle T_1 e_k, H e_\ell \rangle - \langle H e_k, T_1 e_\ell \rangle = -i(1 - \delta_{k\ell}), \quad k, \ell \in \mathbb{N}. \quad (2.19)$$

This means that (T_1, H) does not satisfy the CCR in the sense of sesquilinear form on \mathcal{D}_0 (a weak form of the CCR), either. These facts suggest that the pair (T_1, H) is very sensitive to the domain on which their commutation relation is applied.

In concluding this section we discuss shortly non-uniqueness of time operators of H . We introduce a set of symmetric operators associated with H :

$$\begin{aligned} \{H\}'_{\mathcal{D}_c} &:= \{S | S \text{ is a symmetric operator on } \mathcal{H} \text{ such that} \\ &\quad \mathcal{D}_c \subset D(SH) \cap D(HS) \text{ and } SH\psi = HS\psi, \forall \psi \in \mathcal{D}_c\}, \end{aligned} \quad (2.20)$$

which may be viewed as a commutant of $\{H\}$ in a restricted sense. It is easy to see that, for all real-valued continuous function f on \mathbb{R} , the operator $f(H)$ defined via the functional calculus is in $\{H\}'_{\mathcal{D}_c}$.

Proposition 2.6 For all $S \in \{H\}'_{\mathcal{D}_c}$, $\mathcal{D}_c \subset D((T_1 + S)H) \cap D(H(T_1 + S))$ and

$$[T_1 + S, H]\psi = i\psi, \quad \psi \in \mathcal{D}_c. \quad (2.21)$$

Proof. A direct computation using Theorem 2.5 and (2.20). \square

Proposition 2.7 Let T_2 be a time operator of H such that $\mathcal{D}_c \subset D(T_2H) \cap D(HT_2)$ and

$$[T_2, H]\psi = i\psi, \quad \forall \psi \in \mathcal{D}_c.$$

Then $T_2 = T_1 + S$ with some $S \in \{H\}'_{\mathcal{D}_c}$.

Proof. We need only to show that $S := T_2 - T_1$ is in $\{H\}'_{\mathcal{D}_c}$. But this is obvious. \square

3 General Properties

3.1 Closedness of T and symmetry of T^*

Lemma 3.1 $\mathcal{D}_0 \subset D(T^*)$ and $T^*|_{\mathcal{D}_0} = T_1$, i.e., $T_1 \subset T^*$.

Proof. It is enough to show that, for all $k \in \mathbb{N}$, $e_k \in D(T^*)$ and $T^*e_k = Te_k (= T_1e_k)$. It is easy to see that, for all $\psi \in D(T)$,

$$\langle e_k, T\psi \rangle = i \sum_{m \neq k}^{\infty} \frac{\langle e_m, \psi \rangle}{E_k - E_m}. \quad (3.1)$$

By Lemma 2.2, the right hand side is equal to $\langle Te_k, \psi \rangle$. Hence $e_k \in D(T^*)$ and $T^*e_k = Te_k$. \square

Proposition 3.2 The operator T is closed and

$$T^* \subset T. \quad (3.2)$$

In particular, if T is bounded, then T is self-adjoint with $D(T) = \mathcal{H}$.

Proof. Let $\psi_k \in D(T)$, $k \in \mathbb{N}$ and $\psi_k \rightarrow \psi \in \mathcal{H}$, $T\psi_k \rightarrow \phi \in \mathcal{H}$ as $k \rightarrow \infty$. Then $\sup_{k \geq 1} \|T\psi_k\| < \infty$. Hence, by (2.9), there exists a constant $C > 0$ independent of $k \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \left| \sum_{n \neq m} \frac{\langle e_m, \psi_k \rangle}{E_n - E_m} \right|^2 \leq C.$$

By (2.6), we have

$$\lim_{k \rightarrow \infty} \sum_{n \neq m}^{\infty} \frac{\langle e_m, \psi_k \rangle}{E_n - E_m} = \sum_{n \neq m}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m}. \quad (3.3)$$

Hence it follows that

$$\sum_{n=1}^{\infty} \left| \sum_{n \neq m}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2 \leq C.$$

Therefore $\psi \in D(T)$. By (3.1) and (3.3), we have for all $\ell \in \mathbb{N}$

$$\lim_{k \rightarrow \infty} \langle e_\ell, T\psi_k \rangle = \langle e_\ell, T\psi \rangle.$$

Hence $\langle e_\ell, \phi \rangle = \langle e_\ell, T\psi \rangle$, $\ell \in \mathbb{N}$, implying $\phi = T\psi$. Thus T is closed.

To prove (3.2), let $\psi \in D(T^*)$. Putting $\eta = T^*\psi$, we have $\langle \eta, \chi \rangle = \langle \psi, T\chi \rangle$ for all $\chi \in D(T)$. Taking $\chi = e_k$ ($k \in \mathbb{N}$), we have

$$\langle \eta, e_k \rangle = i \sum_{n \neq k}^{\infty} \frac{\langle \psi, e_n \rangle}{E_n - E_k}, \quad (3.4)$$

which implies that

$$\sum_{k=1}^{\infty} \left| \sum_{n \neq k}^{\infty} \frac{\langle \psi, e_n \rangle}{E_n - E_k} \right|^2 = \|\eta\|^2 < \infty.$$

Hence $\psi \in D(T)$. Then, by (3.1), the right hand side of (3.4) is equal to $\langle T\psi, e_k \rangle$. Hence $\eta = T\psi$. Thus (3.2) holds.

Let T be bounded. Then, by the denseness of $D(T)$ and the closedness of T , $D(T) = \mathcal{H}$. Hence $D(T^*) = \mathcal{H}$. Thus, by (3.2), $T^* = T$, i.e., T is self-adjoint. \square

Corollary 3.3 *The operator T^* is symmetric.*

Proof. By Lemma 3.1, T^* is densely defined. Hence, by Proposition 3.2, $T^* \subset T = (T^*)^*$. Thus T^* is symmetric. \square

Thus we have

$$T_1 \subset T^* \subset T. \quad (3.5)$$

Corollary 3.3 shows that T^* also is a time operator of H .

For a closable operator A on a Hilbert space, we denote its closure by \bar{A} .

Proposition 3.4 $\bar{T}_1 = T^*$.

Proof. Note that $\bar{T}_1 = T^*$ if and only if $T_1^* = T$. By (3.5), we have $\bar{T}_1 \subset T^*$. Hence $T \subset T_1^*$. Thus it is enough to show that $D(T_1^*) \subset D(T)$. For all $\psi \in D(T_1^*)$, we have

$$\langle T_1^*\psi, e_l \rangle = \langle \psi, T_1 e_l \rangle = i \sum_{n \neq l}^{\infty} \frac{\langle \psi, e_n \rangle}{E_n - E_l}.$$

Hence we obtain

$$\infty > \|T_1^*\psi\|^2 = \sum_{l=1}^{\infty} |\langle T_1^*\psi, e_l \rangle|^2 = \sum_{l=1}^{\infty} \left| \sum_{n \neq l}^{\infty} \frac{\langle \psi, e_n \rangle}{E_n - E_l} \right|^2,$$

implying that $\psi \in D(T)$. Thus $D(T_1^*) \subset D(T)$. \square

3.2 Absence of invariant dense domains for T under some condition

We first note the following general fact:

Proposition 3.5 *Let Q be a bounded self-adjoint operator on \mathcal{H} and P be a self-adjoint operator on \mathcal{H} . Suppose that there is a dense subspace \mathcal{D} in \mathcal{H} such that the following (i)–(iii) hold:*

- (i) $Q\mathcal{D} \subset \mathcal{D} \subset D(P)$.
- (ii) \mathcal{D} is a core of P .
- (iii) The pair (Q, P) obeys the CCR on \mathcal{D} : $[Q, P]\psi = i\psi, \forall \psi \in \mathcal{D}$.

Then $\sigma(P) = \mathbb{R}$.

Proof. Since Q is a bounded self-adjoint operator, we have for all $t \in \mathbb{R}$

$$e^{itQ} = \sum_{k=0}^{\infty} \frac{(itQ)^k}{k!}$$

in operator norm. Conditions (i) and (iii) imply that, for all $k \in \mathbb{N}$ and $\psi \in \mathcal{D}$

$$Q^k P\psi - PQ^k\psi = ikQ^{k-1}\psi.$$

Hence, for all $t \in \mathbb{R}$ and vectors ψ in \mathcal{D} , we have

$$\begin{aligned} e^{itQ}P\psi &= P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} Q^k P\psi \\ &= P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} (PQ^k + ikQ^{k-1})\psi \\ &= P\psi + \sum_{k=1}^{\infty} \left\{ P \frac{(itQ)^k}{k!} \psi - t \frac{(itQ)^{k-1}}{(k-1)!} \psi \right\}. \end{aligned}$$

It follows from the closedness of P that $e^{itQ}\psi$ is in $D(P)$ and

$$Pe^{itQ}\psi = e^{itQ}P\psi + te^{itQ}\psi. \quad (3.6)$$

By condition (ii), this equality extends to all $\psi \in D(P)$ with $e^{itQ}\psi \in D(P), \forall t \in \mathbb{R}, \forall \psi \in D(P)$. Hence the operator equality $e^{-itQ}Pe^{itQ} = P + t$ follows. Thus $\sigma(P) = \sigma(P + t)$ for all $t \in \mathbb{R}$. This implies that $\sigma(P) = \mathbb{R}$. \square

Theorem 3.6 *If T is bounded (hence self-adjoint by Proposition 3.2), then there is no dense subspace \mathcal{D} in \mathcal{H} such that the following (i)–(iii) hold:*

- (i) $T\mathcal{D} \subset \mathcal{D} \subset D(H)$.

(ii) \mathcal{D} is a core of H .

(iii) The pair (T, H) obeys the CCR on \mathcal{D} ,

Proof. Suppose that there were such a dense subspace \mathcal{D} as stated above. Then we can apply Proposition 3.5 with $(Q, P) = (T, H)$ to conclude that $\sigma(H) = \mathbb{R}$. But this is a contradiction. \square

Remark 3.1 A special case of this theorem was established in [7, Theorem 9.5].

3.3 Reflection symmetry of the spectrum of T_1, T^* and T

We first recall the definition of the spectrum of a general linear operator (not necessarily closed). For a linear operator A on a Hilbert space \mathcal{K} , the resolvent set of A , denoted $\rho(A)$, is defined by

$$\rho(A) := \{z \in \mathbb{C} \mid \text{Ran}(A - z) \text{ (the range of } A - z) \text{ is dense in } \mathcal{K} \text{ and } A - z \text{ is injective with } (A - z)^{-1} \text{ bounded}\}.$$

Then the set

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

is called the spectrum of A . The set of eigenvalues of A , called the point spectrum of A , is denoted $\sigma_p(A)$.

We denote by $T^\#$ any of T_1, T^* and T .

We define a conjugation J on \mathcal{H} by

$$J\psi := \sum_{n=1}^{\infty} \langle \psi, e_n \rangle e_n, \quad \psi \in \mathcal{H}. \quad (3.7)$$

Proposition 3.7 *The spectrum $\sigma(T^\#)$ of $T^\#$ is reflection symmetric with respect to the imaginary axis, i.e., if $z \in \sigma(T^\#)$, then $-z^* \in \sigma(T^\#)$. In particular, if T is self-adjoint, then $\sigma(T)$ is reflection symmetric with respect to the origin of the real axis. Moreover, for all $z \in \sigma_p(T^\#)$, $-z^*$ is in $\sigma_p(T^\#)$ and*

$$J \ker(T^\# - z) = \ker(T^\# + z^*), \quad \forall z \in \sigma_p(T^\#). \quad (3.8)$$

Proof. It is easy to see that operator equality $JT^\#J = -T^\#$ holds ($JD(T^\#) = D(T^\#)$). Hence, for all $z \in \mathbb{C}$, we have $J(T^\# - z)J = -(T^\# + z^*) \cdots (*)$. This implies that, if $z \in \rho(T^\#)$, then $-z^* \in \rho(T^\#)$. Thus the same holds for the spectrum $\sigma(T^\#) = \mathbb{C} \setminus \rho(T^\#)$. Equation (3.8) follows from $(*)$. \square

4 Boundedness of T

In this section we present a general criterion for the operator T to be bounded. For mathematical generality and for later use, we consider a more general class of operators than that of T . Let $b := \{b_{nm}\}_{n,m=1}^{\infty}$ be a double sequence of complex numbers such that

$$\|b\|_{\infty} := \sup_{n,m \geq 1} |b_{nm}| < \infty. \quad (4.1)$$

Then, in the same way as in Lemma 2.1-(ii), for all $\psi \in \mathcal{H}$, the infinite series

$$\sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle$$

absolutely converges. Hence one can define a linear operator T_b on \mathcal{H} as follows:

$$D(T_b) := \left\{ \psi \in \mathcal{H} \mid \sum_{n=1}^{\infty} \left| \sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right\}, \quad (4.2)$$

$$T_b \psi := i \sum_{n=1}^{\infty} \left(\sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right) e_n, \quad \psi \in D(T_b). \quad (4.3)$$

Obviously $T = T_b$ with b satisfying $b_{nm} = 1$ for all $n, m \in \mathbb{N}$. In the same way as in the case of T , one can prove the following fact:

Lemma 4.1 *The operator T_b is closed.*

The following lemma is probably well known (but, for the completeness, we give a proof):

Lemma 4.2 *Let A be a densely defined linear operator on a Hilbert space \mathcal{K} . Suppose that there exist a dense subspace \mathcal{D} in \mathcal{K} and a constant $C > 0$ such that $\mathcal{D} \subset D(A)$ and*

$$|\langle \psi, A\psi \rangle| \leq C \|\psi\|^2, \quad \psi \in \mathcal{D}.$$

Then A is bounded with $\|\bar{A}\| \leq 2C$, where \bar{A} is the closure of A .

If A is symmetric in addition, then $\|\bar{A}\| \leq C$.

Proof. Let $\psi, \phi \in \mathcal{D}$. Then, by the polarization identity

$$\begin{aligned} \langle \psi, A\phi \rangle &= \frac{1}{4} (\langle \psi + \phi, A(\psi + \phi) \rangle - \langle \psi - \phi, A(\psi - \phi) \rangle \\ &\quad + i \langle \psi - i\phi, A(\psi - i\phi) \rangle - i \langle \psi + i\phi, A(\psi + i\phi) \rangle), \end{aligned}$$

we have

$$|\langle \psi, A\phi \rangle| \leq \frac{C}{4} (\|\psi + \phi\|^2 + \|\psi - \phi\|^2 + \|\psi - i\phi\|^2 + \|\psi + i\phi\|^2) = C(\|\psi\|^2 + \|\phi\|^2).$$

Replacing $\psi \neq 0$ by $\|\phi\|\psi/\|\psi\|$ we have

$$|\langle \psi, A\phi \rangle| \leq 2C\|\psi\|\|\phi\|.$$

For $\psi = 0$, this inequality trivially holds. Since \mathcal{D} is dense, it follows from the Riesz representation theorem that $\|A\phi\| \leq 2C\|\phi\|$, $\phi \in \mathcal{D}$. Thus the first half of the lemma follows.

Let A be symmetric. Then, $\langle \psi, A\psi \rangle \in \mathbb{R}$ for all $\psi \in D(A)$. Hence

$$|\Re \langle \psi, A\phi \rangle| = \frac{1}{4} |\langle \psi + \phi, A(\psi + \phi) \rangle - \langle \psi - \phi, A(\psi - \phi) \rangle| \leq \frac{C}{2} (\|\psi\|^2 + \|\phi\|^2), \quad \psi \in \mathcal{D}.$$

We write $\langle \psi, A\phi \rangle = |\langle \psi, A\phi \rangle| e^{i\theta}$ with $\theta \in \mathbb{R}$. Then $|\langle \psi, A\phi \rangle| = \langle e^{i\theta}\psi, A\phi \rangle$. Hence

$$\begin{aligned} |\langle \psi, A\phi \rangle| &= \Re \langle e^{i\theta}\psi, A\phi \rangle \leq \frac{C}{2} (\|e^{i\theta}\psi\|^2 + \|\phi\|^2) \\ &= \frac{C}{2} (\|\psi\|^2 + \|\phi\|^2). \end{aligned}$$

Thus, in the same manner as above, we can obtain $|\langle \psi, A\phi \rangle| \leq C\|\psi\|\|\phi\|$, $\psi, \phi \in \mathcal{D}$. \square

The next lemma is easily proven by elementary calculus. Therefore we omit proof of it.

Lemma 4.3 *For all $s > 1$ and $n \geq 2$,*

$$\sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \leq \frac{\log n}{n^{s-1}} + \frac{1}{s(n-1)^{s-1}}. \quad (4.4)$$

Lemma 4.4 *Let $s > 1$. Then*

$$\sup_{n \geq 2} \left(\sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \right) < \infty \quad (4.5)$$

and

$$\sup_{m \geq 1} \left(\sum_{n=m+1}^{\infty} \frac{1}{n^s - m^s} \right) < \infty. \quad (4.6)$$

Proof. Property (4.5) follows from Lemma 4.3.

To prove (4.6), we write

$$\sum_{n=m+1}^{\infty} \frac{1}{n^s - m^s} \leq \int_{m+1}^{\infty} \frac{1}{x^s - m^s} dx + \frac{1}{(m+1)^s - m^s}.$$

We fix a constant $R > 2(\geq (m+1)/m)$. By the change of variable $x = my$, we have

$$\int_{m+1}^{\infty} \frac{1}{x^s - m^s} dx = \frac{1}{m^{s-1}} \left(\int_{(m+1)/m}^R \frac{1}{y^s - 1} ds + C_R \right),$$

where

$$C_R := \int_R^\infty \frac{1}{y^s - 1} ds < \infty.$$

Using the well known inequality

$$x^s - 1 \geq s(x - 1), \quad x > 0, s \geq 1, \quad (4.7)$$

we have

$$\begin{aligned} \int_{(m+1)/m}^R \frac{1}{y^s - 1} dy &\leq \int_{(m+1)/m}^R \frac{1}{s(y - 1)} dy \\ &= \frac{1}{s} (\log(R - 1) + \log m). \end{aligned}$$

Hence

$$\int_{m+1}^\infty \frac{1}{x^s - m^s} dx \leq \frac{\log m}{sm^{s-1}} + \frac{\log(R - 1)}{sm^{s-1}} + \frac{1}{m^{s-1}} C_R.$$

Thus (4.6) follows. \square

Let

$$c_H(n) := \sum_{m=1}^{n-1} \frac{E_n}{(E_n - E_m)E_m}, \quad n \geq 2, \quad (4.8)$$

$$d_H(m) := \sum_{n=m+1}^\infty \frac{E_m}{(E_n - E_m)E_n}, \quad m \geq 1. \quad (4.9)$$

Since $c_H(n)$ and $d_H(m)$ are positive (recall that $E_n > 0, \forall n \in \mathbb{N}$), one can define constants

$$c_H := \sup_{n \geq 2} c_H(n), \quad (4.10)$$

$$d_H := \sup_{m \geq 1} d_H(m), \quad (4.11)$$

which are finite or infinite.

Theorem 4.5 *Suppose that there exist constants $\alpha > 1$, $C > 0$ and $a > 0$ such that*

$$E_n - E_m \geq C(n^\alpha - m^\alpha), \quad n > m > a. \quad (4.12)$$

Then T_b is a bounded operator with $D(T_b) = \mathcal{H}$ and

$$\|T_b\| \leq 4\|b\|_\infty \sqrt{c_H d_H}. \quad (4.13)$$

Moreover, if $b_{nm}^ = b_{mn}$ for all $m, n \in \mathbb{N}$, then T_b is a bounded self-adjoint operator with $D(T_b) = \mathcal{H}$ and*

$$\|T_b\| \leq 2\|b\|_\infty \sqrt{c_H d_H}. \quad (4.14)$$

In particular, T is a bounded self-adjoint operator with $D(T) = \mathcal{H}$.

Proof. By Lemma 4.2, it is enough to show that c_H and d_H are finite and

$$|\langle \psi, T_b \psi \rangle| \leq 2 \|b\|_\infty \sqrt{c_H d_H} \|\psi\|^2, \quad \psi \in \mathcal{D}_0. \quad (4.15)$$

Then T_b is bounded with (4.13). Since T_b is densely defined and closed, it follows that $D(T_b) = \mathcal{H}$. As in the case of T , one can show that, if $b_{nm}^* = b_{mn}$ for all $m, n \in \mathbb{N}$, then $T_b|_{\mathcal{D}_0}$ is symmetric and hence T_b is a bounded self-adjoint operator with $D(T_b) = \mathcal{H}$ and (4.14) holds. Therefore the desired result follows.

To prove (4.15), we first note that, for $\psi \in \mathcal{D}_0$,

$$\langle \psi, T_b \psi \rangle = i \sum_{m, n=1, m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle \psi, e_n \rangle \langle e_m, \psi \rangle.$$

Hence

$$|\langle \psi, T_b \psi \rangle| \leq 2 \|b\|_\infty A(\psi),$$

where

$$A(\psi) := \sum_{n > m \geq 1} \frac{|\langle e_m, \psi \rangle| |\langle \psi, e_n \rangle|}{E_n - E_m}.$$

Inserting $1 = \sqrt{E_m/E_n} \cdot \sqrt{E_n/E_m}$ into the summand on the right hand side and using the Cauchy-Schwarz inequality, we have

$$A(\psi)^2 \leq B(\psi)C(\psi)$$

with

$$\begin{aligned} B(\psi) &= \sum_{n > m \geq 1} \frac{|\langle e_n, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_n}{E_m}, \\ C(\psi) &= \sum_{n > m \geq 1} \frac{|\langle e_m, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_m}{E_n}. \end{aligned}$$

One can rewrite and estimate $B(\psi)$ as follows:

$$\begin{aligned} B(\psi) &= \sum_{n=2}^{\infty} |\langle e_n, \psi \rangle|^2 c_H(n) \\ &\leq \|\psi\|^2 c_H. \end{aligned}$$

Similarly we have

$$C(\psi) \leq \|\psi\|^2 d_H. \quad (4.16)$$

Hence

$$|\langle \psi, T_b \psi \rangle| \leq 2 \|b\|_\infty \sqrt{c_H d_H} \|\psi\|^2. \quad (4.17)$$

Therefore we need only to prove that c_H and d_H are finite.

We can write

$$c_H(n) = \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \sum_{m=1}^{n-1} \frac{1}{E_m}.$$

By assumption (4.12), we have

$$\frac{1}{E_n - E_m} \leq \frac{1}{C(n^\alpha - m^\alpha)}, \quad n > m > a. \quad (4.18)$$

Since we have

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

it follows that

$$\varepsilon_1 := \sum_{m=1}^{\infty} \frac{1}{E_m} < \infty.$$

Thus

$$c_H(n) \leq \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \varepsilon_1.$$

Let $n_0 \geq 2$ be a natural number such that $n_0 > a$. Then, for all $n > n_0$

$$\sum_{m=1}^{n-1} \frac{1}{E_n - E_m} \leq \sum_{m=1}^{n_0-1} \frac{1}{E_n - E_m} + \frac{1}{C} \sum_{m=n_0}^{n-1} \frac{1}{n^\alpha - m^\alpha}.$$

By (4.4), the right hand side is uniformly bounded in n . Thus we have $c_H < \infty$.

To prove $d_H < \infty$, we write for $m > a$

$$\begin{aligned} d_H(m) &= \sum_{n=m+1}^{\infty} \frac{1}{(E_n - E_m)} - \sum_{n=m+1}^{\infty} \frac{1}{E_n} \\ &\leq \sum_{n=m+1}^{\infty} \frac{1}{(E_n - E_m)} \\ &\leq \frac{1}{C} \sum_{n=m+1}^{\infty} \frac{1}{n^\alpha - m^\alpha}. \end{aligned}$$

Hence, by (4.6) in Lemma 4.4, we have

$$\sup_{m>a} d_H(m) < \infty.$$

Thus it follows that $d_H < \infty$. □

Example 4.1 Let $\lambda > 0, \alpha > 1$ and $P(x)$ be a real polynomial of $x \in \mathbb{R}$ with degree $p < \alpha$. Then it is easy to see that the sequence $\{E_n\}_{n=1}^{\infty}$ defined by

$$E_n := \lambda n^\alpha + P(n)$$

satisfies the assumptions (H.1), (H.2) and (4.12). Thus, by Theorem 4.5, in the present example, T_b is bounded.

We remark that Theorem 4.5 does not cover the case $E_n = \lambda n + \mu$ with constants $\lambda > 0$ and $\mu \in \mathbb{R}$. For this case, we have the following theorem:

Theorem 4.6 *Suppose that there exist constants $\lambda > 0, \mu \in \mathbb{R}$ and $a > 0$ such that*

$$E_n = \lambda n + \mu, \quad n > a. \quad (4.19)$$

Then T is a bounded self-adjoint operator with $D(T) = \mathcal{H}$.

Proof. Let k_0 be the greatest integer such that $k_0 \leq a$. Let $a_n := \langle e_n, \psi \rangle$ ($\psi \in \mathcal{H}$). Then, by the Parseval equality, we have $\sum_{n=1}^{\infty} |a_n|^2 = \|\psi\|^2$. Let $\psi \in \mathcal{D}_0$. Then we can write:

$$\langle \psi, T\psi \rangle = S_I + S_2 + S_3 + S_4,$$

where

$$\begin{aligned} S_1 &:= i \sum_{n=1}^{k_0} \sum_{m \neq n}^{k_0} \frac{a_n^* a_m}{E_n - E_m}, \\ S_2 &:= i \sum_{n=1}^{k_0} \sum_{m \geq k_0+1}^{\infty} \frac{a_n^* a_m}{E_n - E_m}, \\ S_3 &:= i \sum_{n \geq k_0+1}^{\infty} \sum_{m=1}^{k_0} \frac{a_n^* a_m}{E_n - E_m}, \\ S_4 &:= i \frac{1}{\lambda} \sum_{n=k_0+1}^{\infty} \sum_{m \neq n, m \geq k_0+1}^{\infty} \frac{a_n^* a_m}{n - m}. \end{aligned}$$

By the Schwarz inequality, we have

$$|S_j| \leq C_j \|\psi\|^2, \quad j = 1, 2, 3,$$

where $C_j > 0$ is a constant. To estimate $|S_4|$, we use the following well known inequality [11, Theorem 294]:

$$\left| \sum_{n,m=1, n \neq m}^{\infty} \frac{x_n y_m}{n - m} \right| \leq \pi \sqrt{\sum_{n=1}^{\infty} x_n^2} \sqrt{\sum_{m=1}^{\infty} y_m^2}$$

for all real sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$. Hence

$$|S_4| \leq \pi \|\psi\|^2.$$

Therefore it follows that $|\langle \psi, T\psi \rangle| \leq \text{const.} \|\psi\|^2$. Thus T is bounded. \square

Example 4.2 A physically interesting case is the case where $E_n = \omega(n + \frac{1}{2})$, $n \in \{0\} \cup \mathbb{N}$ with a constant $\omega > 0$. In this case, by Theorem 4.6, T is a bounded self-adjoint operator with $D(T) = \mathcal{H}$ and takes the form

$$T\psi = \frac{i}{\omega} \sum_{n=1}^{\infty} \left(\sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{n - m} \right) e_n, \quad \psi \in \mathcal{H}.$$

Moreover, one can prove that

$$\sigma(T) = [-\pi/\omega, \pi/\omega]$$

([4, Theorem 2.1]).

Let $\hat{N} := \omega^{-1}H - 1/2$ and $\hat{\theta} := \omega T$. Then it follows that

$$\sigma(\hat{N}) = \{0\} \cup \mathbb{N}, \quad \sigma(\hat{\theta}) = [-\pi, \pi], \quad (4.20)$$

$$[\hat{\theta}, \hat{N}]\psi = i\psi, \quad \psi \in \mathcal{D}_c. \quad (4.21)$$

As is well known, in the context of quantum mechanics, the sequence $\{\omega(n + \frac{1}{2})\}_{n=1}^{\infty}$ appears as the spectrum of the one-dimensional quantum harmonic oscillator Hamiltonian with mass $m > 0$

$$H_{\text{os}} := \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

in the Schrödinger representation (q, p) of the CCR, where $p := -iD$ with D being the generalized partial differential operator on $L^2(\mathbb{R})$ and q is the multiplication operator by the variable $x \in \mathbb{R}$. In this context, the operator \hat{N} is called the number operator and, in view of (4.20) and (4.21), the operator $\hat{\theta}$ is interpreted as a *phase operator* [7].

5 Unboundedness of T

As for the unboundedness of T , we have the following fact:

Theorem 5.1 *If $\{E_n\}_{n=1}^{\infty}$ satisfies*

$$\inf_{n \in \mathbb{N}} (E_{n+1} - E_n) = 0, \quad (5.1)$$

then T is unbounded.

Proof. By (5.1), there exists a subsequence $\{E_{p_k}\}_{k=1}^{\infty}$ of $\{E_p\}_{p=1}^{\infty}$ such that

$$\lim_{k \rightarrow \infty} (E_{p_{k+1}} - E_{p_k}) = 0. \quad (5.2)$$

Hence we have

$$\begin{aligned} \|Te_{p_k}\|^2 &= \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{\langle e_m, e_{p_k} \rangle}{E_n - E_m} \right|^2 = \sum_{n \neq p_k} \left| \frac{1}{E_n - E_{p_k}} \right|^2 \\ &\geq \left| \frac{1}{E_{p_{k+1}} - E_{p_k}} \right|^2 \rightarrow \infty \quad (k \rightarrow \infty). \end{aligned}$$

Thus T is unbounded. □

Example 5.1 Let

$$E_n = \lambda n^\alpha + \mu$$

with constants $\lambda > 0$, $\alpha > 1/2$ and $\mu \in \mathbb{R}$. Then $\{E_n\}_{n=1}^{\infty}$ satisfies the assumptions (H.1) and (H.2). As we have already seen, T is bounded if $\alpha \geq 1$

Let $1/2 < \alpha < 1$. Then one easily sees that

$$\lim_{n \rightarrow \infty} (E_{n+1} - E_n) = 0.$$

Hence $\inf_{n \in \mathbb{N}} (E_{n+1} - E_n) = 0$. Therefore, in this case, T is unbounded. Thus T is bounded if and only if $\alpha \geq 1$.

6 Hilbert-Schmidtness of T

In this section we investigate Hilbert-Schmidtness of the operator T .

Proposition 6.1 *The operator T is Hilbert-Schmidt if and only if*

$$\sum_{n=1}^{\infty} \sum_{m \neq n}^{\infty} \frac{1}{(E_n - E_m)^2} < \infty. \quad (6.1)$$

In that case, T is self-adjoint with

$$\|T\|_2^2 = \sum_{n=1}^{\infty} \sum_{m \neq n}^{\infty} \frac{1}{(E_n - E_m)^2}, \quad (6.2)$$

where $\|\cdot\|_2$ denotes Hilbert-Schmidt norm. In particular, there exist a C.O.N.S. $\{f_n\}_{n=1}^{\infty}$ of \mathcal{H} and real numbers t_n , $n \in \mathbb{N}$ such that $Tf_n = t_n f_n$ and $t_n \rightarrow 0$ ($n \rightarrow \infty$).

Proof. Suppose that T is Hilbert-Schmidt. Then $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$. On the other hand, we have

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty} \sum_{m \neq n}^{\infty} \frac{1}{(E_n - E_m)^2} \quad (6.3)$$

Hence (6.1) follows with (6.2).

Conversely, (6.1) holds. Hence, by (6.3), $\sum_{n=1}^{\infty} \|Te_n\|^2 < \infty$. Therefore T is Hilbert-Schmidt. The last statement follows from the Hilbert-Schmidt theorem (e.g., [14, Theorem VI.16]). \square

Remark 6.1 In Proposition 6.1, the number $t_n \neq 0$ is an eigenvalue of T with a finite multiplicity. Since T is self-adjoint in the present case, it may be an observable in the context of quantum mechanics. If this is the case, then Proposition 6.1 shows that the observable described by T (“time” in any sense ?) is quantized (discretized) in the quantum system whose Hamiltonian is H with eigenvalues $\{E_n\}_{n=1}^{\infty}$ satisfying (6.1).

The next theorem gives a class of H such that T is Hilbert-Schmidt:

Theorem 6.2 *Suppose that (4.12) in Theorem 4.5 holds with $\alpha > 3/2$. Then T is Hilbert-Schmidt and self-adjoint.*

Proof. Since $1/(E_n - E_m)^2$ is symmetric in n and m , it is sufficient to show that $\sum_{n>m\geq 1} 1/(E_n - E_m)^2 < \infty$. By the present assumption, we need only to show that

$$\Sigma := \sum_{n>m\geq 1} \frac{1}{(n^\alpha - m^\alpha)^2} < \infty$$

for all $\alpha > 3/2$. We have

$$\begin{aligned} \Sigma &= \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{1}{(n^\alpha - m^\alpha)^2} \\ &\leq \sum_{n=2}^{\infty} \frac{1}{n^\alpha - (n-1)^\alpha} \cdot \sum_{m=1}^{n-1} \frac{1}{(n^\alpha - m^\alpha)}. \end{aligned}$$

Using (4.4) and the elementary inequality

$$\frac{1}{n^\alpha - (n-1)^\alpha} \leq \frac{1}{\alpha(n-1)^{\alpha-1}},$$

we obtain

$$\Sigma \leq \sum_{n=2}^{\infty} \frac{1}{\alpha(n-1)^{\alpha-1}} \frac{\log n}{n^{\alpha-1}} + \sum_{n=2}^{\infty} \frac{1}{\alpha^2(n-1)^{2(\alpha-1)}}.$$

Each infinite series on the right hand side converges for all $\alpha > 3/2$. Thus the desired result follows. \square

7 The Galapon Time Operator as a Generalized Time Operator

It is shown that every self-adjoint operator which has a strong time operator is absolutely continuous [12, 17]. Hence the Galapon time operator T_1 is not a strong time operator of H . But it may be a generalized time operator of H . In this section we investigate this aspect.

7.1 An operator-valued function on \mathbb{R}

In the same way as in Lemma 2.1-(ii), one can show that, for all $\psi \in \mathcal{H}$, $n \in \mathbb{N}$ and all $t \in \mathbb{R}$, the infinite series

$$\sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle$$

absolutely converges. Hence, for each $t \in \mathbb{R}$, one can define a linear operator $K(t)$ as follows:

$$D(K(t)) := \left\{ \psi \in \mathcal{H} \left| \sum_{n=1}^{\infty} \left| \sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right. \right\}, \quad (7.1)$$

$$K(t)\psi := i \sum_{n=1}^{\infty} \left(\sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle \right) e_n, \quad \psi \in D(K(t)). \quad (7.2)$$

It is easy to see that, for all $t \in \mathbb{R}$,

$$\mathcal{D}_0 \subset D(K(t)) \quad (7.3)$$

and

$$K(t)e_k = i \sum_{n \neq k}^{\infty} \frac{e^{it(E_n - E_k)} - 1}{E_n - E_k} e_n, \quad k \in \mathbb{N}. \quad (7.4)$$

The correspondence $K : \mathbb{R} \ni t \mapsto K(t)$ gives an operator-valued function on \mathbb{R} . In the notation in Section 4, $K(t)$ is the operator T_b with $b_{nm} = e^{it(E_n - E_m)} - 1$, $n, m \in \mathbb{N}$.

Remark 7.1 Equation (7.4) shows that $K(t) \neq tI|_{\mathcal{D}_0}$. Hence T cannot be a strong time operator of H , as already remarked based on the general theory of strong time operators.

Proposition 7.1 *For all $t \in \mathbb{R}$, $K(t)$ is a densely defined closed operator.*

Proof. Similar to the proof of Proposition 3.2. □

Proposition 7.2 *For all $t \in \mathbb{R}$, $K(t)|_{\mathcal{D}_0}$ is symmetric.*

Proof. Similar to the proof of Lemma 2.3. □

Theorem 7.3 *For all $\psi \in D(T_1)(= \mathcal{D}_0)$ and $t \in \mathbb{R}$, $e^{-itH}\psi \in D(T_1)$ and*

$$T_1 e^{-itH}\psi = e^{-itH}(T_1 + K(t))\psi. \quad (7.5)$$

Proof. We need only to prove the statement in the case $\psi = e_k$ ($\forall k \in \mathbb{N}$). Since $e^{-itH}e_k = e^{-itE_k}e_k$, it follows that $e^{-itH}e_k \in D(T_1)$ with

$$T_1 e^{-itH}e_k = e^{-itE_k} \sum_{n \neq k}^{\infty} \frac{i}{E_n - E_k} e_n.$$

We have

$$e^{-itH}T_1 e_k = i \sum_{n \neq k}^{\infty} \frac{e^{-itE_n}}{E_n - E_k} e_n.$$

It follows from these equations that

$$T_1 e^{-itH}e_k - e^{-itH}T_1 e_k = e^{-itH}K(t)e_k.$$

Thus the desired result follows. □

Corollary 7.4 *Suppose that, for all $t \in \mathbb{R}$, $K(t)$ is bounded. Then T_1 is a generalized time operator of H with commutation factor K .*

Proof. This follows from Theorem 7.3, Proposition 7.1 and Proposition 7.2. \square

In view of Corollary 7.4, we need to investigate conditions for $K(t)$ to be bounded.

Proposition 7.5 *Suppose that (4.12) holds with $\alpha > 1$. Then, for all $t \in \mathbb{R}$, $K(t)$ is a bounded self-adjoint operator with $D(K(t)) = \mathcal{H}$.*

Proof. This follows from an application of Theorem 4.5 to the case where $b_{nm} = e^{it(E_n - E_m)} - 1$, $n, m \in \mathbb{N}$. \square

Proposition 7.6 *Suppose that (6.1) holds. Then, for all $t \in \mathbb{R}$, $K(t)$ is Hilbert-Schmidt and self-adjoint with*

$$\|K(t)\|_2^2 = \sum_{k=1}^{\infty} \sum_{n \neq k}^{\infty} \left| \frac{e^{it(E_n - E_k)} - 1}{E_n - E_k} \right|^2. \quad (7.6)$$

Proof. Similar to the proof of Proposition 6.1. \square

7.2 Non-differentiability of K

From the view-point of the theory of generalized time operators [2], it is interesting to examine differentiability of the operator-valued function K .

Proposition 7.7 *For all $k \in \mathbb{N}$, the \mathcal{H} -valued function $: \mathbb{R} \ni t \mapsto K(t)e_k$ is not strongly differentiable on \mathbb{R} .*

Proof. We first show that $K(t)e_k$ is not strongly differentiable at $t = 0$. Since $K(0)e_k = 0$, we have for all $t \in \mathbb{R} \setminus \{0\}$ and $N > k$

$$\begin{aligned} \left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 &= \sum_{n \neq k}^{\infty} \frac{|e^{it(E_n - E_k)} - 1|^2}{t^2 |E_n - E_k|^2} \\ &\geq \sum_{n \neq k}^{N+1} \frac{|e^{it(E_n - E_k)} - 1|^2}{t^2 |E_n - E_k|^2}. \end{aligned}$$

Hence

$$\liminf_{t \rightarrow 0} \left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 \geq \sum_{n \neq k}^{N+1} 1 = N.$$

Since $N > k$ is arbitrary, it follows that

$$\lim_{t \rightarrow 0} \left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 = +\infty.$$

This implies that $K(t)e_k$ is not strongly differentiable at $t = 0$.

We next show that $K(t)e_k$ is not strongly differentiable at each $t \neq 0$. By (7.5), we have for all $s \in \mathbb{R} \setminus \{0\}$

$$\frac{K(t+s)e_k - K(t)e_k}{s} = e^{it(H-E_k)} \frac{K(s)e_k}{s}.$$

Hence

$$\left\| \frac{K(t+s)e_k - K(t)e_k}{s} \right\| = \left\| \frac{K(s)e_k}{s} \right\|.$$

By the preceding result, the right hand side diverges to $+\infty$ as $s \rightarrow 0$. Therefore $K(t)e_k$ is not strongly differentiable at t . \square

Remark 7.2 We have

$$\langle e_\ell, K(t)e_k \rangle = \begin{cases} i \frac{e^{it(E_\ell - E_k)} - 1}{E_\ell - E_k} & ; \ell \neq k \\ 0 & ; \ell = k \end{cases} \quad (7.7)$$

Hence, for all $k, \ell \in \mathbb{N}$, $\langle e_\ell, K(t)e_k \rangle$ is differentiable in $t \in \mathbb{R}$ and

$$\frac{d}{dt} \langle e_\ell, K(t)e_k \rangle = (\delta_{\ell k} - 1) e^{it(E_\ell - E_k)}. \quad (7.8)$$

Proposition 7.7 tells us some singularity of $K(t)$ acting on \mathcal{D}_0 . But, as shown in the next proposition, $K(t)$ restricted to \mathcal{D}_c is strongly differentiable at $t = 0$.

Proposition 7.8 For all $\psi \in \mathcal{D}_c$, the \mathcal{H} -valued function $K(t)\psi$ is strongly differentiable at $t = 0$ with

$$\left. \frac{d}{dt} K(t)\psi \right|_{t=0} = \psi. \quad (7.9)$$

Proof. We need only to prove the statement for ψ of the form $\psi = e_k - e_\ell$ ($k, \ell \in \mathbb{N}, k \neq \ell$). For all $t \in \mathbb{R} \setminus \{0\}$, we have

$$\frac{K(t)(e_k - e_\ell)}{t} = A(t) + B(t),$$

where

$$\begin{aligned} A(t) &:= i \frac{e^{it(E_\ell - E_k)} - 1}{t(E_\ell - E_k)} e_\ell - i \frac{e^{it(E_k - E_\ell)} - 1}{t(E_k - E_\ell)} e_k, \\ B(t) &:= i \sum_{n \neq k, \ell}^{\infty} \left(\frac{e^{it(E_n - E_k)} - 1}{t(E_n - E_k)} - \frac{e^{it(E_n - E_\ell)} - 1}{t(E_n - E_\ell)} \right) e_n. \end{aligned}$$

It is easy to see that

$$\lim_{t \rightarrow 0} A(t) = e_k - e_\ell.$$

As for $B(t)$, we have

$$\|B(t)\|^2 = \sum_{n \neq k, \ell}^{\infty} |F_n(t)|^2,$$

where

$$F_n(t) := \frac{e^{it(E_n - E_k)} - 1}{t(E_n - E_k)} - \frac{e^{it(E_n - E_\ell)} - 1}{t(E_n - E_\ell)}.$$

It is easy to see that

$$\lim_{t \rightarrow 0} F_n(t) = 0.$$

Moreover, one can show that

$$|F_n(t)| \leq \frac{C}{|E_n - E_k|}, \quad n \neq k, \ell,$$

where $C > 0$ is a constant independent of n and t . Since $\sum_{n \neq k}^{\infty} 1/|E_n - E_k|^2 < \infty$, one can apply the dominated convergence theorem to conclude that $\lim_{t \rightarrow 0} \|B(t)\|^2 = 0$. Thus $K(t)(e_k - e_\ell)$ is strongly differentiable at $t = 0$ and (7.9) with $\psi = e_k - e_\ell$ holds. \square

Proposition 7.9 *For all $k, \ell \in \mathbb{N}$ with $k \neq \ell$, the \mathcal{H} -valued function $K(t)(e_k - e_\ell)$ is not strongly differentiable at $t \notin \{2\pi n/(E_k - E_\ell) | n \in \mathbb{Z}\}$.*

Proof. Let $t \neq 2\pi n/(E_k - E_\ell)$ ($n \in \mathbb{Z}$) and $s \in \mathbb{R} \setminus \{0\}$. Then, by (7.5), we have

$$\frac{(K(t+s) - K(t))(e_k - e_\ell)}{s} = e^{itH} \frac{K(s)}{s} e^{-itH} (e_k - e_\ell).$$

Hence

$$\left\| \frac{(K(t+s) - K(t))(e_k - e_\ell)}{s} \right\| = \|u(s)\|$$

with

$$u(s) := \frac{K(s)}{s} (e^{-itE_k} e_k - e^{-itE_\ell} e_\ell).$$

We write

$$u(s) = u_1(s) + u_2(s)$$

with

$$u_1(s) := e^{-itE_k} \frac{K(s)}{s} (e_k - e_\ell), \quad u_2(s) := (e^{-itE_k} - e^{-itE_\ell}) \frac{K(s)}{s} e_\ell.$$

By Proposition 7.8, we have $\lim_{s \rightarrow 0} u_1(s) = e^{-itE_k} (e_k - e_\ell)$. On the other hand, we have from the proof of Proposition 7.7 and the assumed condition for t

$$\lim_{s \rightarrow 0} \|u_2(s)\| = +\infty.$$

Hence $\lim_{s \rightarrow 0} \|u(s)\| = +\infty$. Thus the desired result follows. \square

8 Possible Connections with Regular Perturbation Theory

We consider a perturbation of H by a symmetric operator H_I on \mathcal{H} :

$$H(\lambda) := H + \lambda H_I, \quad (8.1)$$

where $\lambda \in \mathbb{R}$ is a perturbation parameter. For simplicity, we assume that H_I is H -bounded: $D(H) \subset D(H_I)$ and there exist constants $a, b \geq 0$ such that

$$\|H_I \psi\| \leq a \|H \psi\| + b \|\psi\|, \quad \psi \in D(H).$$

Then, by the Kato-Rellich theorem (e.g., [15, Theorem X.12]), for all $\lambda \in \mathbb{R}$ satisfying

$$a|\lambda| < 1, \quad (8.2)$$

$H(\lambda)$ is self-adjoint and bounded below. In what follows, we assume (8.2).

8.1 Eigenvalues of $H(\lambda)$

We fix $n \in \mathbb{N}$ arbitrarily. By a general theorem in regular perturbation theory (e.g., [16, Theorem XII.9]), there exists a constant $c_n > 0$ such that, for all $|\lambda| < c_n$, H has a unique, isolated non-degenerate eigenvalue $E_n(\lambda)$ near E_n . Moreover, $E_n(\lambda)$ is analytic in λ with Taylor expansion

$$E_n(\lambda) = E_n + E_n^{(1)} \lambda + E_n^{(2)} \lambda^2 + \cdots, \quad (8.3)$$

where

$$E_n^{(1)} := \langle e_n, H_I e_n \rangle, \quad E_n^{(2)} := \sum_{m \neq n}^{\infty} \frac{|\langle e_n, H_I e_m \rangle|^2}{E_n - E_m}. \quad (8.4)$$

As an eigenvector of $H(\lambda)$ with eigenvalue $E_n(\lambda)$, one can take a vector $\psi_n(\lambda)$ analytic in λ with Taylor expansion

$$\psi_n(\lambda) = e_n + e_n^{(1)} \lambda + \cdots, \quad (8.5)$$

where

$$e_n^{(1)} := \sum_{m \neq n}^{\infty} \frac{\langle e_m, H_I e_n \rangle}{E_n - E_m} e_m. \quad (8.6)$$

By Lemma 2.2, we have

$$\langle e_n, T e_m \rangle = \begin{cases} \frac{i}{E_n - E_m} & ; n \neq m \\ 0 & ; n = m \end{cases}$$

Hence $E_n^{(2)}$ can be written

$$E_n^{(2)} = (-i) \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 \langle e_n, T e_m \rangle. \quad (8.7)$$

To rewrite the right hand side only in terms of e_n and linear operators on \mathcal{H} , we note that

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 = \|H_I e_n\|^2 < \infty$$

by the Parseval equality. Hence

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^4 < \infty.$$

Therefore the infinite series

$$f_n := \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 e_m \quad (8.8)$$

strongly converges and defines a vector in \mathcal{H} . Thus we can define a linear operator V on \mathcal{H} as follows:

$$D(V) := \mathcal{D}_0, \quad (8.9)$$

$$V\psi := -i \sum_{n=1}^{\infty} \langle e_n, \psi \rangle f_n, \quad \psi \in \mathcal{D}_0 \quad (8.10)$$

where the right hand side of (8.10) is a sum over a finite term. It is easy to see that V is a symmetric operator.

Proposition 8.1 For all $n \in \mathbb{N}$,

$$E_n^{(2)} = \langle T e_n, V e_n \rangle. \quad (8.11)$$

Proof. We have $V e_n = -i f_n$. Hence $\langle T e_n, V e_n \rangle = -i \langle T e_n, f_n \rangle$, which is equal to the right hand side of (8.7). \square

This proposition suggests some role of the time operator $T_1 = T|_{\mathcal{D}_0}$ in the perturbation expansions of the eigenvalues of H .

As for the first order term $e_n^{(1)} \lambda$ of the eigenvector $\psi_n(\lambda)$, we have

$$e_n^{(1)} = (-i) \sum_{m=1}^{\infty} \langle e_m, H_I e_n \rangle \langle e_n, T e_m \rangle e_m. \quad (8.12)$$

8.2 Transition probability amplitudes

In the context of quantum mechanics where $H(\lambda)$ is the Hamiltonian of a quantum system, the complex number $\langle \phi, e^{-itH(\lambda)} \psi \rangle$ with unit vectors $\phi, \psi \in \mathcal{H}$ is called the *transition probability amplitude* for the probability such that the state of the quantum system at time t is found in the state ϕ under the condition that the state of the quantum system at time zero is ψ .

Lemma 8.2 Let $\phi, \psi \in D(H)$. Then, for all $t \in \mathbb{R}$,

$$\langle \phi, e^{-itH(\lambda)} \psi \rangle = \langle \phi, e^{-itH} \psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H} \phi, H_I e^{-isH} \psi \rangle ds + O(\lambda^2). \quad (8.13)$$

Proof. By a simple application of a general formula for the unitary group generated by a self-adjoint operator ([5, Lemma 5.9]), we have

$$e^{-itH(\lambda)} \psi = e^{-itH} \psi - i\lambda \int_0^t e^{-i(t-s)H(\lambda)} H_I e^{-isH} \psi ds, \quad (8.14)$$

where the integral is taken in the strong sense. Hence

$$\begin{aligned} \langle \phi, e^{-itH(\lambda)} \psi \rangle &= \langle \phi, e^{-itH} \psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H(\lambda)} \phi, H_I e^{-isH} \psi \rangle ds \\ &= \langle \phi, e^{-itH} \psi \rangle - i\lambda \int_0^t \langle e^{i(t-s)H} \phi, H_I e^{-isH} \psi \rangle ds + R(\lambda), \end{aligned}$$

where

$$R(\lambda) := -i\lambda \int_0^t \langle (e^{i(t-s)H(\lambda)} - e^{i(t-s)H}) \phi, H_I e^{-isH} \psi \rangle ds.$$

Using (8.14) again, we have

$$R(\lambda) = -\lambda^2 \int_0^t ds \int_0^{-(t-s)} ds' \langle e^{i(t-s+s')H(\lambda)} H_I e^{-is'H} \phi, H_I e^{-isH} \psi \rangle.$$

Hence

$$|R(\lambda)| \leq \lambda^2 \int_0^{|t|} ds \int_0^{|t-s|} ds' \|H_I e^{-is'H} \phi\| \|H_I e^{-isH} \psi\|$$

Therefore $R(\lambda) = O(\lambda^2)$. Thus (8.13) holds. \square

Applying (8.13) with $\phi = e_m$ and $\psi = e_n$ ($n \neq m$), we have

$$\langle e_m, e^{-itH(\lambda)} e_n \rangle = -\lambda \frac{e^{-itE_n} - e^{-itE_m}}{E_m - E_n} \langle e_m, H_I e_n \rangle + O(\lambda^2), \quad (8.15)$$

which, combined with (7.7), gives

$$\langle e_m, e^{-itH(\lambda)} e_n \rangle = i\lambda \langle e_n, e^{-itH} K(t) e_m \rangle \langle e_m, H_I e_n \rangle + O(\lambda^2). \quad m \neq n. \quad (8.16)$$

This suggests a physical meaning of the commutation factor K .

By Theorem 7.3, one can rewrite the first term on the right hand side in terms of T_1 and e^{-itH} , obtaining

$$\langle e_m, e^{-itH(\lambda)} e_n \rangle = i\lambda \langle e_n, [T_1, e^{-itH}] e_m \rangle \langle e_m, H_I e_n \rangle + O(\lambda^2), \quad m \neq n. \quad (8.17)$$

This also is suggestive on physical meaning of the time operator T_1 .

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