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Title	Time Operators of a Hamiltonian with Purely Discrete Spectrum
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Citation	Reviews in Mathematical Physics, 20(8), 951-978 https://doi.org/10.1142/S0129055X08003481
Issue Date	2008-09
Doc URL	http://hdl.handle.net/2115/38104
Rights	Electronic version of an article published as Reviews in Mathematical Physics, Vol. 20, Issue. 8, 2008, pp.951-978, DOI:10.1142/S0129055X08003481 © World Scientific Publishing Company, http://www.worldscinet.com/rmp/rmp.shtml
Туре	article (author version)
File Information	ArMa-rev3.pdf



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Time Operators of a Hamiltonian with Purely Discrete Spectrum

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Abstract

We develop a mathematical theory of time operators of a Hamiltonian with purely discrete spectrum. The main results include boundedness, unboundedness and spectral properties of them. In addition, possible connections of a time operator of H with regular perturbation theory are discussed.

Keywords. canonical commutation relation, Hamiltonian, time operator, time-energy uncertainty relation, phase operator, spectrum, regular perturbation theory.

Mathematics Subject Classification (2000). 81Q10, 47N50.

1 Introduction

This paper is concerned with mathematical theory of time operators in quantum mechanics [2, 3, 4, 6, 12]. There are some types of time operators which are not necessarily equivalent each other. For the reader's convenience, we first recall the definitions of them with comments.

Let \mathcal{H} be a complex Hilbert space. We denote the inner product and the norm of \mathcal{H} by $\langle \cdot, \cdot \rangle$ (antilinear in the first variable) and $\|\cdot\|$ respectively. For a linear operator A on a Hilbert space, D(A) denotes the domain of A.

Let H be a self-adjoint operator on \mathcal{H} and T be a symmetric operator on \mathcal{H} .

The operator T is called a *time operator* of H if there is a (not necessarily dense) subspace $\mathcal{D} \neq \{0\}$ of \mathcal{H} such that $\mathcal{D} \subset D(TH) \cap D(HT)$ and the canonical commutation relation (CCR)

$$[T, H] := (TH - HT) = i \tag{1.1}$$

^{*}Corresponding author. The work is supported by the Grant-in-Aid No.17340032 for Scientific Research from Japan Society for the Promotion of Science (JSPS).

holds on \mathcal{D} (i.e., $[T, H]\psi = i\psi$, $\forall \psi \in \mathcal{D}$), where *i* is the imaginary unit. In this case, *T* is called a *canonical conjugate* to *H* too.

The name "time operator" for the operator T comes from the quantum mechanical context where H is taken to be the Hamiltonian of a quantum system and the heuristic classical-quantum correspondence based on the structure that, in the classical relativistic mechanics, time is a canonical conjugate variable to energy in each Lorentz frame of coordinates. Note also that the dimension of T is that of time if the dimension of H is that of energy in the original unit system where the right hand side of (1.1) takes the form $i\hbar$ with \hbar being the Planck constant h divided by 2π . We remark, however, that this name is somewhat misleading, because, in the framework of the standard quantum mechanics, time is *not an observable*, but just a parameter assigning the time when a quantum event is observed. But we follow the convention in this respect. By the same reason as just remarked, T is not necessarily (essentially) self-adjoint. But this does not mean that it is "unphysical" [2, 12]. Note also that we do not require the denseness of the subspace \mathcal{D} in the definition stated above. This is more general. In fact, there is an example of the pair (T, H) satisfying (1.1) on a non-dense subsapce \mathcal{D} [8, 10].

From a representation theoretic point of view, the pair (T, H) is a symmetric representation of the CCR with one degree of freedom. But one should remember that, as for this original form of representation of the CCR, the von Neumann uniqueness theorem ([13], [14, Theorem VIII.14]) does not necessarily hold. In other words, (T, H) is not necessarily unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom. Indeed, for example, it is obvious that, if T or H is bounded below or bounded above, then (T, H) cannot be unitarily equivalent to a direct sum of the Schrödinger representation of the CCR with one degree of freedom.

A classification of pairs (T, H) with T being a bounded self-adjoint operator has been done by G. Dorfmeister and J. Dorfmeister [7]. We remark, however, that the class discussed in [7] does not cover the pairs (T, H) considered in this paper, because the paper [7] treats only the case where T is bounded and absolutely continuous.

A weak form of time operator is defined as follows. We say that a symmetric operator T is a *weak time operator* of H if there is a subspace $\mathcal{D}_{w} \neq \{0\}$ of \mathcal{H} such that $\mathcal{D}_{w} \subset D(T) \cap D(H)$ and

$$\langle T\psi, H\phi \rangle - \langle H\psi, T\phi \rangle = \langle \psi, i\phi \rangle, \quad \psi, \phi \in \mathcal{D}_{w},$$

i.e., (T, H) satisfies the CCR in the sense of sesquilinear form on \mathcal{D}_{w} . Obviously a time operator T of H is a weak time operator of H. But the converse is not true (it is easy to see, however, that, if T is a weak time operator of H and $\mathcal{D}_{w} \subset D(TH) \cap D(HT)$, then T is a time operator). An important aspect of a weak time operator T of H is that a *time-energy uncertainty relation* is naturally derived [2, Proposition 4.1]: for all unit vectors ψ in $\mathcal{D}_{w} \subset D(T) \cap D(H)$,

$$(\Delta T)_{\psi}(\Delta H)_{\psi} \ge \frac{1}{2},$$

where, for a linear operator A on \mathcal{H} and $\phi \in D(A)$ with $\|\phi\| = 1$,

$$(\Delta A)_{\phi} := \| (A - \langle \phi, A\phi \rangle) \phi \|_{2}$$

called the uncertainty of A in the vector ϕ .

In contrast to the weak form of time operator, there is a strong form. We say that T is a strong time operator of H if, for all $t \in \mathbb{R}$, $e^{-itH}D(T) \subset D(T)$ and

$$Te^{-itH}\psi = e^{-itH}(T+t)\psi, \quad \psi \in D(T).$$
(1.2)

We call (1.2) the weak Weyl relation [2]. From a representation theoretic point of view, we call a pair (T, H) obeying the weak Weyl relation a weak Weyl representation of the CCR. This type of representation of the CCR was extensively studied by Schmüdgen [17, 18]. It is shown that a strong time operator of H is a time operator of H [12]. But the converse is not true. In fact, the time operators considered in the present paper are not strong ones.

There is a generalized version of strong time operator [2]. We say that T is a generalized time operator of H if, for each $t \in \mathbb{R}$, there is a bounded self-adjoint operator K(t) on \mathcal{H} with $D(K(t)) = \mathcal{H}, e^{-itH}D(T) \subset D(T)$ and a generalized weak Weyl relation (GWWR)

$$Te^{-itH}\psi = e^{-itH}(T + K(t))\psi \quad (\forall \psi \in D(T))$$
(1.3)

holds. In this case, the bounded operator-valued function K(t) of $t \in \mathbb{R}$ is called the *commutation factor* of the GWWR under consideration.

We now come to the subject of the present paper. In his interesting paper [9], Galapon showed by an explicit construction that, for every self-adjoint operator H (a Hamiltonian) on an abstract Hilbert space \mathcal{H} which is *bounded below* and has purely discrete spectrum with some growth condition, there is a time operator T_1 on \mathcal{H} , which is a bounded selfadjoint operator under an additional condition (for the definition of T_1 , see (2.12) below). To be definite, we call the operator T_1 introduced in [9] the *Galapon time operator*.

An important point of Galapon's work [9] is in that it disproved the long-standing belief or folklore among physicists that there is no self-adjoint operator canonically conjugate to a Hamiltonian which is bounded below (for a historical survey, see Introduction of [9]).

Motivated by work of Galapon [9], we investigate, in this paper, properties of time operators of a self-adjoint operator H with purely discrete spectrum. In Section 2, we introduce a densely defined linear operator T whose restriction to a subspace yields the Galapon time operator T_1 and prove basic properties of T and T_1 , in particular the closedness of T. It follows that, if T is bounded, then T is self-adjoint with $D(T) = \mathcal{H}$ and a time operator of H. We denote by $T^{\#}$ one of T_1 , T and T^* (the adjoint of T). In Section 3, we discuss some general properties of $T^{\#}$. Moreover the reflection symmetry of the spectrum of $T^{\#}$ with respect to the imaginary axis is proved. Sections 4–6 are the main parts of this paper. In Section 4, we present a general criterion for T to be bounded with $D(T) = \mathcal{H}$, while, in Section 5, we give a sufficient condition for T to be unbounded. In Section 6, we present a necessary and sufficient condition for T to be Hilbert-Schmidt. In Section 7, we show that, under some condition, the Galapon time operator is a generalized time operator of H, too. We also discuss non-differentiability of the commutation factor K in the GWWR for (T_1, H) . In the last section, we consider a perturbation of H by a symmetric operator and try to draw out physical meanings of T_1 and K in the context of regular perturbation theory.

2 Time Operators

In this section, we recapitulate some basic aspects of the Galapon time operator in more apparent manner than in [9].

Let \mathcal{H} be a complex Hilbert space and

H be a self-adjoint operator on \mathcal{H} which has the following properties (H.1) and (H.2):

(H.1) The spectrum of H, denoted $\sigma(H)$, is purely discrete with $\sigma(H) = \{E_n\}_{n=1}^{\infty}$, where each eigenvalue E_n of H is simple and

$$0 < E_n < E_{n+1}$$

for all $n \in \mathbb{N}$ (the set of positive integers).

(H.2)
$$\sum_{n=1}^{\infty} \frac{1}{E_n^2} < \infty.$$

Throughout the present paper we assume (H.1) and (H.2).

Remark 2.1 The positivity condition $E_n > 0$ for the eigenvalues of H does not lose generality, because, if H is bounded below, but not strictly positive, then one needs only to consider, instead of H, $\tilde{H} := H + c$ with a constant $c > -\inf \sigma(H)$, which is a strictly positive self-adjoint operator.

Property (H.2) implies that

$$E_n \to \infty \quad (n \to \infty).$$
 (2.1)

Let e_n be a normalized eigenvector of H belonging to eigenvalue E_n :

$$He_n = E_n e_n, \quad n \in \mathbb{N}.$$

Then, by property (H.1), the set $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal system (C.O.N.S.) of \mathcal{H} .

Lemma 2.1

(i) For all $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{(E_n - E_m)^2} < \infty.$$
(2.3)

In particular, for each $m \in \mathbb{N}$,

$$\sum_{n \neq m}^{\infty} \frac{1}{E_n - E_m} e_n$$

converges in \mathcal{H} .

(ii) For all $n \in \mathbb{N}$ and vectors ψ in \mathcal{H} , the infinite series

$$\sum_{m\neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \tag{2.4}$$

absolutely converges.

Proof. (i) By (2.1), we have

$$C_m := \sup_{n \neq m} \frac{1}{\left(1 - \frac{E_m}{E_n}\right)^2} < \infty.$$
(2.5)

Hence we have

$$\sum_{n \neq m}^{\infty} \frac{1}{|E_n - E_m|^2} \le C_m \sum_{n \neq m}^{\infty} \frac{1}{E_n^2} < \infty.$$

(ii) By the Cauchy-Schwarz inequality, the Parseval equality and part (i), we have

$$\sum_{m \neq n}^{\infty} \left| \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right| \le \left(\sum_{m \neq n}^{\infty} |\langle e_m, \psi \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{m \neq n}^{\infty} \left| \frac{1}{E_n - E_m} \right|^2 \right)^{\frac{1}{2}} \le \|\psi\| \left(\sum_{m \neq n}^{\infty} \left| \frac{1}{E_n - E_m} \right|^2 \right)^{\frac{1}{2}} < \infty.$$

$$(2.6)$$

By Lemma 2.1-(ii), one can define a linear operator T on $\mathcal H$ as follows:

$$D(T) := \left\{ \psi \in \mathcal{H} \Big| \sum_{n=1}^{\infty} \left| \sum_{m \neq n} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2 < \infty \right\},$$
(2.7)

$$T\psi := i \sum_{n=1}^{\infty} \left(\sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right) e_n, \qquad \psi \in D(T).$$
(2.8)

Note that

$$||T\psi||^2 = \sum_{n=1}^{\infty} \left| \sum_{m\neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2.$$
(2.9)

For a subset $\mathcal{D} \subset \mathcal{H}$, we denote by l.i.h.(\mathcal{D}) the subspace algebraically spanned by the vectors of \mathcal{D} .

The subspace

$$\mathcal{D}_0 := \text{l.i.h.}(\{e_n\}_{n=1}^{\infty}) \tag{2.10}$$

is dense in \mathcal{H} .

Lemma 2.2 The operator T is densely defined with $\mathcal{D}_0 \subset D(T)$ and

$$Te_k = i \sum_{n \neq k}^{\infty} \frac{1}{E_n - E_k} e_n, \quad k \in \mathbb{N}.$$
(2.11)

Proof. To prove $\mathcal{D}_0 \subset D(T)$, it is sufficient to show that $e_k \in D(T), k \in \mathbb{N}$. Putting

$$c_n(k) := \sum_{m \neq n}^{\infty} \frac{\langle e_m, e_k \rangle}{E_n - E_m},$$

we have $c_k(k) = 0$ and $c_n(k) = 1/(E_n - E_k)$ for $n \neq k$. Hence, by Lemma 2.1-(i), we have

$$\sum_{n=1}^{\infty} |c_n(k)|^2 = \sum_{n \neq k}^{\infty} \frac{1}{(E_n - E_k)^2} < \infty.$$

Hence $e_k \in D(T)$ and (2.11) holds.

In general, it is not clear whether or not T is a symmetric operator. But a restriction of T to a smaller subspace gives a symmetric operator. Indeed, we have the following fact:

Lemma 2.3 ([9]) The operator

$$T_1 := T | \mathcal{D}_0 \tag{2.12}$$

(the restriction of T to \mathcal{D}_0) is symmetric.

Proof. It is enough to show that, for all $\psi \in \mathcal{D}_0$, $\langle \psi, T\psi \rangle$ is real. For a complex number $z \in \mathbb{C}$ (the set of complex numbers), we denote its complex conjugate by z^* . We have

$$\langle \psi, T\psi \rangle = i \sum_{n=1}^{\infty} \langle \psi, e_n \rangle \sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m}.$$

Hence

$$\langle \psi, T\psi \rangle^* = i \sum_{n=1}^{\infty} \langle e_n, \psi \rangle \sum_{m \neq n}^{\infty} \frac{\langle \psi, e_m \rangle}{E_m - E_n}.$$

Since ψ is in \mathcal{D}_0 , the double sum on m, n with $m \neq n$ is a sum consisting of a finite term. Hence we can exchange the sum on n and that on m to obtain

$$\langle \psi, T\psi \rangle^* = i \sum_{m=1}^{\infty} \langle \psi, e_m \rangle \sum_{n \neq m}^{\infty} \frac{\langle e_n, \psi \rangle}{E_m - E_n} = \langle \psi, T\psi \rangle.$$

Hence $\langle \psi, T\psi \rangle$ is real.

The operator T_1 defined by (2.12) is the time operator introduced by Galapon in [9]. Obviously we have

$$T_1 \subset T. \tag{2.13}$$

Hence

$$T^* \subset T_1^*. \tag{2.14}$$

Remark 2.2 It is asserted in [9] that T_1 is essentially self-adjoint without additional conditions. But, unfortunately, we find that this is not conclusive, because the proof of it given in [9] (pp.2678–2679) has some gap: the interchange of the double sum in Equation (2.30) on p.2678 in [9] may not be justified, at least, by the reasoning given there. The assertion is true in the case where T_1 becomes a bounded operator under an additional condition for $\{E_n\}_{n=1}^{\infty}$, as we show below in the present paper. But, in the case where T_1 is unbounded, it seems to be very difficult to prove or disprove the essential self-adjointness of T_1 . We leave this problem for future study.

Lemma 2.4 The subspace

$$\mathcal{D}_{c} := \text{l.i.h.}(\{e_{n} - e_{m} \in \mathcal{H} | n, m \ge 1\}).$$

$$(2.15)$$

is dense in the Hilbert space H. Moreover

$$\mathcal{D}_{c} \subset \mathcal{D}_{0} \subset D(T). \tag{2.16}$$

Proof. Let $\psi \in \mathcal{D}_{c}^{\perp}$ (the orthogonal complement of \mathcal{D}_{c}). Then, for all $m, n \geq 1$, $|\langle e_{n}, \psi \rangle|^{2} = |\langle e_{m}, \psi \rangle|^{2}$. By the Parseval equality, $\|\psi\|^{2} = \lim_{N \to \infty} N |\langle e_{m}, \psi \rangle|^{2}$. This implies that $|\langle e_{m}, \psi \rangle|^{2} = 0$ for all $m \geq 1$ and $\|\psi\| = 0$. Hence $\psi = 0$. Thus \mathcal{D}_{c} is dense in \mathcal{H} . Inclusion relation (2.16) is obvious.

Theorem 2.5 ([9]) It holds that

$$\mathcal{D}_{c} \subset D(T_{1}H) \cap D(HT_{1}) \tag{2.17}$$

and

$$[T_1, H]\psi = i\psi, \qquad \psi \in \mathcal{D}_{c}.$$
(2.18)

Theorem 2.5 shows that T_1 is a time operator of H.

Remark 2.3 It is easy to see that, for all $k \in \mathbb{N}$, $T_1 e_k \notin D(H)$. Hence $\mathcal{D}_0 \notin D(HT_1)$. Therefore one can not consider the commutation relation $[T_1, H]$ on \mathcal{D}_0 . Moreover, by direct computation, we have

$$\langle T_1 e_k, H e_\ell \rangle - \langle H e_k, T_1 e_\ell \rangle = -i(1 - \delta_{k\ell}), \quad k, \ell \in \mathbb{N}.$$
(2.19)

This means that (T_1, H) does not satisfy the CCR in the sense of sesquilinear form on \mathcal{D}_0 (a weak form of the CCR), either. These facts suggest that the pair (T_1, H) is very sensitive to the domain on which their commutation relation is applied.

In concluding this section we discuss shortly non-uniqueness of time operators of H. We introduce a set of symmetric operators associated with H:

$$\{H\}'_{\mathcal{D}_c} := \{S|S \text{ is a symmetric operator on } \mathcal{H} \text{ such that} \\ \mathcal{D}_c \subset D(SH) \cap D(HS) \text{ and } SH\psi = HS\psi, \forall \psi \in \mathcal{D}_c\}, \quad (2.20)$$

which may be viewed as a commutant of $\{H\}$ in a restricted sense. It is easy to see that, for all real-valued continuous function f on \mathbb{R} , the operator f(H) defined via the functional calculus is in $\{H\}'_{\mathcal{D}_c}$.

Proposition 2.6 For all $S \in \{H\}'_{\mathcal{D}_c}$, $\mathcal{D}_c \subset D((T_1 + S)H) \cap D(H(T_1 + S))$ and

$$[T_1 + S, H]\psi = i\psi, \quad \psi \in \mathcal{D}_c.$$
(2.21)

Proof. A direct computation using Theorem 2.5 and (2.20).

Proposition 2.7 Let T_2 be a time operator of H such that $\mathcal{D}_c \subset D(T_2H) \cap D(HT_2)$ and

$$[T_2, H]\psi = i\psi, \quad \forall \psi \in \mathcal{D}_c.$$

Then $T_2 = T_1 + S$ with some $S \in \{H\}'_{\mathcal{D}_c}$.

Proof. We need only to show that $S := T_2 - T_1$ is in $\{H\}'_{\mathcal{D}_c}$. But this is obvious. \Box

3 General Properties

3.1 Closedness of T and symmetry of T^*

Lemma 3.1 $\mathcal{D}_0 \subset D(T^*)$ and $T^*|\mathcal{D}_0 = T_1$, *i.e.*, $T_1 \subset T^*$.

Proof. It is enough to show that, for all $k \in \mathbb{N}$, $e_k \in D(T^*)$ and $T^*e_k = Te_k (=T_1e_k)$. It is easy to see that, for all $\psi \in D(T)$,

$$\langle e_k, T\psi \rangle = i \sum_{m \neq k}^{\infty} \frac{\langle e_m, \psi \rangle}{E_k - E_m}.$$
 (3.1)

By Lemma 2.2, the right hand side is equal to $\langle Te_k, \psi \rangle$. Hence $e_k \in D(T^*)$ and $T^*e_k = Te_k$.

Proposition 3.2 The operator T is closed and

$$T^* \subset T. \tag{3.2}$$

In particular, if T is bounded, then T is self-adjoint with $D(T) = \mathcal{H}$.

Proof. Let $\psi_k \in D(T)$, $k \in \mathbb{N}$ and $\psi_k \to \psi \in \mathcal{H}, T\psi_k \to \phi \in \mathcal{H}$ as $k \to \infty$. Then $\sup_{k\geq 1} ||T\psi_k|| < \infty$. Hence, by (2.9), there exists a constant C > 0 independent of $k \in \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \left| \sum_{n \neq m} \frac{\langle e_m, \psi_k \rangle}{E_n - E_m} \right|^2 \le C.$$

By (2.6), we have

$$\lim_{k \to \infty} \sum_{n \neq m}^{\infty} \frac{\langle e_m, \psi_k \rangle}{E_n - E_m} = \sum_{n \neq m}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m}.$$
(3.3)

Hence it follows that

$$\sum_{n=1}^{\infty} \left| \sum_{n \neq m}^{\infty} \frac{\langle e_m, \psi \rangle}{E_n - E_m} \right|^2 \le C.$$

Therefore $\psi \in D(T)$. By (3.1) and (3.3), we have for all $\ell \in \mathbb{N}$

$$\lim_{k \to \infty} \left\langle e_{\ell}, T\psi_k \right\rangle = \left\langle e_{\ell}, T\psi \right\rangle.$$

Hence $\langle e_{\ell}, \phi \rangle = \langle e_{\ell}, T\psi \rangle$, $\ell \in \mathbb{N}$, implying $\phi = T\psi$. Thus T is closed.

To prove (3.2), let $\psi \in D(T^*)$. Putting $\eta = T^*\psi$, we have $\langle \eta, \chi \rangle = \langle \psi, T\chi \rangle$ for all $\chi \in D(T)$. Taking $\chi = e_k$ ($k \in \mathbb{N}$), we have

$$\langle \eta, e_k \rangle = i \sum_{n \neq k}^{\infty} \frac{\langle \psi, e_n \rangle}{E_n - E_k},$$
(3.4)

which implies that

$$\sum_{k=1}^{\infty} \left| \sum_{n \neq k}^{\infty} \frac{\langle \psi, e_n \rangle}{E_n - E_k} \right|^2 = \|\eta\|^2 < \infty.$$

Hence $\psi \in D(T)$. Then, by (3.1), the right hand side of (3.4) is equal to $\langle T\psi, e_k \rangle$. Hence $\eta = T\psi$. Thus (3.2) holds.

Let T be bounded. Then, by the denseness of D(T) and the closedness of T, $D(T) = \mathcal{H}$. Hence $D(T^*) = \mathcal{H}$. Thus, by (3.2), $T^* = T$, i.e., T is self-adjoint.

Corollary 3.3 The operator T^* is symmetric.

Proof. By Lemma 3.1, T^* is densely defined. Hence, by Proposition 3.2, $T^* \subset T = (T^*)^*$. Thus T^* is symmetric.

Thus we have

$$T_1 \subset T^* \subset T. \tag{3.5}$$

Corollary 3.3 shows that T^* also is a time operator of H.

For a closable operator A on a Hilbert space, we denote its closure by \overline{A} .

Proposition 3.4 $\overline{T}_1 = T^*$.

Proof. Note that $\overline{T}_1 = T^*$ if and only if $T_1^* = T$. By (3.5), we have $\overline{T}_1 \subset T^*$. Hence $T \subset T_1^*$. Thus it is enough to show that $D(T_1^*) \subset D(T)$. For all $\psi \in D(T_1^*)$, we have

$$\langle T_1^*\psi, e_l \rangle = \langle \psi, T_1 e_l \rangle = i \sum_{n \neq l} \frac{\langle \psi, e_n \rangle}{E_n - E_l}.$$

Hence we obtain

$$\infty > ||T_1^*\psi||^2 = \sum_{l=1}^{\infty} |\langle T_1^*\psi, e_l\rangle|^2 = \sum_{l=1}^{\infty} \left|\sum_{n\neq l} \frac{\langle \psi, e_n\rangle}{E_n - E_l}\right|^2,$$

implying that $\psi \in D(T)$. Thus $D(T_1^*) \subset D(T)$.

3.2 Absence of invariant dense domains for *T* under some condition

We first note the following general fact:

Proposition 3.5 Let Q be a bounded self-adjoint operator on \mathcal{H} and P be a self-adjoint operator on \mathcal{H} . Suppose that there is a dense subspace \mathcal{D} in \mathcal{H} such that the following (i)–(iii) hold:

- (i) $Q\mathcal{D} \subset \mathcal{D} \subset D(P)$.
- (ii) \mathcal{D} is a core of P.
- (iii) The pair (Q, P) obeys the CCR on \mathcal{D} : $[Q, P]\psi = i\psi, \forall \psi \in \mathcal{D}$.

Then $\sigma(P) = \mathbb{R}$.

Proof. Since Q is a bounded self-adjoint operator, we have for all $t \in \mathbb{R}$

$$e^{itQ} = \sum_{k=0}^{\infty} \frac{(itQ)^k}{k!}$$

in operator norm. Conditions (i) and (iii) imply that, for all $k \in \mathbb{N}$ and $\psi \in \mathcal{D}$

$$Q^k P \psi - P Q^k \psi = ik Q^{k-1} \psi.$$

Hence, for all $t \in \mathbb{R}$ and vectors ψ in \mathcal{D} , we have

$$\begin{split} e^{itQ} P\psi &= P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} Q^k P\psi \\ &= P\psi + \sum_{k=1}^{\infty} \frac{(it)^k}{k!} (PQ^k + ikQ^{k-1})\psi \\ &= P\psi + \sum_{k=1}^{\infty} \left\{ P \frac{(itQ)^k}{k!} \psi - t \frac{(itQ)^{k-1}}{(k-1)!} \psi \right\}. \end{split}$$

It follows from the closedness of P that $e^{itQ}\psi$ is in D(P) and

$$Pe^{itQ}\psi = e^{itQ}P\psi + te^{itQ}\psi.$$
(3.6)

By condition (ii), this equality extends to all $\psi \in D(P)$ with $e^{itQ}\psi \in D(P), \forall t \in \mathbb{R}, \forall \psi \in D(P)$. Hence the operator equality $e^{-itQ}Pe^{itQ} = P + t$ follows. Thus $\sigma(P) = \sigma(P+t)$ for all $t \in \mathbb{R}$. This implies that $\sigma(P) = \mathbb{R}$.

Theorem 3.6 If T is bounded (hence self-adjoint by Proposition 3.2), then there is no dense subspace \mathcal{D} in \mathcal{H} such that the following (i)–(iii) hold:

(i) $T\mathcal{D} \subset \mathcal{D} \subset D(H)$.

- (ii) \mathcal{D} is a core of H.
- (iii) The pair (T, H) obeys the CCR on \mathcal{D} ,

Proof. Suppose that there were such a dense subspace \mathcal{D} as stated above. Then we can apply Proposition 3.5 with (Q, P) = (T, H) to conclude that $\sigma(H) = \mathbb{R}$. But this is a contradiction.

Remark 3.1 A special case of this theorem was established in [7, Theorem 9.5].

3.3 Reflection symmetry of the spectrum of T_1, T^* and T

We first recall the definition of the spectrum of a general linear operator (not necessarily closed). For a linear operator A on a Hilbert space \mathcal{K} , the resolvent set of A, denoted $\rho(A)$, is defined by

 $\rho(A) := \{ z \in \mathbb{C} | \operatorname{Ran}(A - z) \text{ (the range of } A - z) \text{ is dense in } \mathcal{K} \\ \text{and } A - z \text{ is injective with } (A - z)^{-1} \text{ bounded} \}.$

Then the set

$$\sigma(A) := \mathbb{C} \setminus \rho(A)$$

is called the spectrum of A. The set of eigenvalues of A, called the point spectrum of A, is denoted $\sigma_{\rm p}(A)$.

We denote by $T^{\#}$ any of T_1, T^* and T.

We define a conjugation J on \mathcal{H} by

$$J\psi := \sum_{n=1}^{\infty} \langle \psi, e_n \rangle e_n, \quad \psi \in \mathcal{H}.$$
(3.7)

Proposition 3.7 The spectrum $\sigma(T^{\#})$ of $T^{\#}$ is reflection symmetric with respect to the imaginary axis, i.e., if $z \in \sigma(T^{\#})$, then $-z^* \in \sigma(T^{\#})$. In particular, if T is self-adjoint, then $\sigma(T)$ is reflection symmetric with respect to the origin of the real axis. Moreover, for all $z \in \sigma_p(T^{\#})$, $-z^*$ is in $\sigma_p(T^{\#})$ and

$$J \ker(T^{\#} - z) = \ker(T^{\#} + z^{*}), \quad \forall z \in \sigma_{p}(T^{\#}).$$
(3.8)

Proof. It is easy to see that operator equality $JT^{\#}J = -T^{\#}$ holds $(JD(T^{\#}) = D(T^{\#}))$. Hence, for all $z \in \mathbb{C}$, we have $J(T^{\#} - z)J = -(T^{\#} + z^{*})\cdots(*)$. This implies that, if $z \in \rho(T^{\#})$, then $-z^{*} \in \rho(T^{\#})$. Thus the same holds for the spectrum $\sigma(T^{\#}) = \mathbb{C} \setminus \rho(T^{\#})$. Equation (3.8) follows from (*).

4 Boundedness of T

In this section we present a general criterion for the operator T to be bounded. For mathematical generality and for later use, we consider a more general class of operators than that of T. Let $b := \{b_{nm}\}_{n,m=1}^{\infty}$ be a double sequence of complex numbers such that

$$||b||_{\infty} := \sup_{n,m \ge 1} |b_{nm}| < \infty.$$
(4.1)

Then, in the same way as in Lemma 2.1-(ii), for all $\psi \in \mathcal{H}$, the infinite series

$$\sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \left\langle e_m, \psi \right\rangle$$

absolutely converges. Hence one can define a linear operator T_b on \mathcal{H} as follows:

$$D(T_b) := \left\{ \psi \in \mathcal{H} \Big| \sum_{n=1}^{\infty} \left| \sum_{m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right\},$$
(4.2)

$$T_b\psi := i\sum_{n=1}^{\infty} \left(\sum_{m\neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle e_m, \psi \rangle \right) e_n, \quad \psi \in D(T_b).$$
(4.3)

Obviously $T = T_b$ with b satisfying $b_{nm} = 1$ for all $n, m \in \mathbb{N}$. In the same way as in the case of T, one can prove the following fact:

Lemma 4.1 The operator T_b is closed.

The following lemma is probably well known (but, for the completeness, we give a proof):

Lemma 4.2 Let A be a densely defined linear operator on a Hilbert space \mathcal{K} . Suppose that there exist a dense subspace \mathcal{D} in \mathcal{K} and a constant C > 0 such that $\mathcal{D} \subset D(A)$ and

$$|\langle \psi, A\psi \rangle| \le C \|\psi\|^2, \quad \psi \in \mathcal{D}.$$

Then A is bounded with $\|\bar{A}\| \leq 2C$, where \bar{A} is the closure of A.

If A is symmetric in addition, then $\|\bar{A}\| \leq C$.

Proof. Let $\psi, \phi \in \mathcal{D}$. Then, by the polarization identity

$$\langle \psi, A\phi \rangle = \frac{1}{4} (\langle \psi + \phi, A(\psi + \phi) \rangle - \langle \psi - \phi, A(\psi - \phi) \rangle + i \langle \psi - i\phi, A(\psi - i\phi) \rangle - i \langle \psi + i\phi, A(\psi + i\phi) \rangle)$$

we have

$$|\langle \psi, A\phi \rangle| \le \frac{C}{4} (\|\psi + \phi\|^2 + \|\psi - \phi\|^2 + \|\psi - i\phi\|^2 + \|\psi + i\phi\|^2) = C(\|\psi\|^2 + \|\phi\|^2).$$

Replacing $\psi \neq 0$ by $\|\phi\|\psi/\|\psi\|$ we have

$$|\langle \psi, A\phi \rangle| \le 2C \|\psi\| \|\phi\|.$$

For $\psi = 0$, this inequality trivially holds. Since \mathcal{D} is dense, it follows from the Riesz representation theorem that $||A\phi|| \leq 2C||\phi||, \phi \in \mathcal{D}$. Thus the first half of the lemma follows.

Let A be symmetric. Then, $\langle \psi, A\psi \rangle \in \mathbb{R}$ for all $\psi \in D(A)$. Hecne

$$|\Re\langle\psi,A\phi\rangle| = \frac{1}{4} \left|\langle\psi+\phi,A(\psi+\phi)\rangle - \langle\psi-\phi,A(\psi-\phi)\rangle\right| \le \frac{C}{2} (\|\psi\|^2 + \|\phi\|^2), \quad \psi \in \mathcal{D}.$$

We write $\langle \psi, A\phi \rangle = |\langle \psi, A\phi \rangle | e^{i\theta}$ with $\theta \in \mathbb{R}$. Then $|\langle \psi, A\phi \rangle | = \langle e^{i\theta}\psi, A\phi \rangle$. Hence

$$\begin{aligned} |\langle \psi, A\phi \rangle| &= \Re \left\langle e^{i\theta}\psi, A\phi \right\rangle \leq \frac{C}{2} (\|e^{i\theta}\psi\|^2 + \|\phi\|^2) \\ &= \frac{C}{2} (\|\psi\|^2 + \|\phi\|^2). \end{aligned}$$

Thus, in the same manner as above, we can obtain $|\langle \psi, A\phi \rangle| \leq C ||\psi|| ||\phi||, \ \psi, \phi \in \mathcal{D}$. \Box

The next lemma is easily proven by elementary calculus. Therefore we omit proof of it.

Lemma 4.3 For all s > 1 and $n \ge 2$,

$$\sum_{m=1}^{n-1} \frac{1}{n^s - m^s} \le \frac{\log n}{n^{s-1}} + \frac{1}{s(n-1)^{s-1}}.$$
(4.4)

Lemma 4.4 Let s > 1. Then

$$\sup_{n\geq 2} \left(\sum_{m=1}^{n-1} \frac{1}{n^s - m^s}\right) < \infty \tag{4.5}$$

and

$$\sup_{m\geq 1} \left(\sum_{n=m+1}^{\infty} \frac{1}{n^s - m^s} \right) < \infty.$$

$$(4.6)$$

Proof. Property (4.5) follows from Lemma 4.3. To prove (4.6), we write

$$\sum_{n=m+1}^{\infty} \frac{1}{n^s - m^s} \le \int_{m+1}^{\infty} \frac{1}{x^s - m^s} dx + \frac{1}{(m+1)^s - m^s}.$$

We fix a constant $R > 2 (\geq (m+1)/m)$. By the change of variable x = my, we have

$$\int_{m+1}^{\infty} \frac{1}{x^s - m^s} dx = \frac{1}{m^{s-1}} \left(\int_{(m+1)/m}^{R} \frac{1}{y^s - 1} ds + C_R \right),$$

where

$$C_R := \int_R^\infty \frac{1}{y^s - 1} ds < \infty.$$

Using the well known inequality

$$x^{s} - 1 \ge s(x - 1), \quad x > 0, s \ge 1,$$
(4.7)

we have

$$\int_{(m+1)/m}^{R} \frac{1}{y^s - 1} dy \leq \int_{(m+1)/m}^{R} \frac{1}{s(y - 1)} dy$$
$$= \frac{1}{s} \left(\log(R - 1) + \log m \right).$$

Hence

$$\int_{m+1}^{\infty} \frac{1}{x^s - m^s} dx \le \frac{\log m}{sm^{s-1}} + \frac{\log(R-1)}{sm^{s-1}} + \frac{1}{m^{s-1}}C_R.$$

Thus (4.6) follows.

Let

$$c_H(n) := \sum_{m=1}^{n-1} \frac{E_n}{(E_n - E_m)E_m}, \quad n \ge 2,$$
(4.8)

$$d_H(m) := \sum_{n=m+1}^{\infty} \frac{E_m}{(E_n - E_m)E_n}, \quad m \ge 1.$$
(4.9)

Since $c_H(n)$ and $d_H(m)$ are positive (recall that $E_n > 0, \forall n \in \mathbb{N}$), one can define constants

$$c_H := \sup_{n \ge 2} c_H(n),$$
 (4.10)

$$d_H := \sup_{m \ge 1} d_H(m), \tag{4.11}$$

which are finite or infinite.

Theorem 4.5 Suppose that there exist constants $\alpha > 1$, C > 0 and a > 0 such that

$$E_n - E_m \ge C(n^\alpha - m^\alpha), \quad n > m > a.$$

$$(4.12)$$

Then T_b is a bounded operator with $D(T_b) = \mathcal{H}$ and

$$||T_b|| \le 4||b||_{\infty}\sqrt{c_H d_H}.$$
(4.13)

Moreover, if $b_{nm}^* = b_{mn}$ for all $m, n \in \mathbb{N}$, then T_b is a bounded self-adjoint operator with $D(T_b) = \mathcal{H}$ and

$$||T_b|| \le 2||b||_{\infty} \sqrt{c_H d_H}.$$
(4.14)

In particular, T is a bounded self-adjoint operator with $D(T) = \mathcal{H}$.

Proof. By Lemma 4.2, it is enough to show that c_H and d_H are finite and

$$|\langle \psi, T_b \psi \rangle| \le 2 \|b\|_{\infty} \sqrt{c_H d_H} \|\psi\|^2, \quad \psi \in \mathcal{D}_0.$$

$$(4.15)$$

Then T_b is bounded with (4.13). Since T_b is densely defined and closed, it follows that $D(T_b) = \mathcal{H}$. As in the case of T, one can show that, if $b_{nm}^* = b_{mn}$ for all $m, n \in \mathbb{N}$, then $T_b | \mathcal{D}_0$ is symmetric and hence T_b is a bound self-adjoint operator with $D(T_b) = \mathcal{H}$ and (4.14) holds. Therefore the desired result follows.

To prove (4.15), we first note that, for $\psi \in \mathcal{D}_0$,

$$\langle \psi, T_b \psi \rangle = i \sum_{m,n=1, m \neq n}^{\infty} \frac{b_{nm}}{E_n - E_m} \langle \psi, e_n \rangle \langle e_m, \psi \rangle.$$

Hence

$$|\langle \psi, T_b \psi \rangle| \le 2 ||b||_{\infty} A(\psi),$$

where

$$A(\psi) := \sum_{n > m \ge 1} \frac{|\langle e_m, \psi \rangle || \langle \psi, e_n \rangle|}{E_n - E_m}$$

Inserting $1 = \sqrt{E_m/E_n} \cdot \sqrt{E_n/E_m}$ into the summand on the right hand side and using the Cauchy-Schwarz inequality, we have

$$A(\psi)^2 \le B(\psi)C(\psi)$$

with

$$B(\psi) = \sum_{n > m \ge 1} \frac{|\langle e_n, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_n}{E_m},$$
$$C(\psi) = \sum_{n > m \ge 1} \frac{|\langle e_m, \psi \rangle|^2}{E_n - E_m} \cdot \frac{E_m}{E_n}.$$

One can rewrite and estimate $B(\psi)$ as follows:

$$B(\psi) = \sum_{n=2}^{\infty} |\langle e_n, \psi \rangle|^2 c_H(n)$$

$$\leq \|\psi\|^2 c_H.$$

Similarly we have

$$C(\psi) \le \|\psi\|^2 d_H.$$
 (4.16)

Hence

$$|\langle \psi, T_b \psi \rangle| \le 2 \|b\|_{\infty} \sqrt{c_H d_H} \|\psi\|^2.$$
(4.17)

Therefore we need only to prove that c_H and d_H are finite.

We can write

$$c_H(n) = \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \sum_{m=1}^{n-1} \frac{1}{E_m}$$

By assumption (4.12), we have

$$\frac{1}{E_n - E_m} \le \frac{1}{C(n^{\alpha} - m^{\alpha})}, \quad n > m > a.$$
(4.18)

Since we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < \infty,$$

it follows that

$$\varepsilon_1 := \sum_{m=1}^{\infty} \frac{1}{E_m} < \infty.$$

Thus

$$c_H(n) \le \sum_{m=1}^{n-1} \frac{1}{E_n - E_m} + \varepsilon_1.$$

Let $n_0 \ge 2$ be a natural number such that $n_0 > a$. Then, for all $n > n_0$

$$\sum_{m=1}^{n-1} \frac{1}{E_n - E_m} \le \sum_{m=1}^{n_0 - 1} \frac{1}{E_n - E_m} + \frac{1}{C} \sum_{m=n_0}^{n-1} \frac{1}{n^\alpha - m^\alpha}.$$

By (4.4), the right hand side is uniformly bounded in n. Thus we have $c_H < \infty$.

To prove $d_H < \infty$, we write for m > a

$$d_H(m) = \sum_{n=m+1}^{\infty} \frac{1}{(E_n - E_m)} - \sum_{n=m+1}^{\infty} \frac{1}{E_n}$$

$$\leq \sum_{n=m+1}^{\infty} \frac{1}{(E_n - E_m)}$$

$$\leq \frac{1}{C} \sum_{n=m+1}^{\infty} \frac{1}{n^{\alpha} - m^{\alpha}}.$$

Hence, by (4.6) in Lemma 4.4, we have

$$\sup_{m>a} d_H(m) < \infty.$$

Thus it follows that $d_H < \infty$.

Example 4.1 Let $\lambda > 0, \alpha > 1$ and P(x) be a real polynomial of $x \in \mathbb{R}$ with degree $p < \alpha$. Then it is easy to see that the sequence $\{E_n\}_{n=1}^{\infty}$ defined by

$$E_n := \lambda n^{\alpha} + P(n)$$

satisfies the assumptions (H.1), (H.2) and (4.12). Thus, by Theorem 4.5, in the present example, T_b is bounded.

We remark that Theorem 4.5 does not cover the case $E_n = \lambda n + \mu$ with constants $\lambda > 0$ and $\mu \in \mathbb{R}$. For this case, we have the following theorem:

Theorem 4.6 Suppose that there exist constants $\lambda > 0, \mu \in \mathbb{R}$ and a > 0 such that

$$E_n = \lambda n + \mu, \quad n > a. \tag{4.19}$$

Then T is a bounded self-adjoint operator with $D(T) = \mathcal{H}$.

Proof. Let k_0 be the greatest integer such that $k_0 \leq a$. Let $a_n := \langle e_n, \psi \rangle$ ($\psi \in \mathcal{H}$). Then, by the Parseval equality, we have $\sum_{n=1}^{\infty} |a_n|^2 = ||\psi||^2$. Let $\psi \in \mathcal{D}_0$. Then we can write:

$$\langle \psi, T\psi \rangle = S_I + S_2 + S_3 + S_4,$$

where

$$S_{1} := i \sum_{n=1}^{k_{0}} \sum_{m \neq n}^{k_{0}} \frac{a_{n}^{*}a_{m}}{E_{n} - E_{m}},$$

$$S_{2} := i \sum_{n=1}^{k_{0}} \sum_{m \geq k_{0}+1}^{\infty} \frac{a_{n}^{*}a_{m}}{E_{n} - E_{m}},$$

$$S_{3} := i \sum_{n \geq k_{0}+1}^{\infty} \sum_{m=1}^{k_{0}} \frac{a_{n}^{*}a_{m}}{E_{n} - E_{m}},$$

$$S_{4} := i \frac{1}{\lambda} \sum_{n=k_{0}+1}^{\infty} \sum_{m \neq n, m \geq k_{0}+1}^{\infty} \frac{a_{n}^{*}a_{m}}{n - m}$$

By the Schwarz inequality, we have

$$|S_j| \le C_j \|\psi\|^2, \quad j = 1, 2, 3,$$

where $C_j > 0$ is a constant. To estimate $|S_4|$, we use the following well known inequality [11, Theorem 294]:

$$\left|\sum_{n,m=1,n\neq m}^{\infty} \frac{x_n y_m}{n-m}\right| \le \pi \sqrt{\sum_{n=1}^{\infty} x_n^2} \sqrt{\sum_{m=1}^{\infty} y_m^2}$$

for all real sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$. Hence

$$|S_4| \le \pi \|\psi\|^2.$$

Therefore it follows that $|\langle \psi, T\psi \rangle| \leq \text{const.} \|\psi\|^2$. Thus T is bounded.

Example 4.2 A physically interesting case is the case where $E_n = \omega(n+\frac{1}{2}), n \in \{0\} \cup \mathbb{N}$ with a constant $\omega > 0$. In this case, by Theorem 4.6, T is a bounded self-adjoint operator with $D(T) = \mathcal{H}$ and takes the form

$$T\psi = \frac{i}{\omega} \sum_{n=1}^{\infty} \left(\sum_{m \neq n}^{\infty} \frac{\langle e_m, \psi \rangle}{n-m} \right) e_n, \quad \psi \in \mathcal{H}.$$

Moreover, one can prove that

$$\sigma(T) = [-\pi/\omega, \pi/\omega]$$

([4, Theorem 2.1]).

Let $\hat{N} := \omega^{-1} H - 1/2$ and $\hat{\theta} := \omega T$. Then it follows that

$$\sigma(\hat{N}) = \{0\} \cup \mathbb{N}, \quad \sigma(\hat{\theta}) = [-\pi, \pi], \tag{4.20}$$

$$[\hat{\theta}, \hat{N}]\psi = i\psi, \quad \psi \in \mathcal{D}_{c}.$$
 (4.21)

As is well known, in the context of quantum mechanics, the sequence $\{\omega(n+\frac{1}{2})\}_{n=1}^{\infty}$ appears as the spectrum of the one-dimensional quantum harmonic oscillator Hamiltonian with mass m > 0

$$H_{\rm os} := \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

in the Schrödinger representation (q, p) of the CCR, where p := -iD with D being the generalized partial differential operator on $L^2(\mathbb{R})$ and q is the multiplication operator by the variable $x \in \mathbb{R}$. In this context, the operator \hat{N} is called the number operator and, in view of (4.20) and (4.21), the operator $\hat{\theta}$ is interpreted as a *phase operator* [7].

5 Unboundedness of T

As for the unboundedness of T, we have the following fact:

Theorem 5.1 If $\{E_n\}_{n=1}^{\infty}$ satisfies

$$\inf_{n \in \mathbb{N}} (E_{n+1} - E_n) = 0, \tag{5.1}$$

then T is unbounded.

Proof. By (5.1), there exists a subsequence $\{E_{p_k}\}_{k=1}^{\infty}$ of $\{E_p\}_{p=1}^{\infty}$ such that

$$\lim_{k \to \infty} (E_{p_k+1} - E_{p_k}) = 0.$$
(5.2)

Hence we have

$$||Te_{p_k}||^2 = \sum_{n=1}^{\infty} \left| \sum_{m\neq n}^{\infty} \frac{\langle e_m, e_{p_k} \rangle}{E_n - E_m} \right|^2 = \sum_{n\neq p_k}^{\infty} \left| \frac{1}{E_n - E_{p_k}} \right|^2$$
$$\geq \left| \frac{1}{E_{p_k+1} - E_{p_k}} \right|^2 \to \infty \ (k \to \infty).$$

Thus T is unbounded.

Example 5.1 Let

$$E_n = \lambda n^{\alpha} + \mu$$

with constants $\lambda > 0$, $\alpha > 1/2$ and $\mu \in \mathbb{R}$. Then $\{E_n\}_{n=1}^{\infty}$ satisfies the assumptions (H.1) and (H.2). As we have already seen, T is bounded if $\alpha \ge 1$

Let $1/2 < \alpha < 1$. Then one easily sees that

$$\lim_{n \to \infty} (E_{n+1} - E_n) = 0.$$

Hence $\inf_{n \in \mathbb{N}} (E_{n+1} - E_n) = 0$. Therefore, in this case, T is unbounded. Thus T is bounded if and only if $\alpha \ge 1$.

6 Hilbert-Schmidtness of T

In this section we investigate Hilbert-Schmidtness of the operator T.

Proposition 6.1 The operator T is Hilbert-Schmidt if and only if

$$\sum_{n=1}^{\infty} \sum_{m \neq n}^{\infty} \frac{1}{(E_n - E_m)^2} < \infty.$$
(6.1)

In that case, T is self-adjoint with

$$||T||_2^2 = \sum_{n=1}^{\infty} \sum_{m \neq n}^{\infty} \frac{1}{(E_n - E_m)^2},$$
(6.2)

where $\|\cdot\|_2$ denotes Hilbert-Schmidt norm. In particular, there exist a C.O.N.S. $\{f_n\}_{n=1}^{\infty}$ of \mathcal{H} and real numbers t_n , $n \in \mathbb{N}$ such that $Tf_n = t_n f_n$ and $t_n \to 0$ $(n \to \infty)$.

Proof. Suppose that T is Hilbert-Schmidt. Then $\sum_{n=1}^{\infty} ||Te_n||^2 < \infty$. On the other hand, we have

$$\sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty} \sum_{m \neq n}^{\infty} \frac{1}{(E_n - E_m)^2}$$
(6.3)

Hence (6.1) follows with (6.2).

Conversely, (6.1) holds. Hence, by (6.3), $\sum_{n=1}^{\infty} ||Te_n||^2 < \infty$. Therefore *T* is Hilbert-Schmidt. The last statement follows from the Hilbert-Schmidt theorem (e.g., [14, Theorem VI.16]).

Remark 6.1 In Proposition 6.1, the number $t_n \neq 0$ is an eigenvalue of T with a finite multiplicity. Since T is self-adjoint in the present case, it may be an observable in the context of quantum mechanics. If this is the case, then Proposition 6.1 shows that the observable described by T ("time" in any sense ?) is quantized (discretized) in the quantum system whose Hamiltonian is H with eigenvalues $\{E_n\}_{n=1}^{\infty}$ satisfying (6.1).

The next theorem gives a class of H such that T is Hilbert-Schmidt:

Theorem 6.2 Suppose that (4.12) in Theorem 4.5 holds with $\alpha > 3/2$. Then T is Hilbert-Schmidt and self-adjoint.

Proof. Since $1/(E_n - E_m)^2$ is symmetric in n and m, it is sufficient to show that $\sum_{n>m\geq 1} 1/(E_n - E_m)^2 < \infty$. By the present assumption, we need only to show that

$$\Sigma := \sum_{n > m \ge 1} \frac{1}{(n^{\alpha} - m^{\alpha})^2} < \infty$$

for all $\alpha > 3/2$. We have

$$\Sigma = \sum_{n=2}^{\infty} \sum_{m=1}^{n-1} \frac{1}{(n^{\alpha} - m^{\alpha})^2} \\ \leq \sum_{n=2}^{\infty} \frac{1}{n^{\alpha} - (n-1)^{\alpha}} \cdot \sum_{m=1}^{n-1} \frac{1}{(n^{\alpha} - m^{\alpha})}$$

Using (4.4) and the elementary inequality

$$\frac{1}{n^{\alpha}-(n-1)^{\alpha}s} \leq \frac{1}{\alpha(n-1)^{\alpha-1}},$$

we obtain

$$\Sigma \le \sum_{n=2}^{\infty} \frac{1}{\alpha (n-1)^{\alpha-1}} \frac{\log n}{n^{\alpha-1}} + \sum_{n=2}^{\infty} \frac{1}{\alpha^2 (n-1)^{2(\alpha-1)}}.$$

Each infinite series on the right hand side converges for all $\alpha > 3/2$. Thus the desired result follows.

7 The Galapon Time Operator as a Generalized Time Operator

It is shown that every self-adjoint operator which has a strong time operator is absolutely continuous [12, 17]. Hence the Galapon time operator T_1 is not a strong time operator of H. But it may be a generalized time operator of H. In this section we investigate this aspect.

7.1 An operator-valued function on \mathbb{R}

In the same way as in Lemma 2.1-(ii), one can show that, for all $\psi \in \mathcal{H}$, $n \in \mathbb{N}$ and all $t \in \mathbb{R}$, the infinite series

$$\sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \left\langle e_m, \psi \right\rangle$$

absolutely converges. Hence, for each $t \in \mathbb{R}$, one can define a linear operator K(t) as follows:

$$D(K(t)) := \left\{ \psi \in \mathcal{H} | \sum_{n=1}^{\infty} \left| \sum_{m \neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle \right|^2 < \infty \right\},$$
(7.1)

$$K(t)\psi := i\sum_{n=1}^{\infty} \left(\sum_{m\neq n}^{\infty} \frac{e^{it(E_n - E_m)} - 1}{E_n - E_m} \langle e_m, \psi \rangle\right) e_n, \quad \psi \in D(K(t)).$$
(7.2)

It is easy to see that, for all $t \in \mathbb{R}$,

$$\mathcal{D}_0 \subset D(K(t)) \tag{7.3}$$

and

$$K(t)e_{k} = i \sum_{n \neq k} \frac{e^{it(E_{n} - E_{k})} - 1}{E_{n} - E_{k}} e_{n}, \quad k \in \mathbb{N}.$$
(7.4)

The correspondence $K : \mathbb{R} \ni t \mapsto K(t)$ gives an operator-valued function on \mathbb{R} . In the notation in Section 4, K(t) is the operator T_b with $b_{nm} = e^{it(E_n - E_m)} - 1$, $n, m \in \mathbb{N}$.

Remark 7.1 Equation (7.4) shows that $K(t) \neq tI | \mathcal{D}_0$. Hence T cannot be a strong time operator of H, as already remarked based on the general theory of strong time operators.

Proposition 7.1 For all $t \in \mathbb{R}$, K(t) is a densely defined closed operator.

Proof. Similar to the proof of Proposition 3.2.

Proposition 7.2 For all $t \in \mathbb{R}$, $K(t)|\mathcal{D}_0$ is symmetric.

Proof. Similar to the proof of Lemma 2.3.

Theorem 7.3 For all $\psi \in D(T_1)(=\mathcal{D}_0)$ and $t \in \mathbb{R}$, $e^{-itH}\psi \in D(T_1)$ and

$$T_1 e^{-itH} \psi = e^{-itH} (T_1 + K(t)) \psi.$$
(7.5)

Proof. We need only to prove the statement in the case $\psi = e_k \; (\forall k \in \mathbb{N})$. Since $e^{-itH}e_k = e^{-itE_k}e_k$, it follows that $e^{-itH}e_k \in D(T_1)$ with

$$T_1 e^{-itH} e_k = e^{-itE_k} \sum_{n \neq k}^{\infty} \frac{i}{E_n - E_k} e_n.$$

We have

$$e^{-itH}T_1e_k = i\sum_{n\neq k}^{\infty} \frac{e^{-itE_n}}{E_n - E_k}e_n.$$

It follows from these equations that

$$T_1 e^{-itH} e_k - e^{-itH} T_1 e_k = e^{-itH} K(t) e_k.$$

Thus the desired result follows.

Corollary 7.4 Suppose that, for all $t \in \mathbb{R}$, K(t) is bounded. Then T_1 is a generalized time operator of H with commutation factor K.

Proof. This follows from Theorem 7.3, Proposition 7.1 and Proposition 7.2. \Box

In view of Corollary 7.4, we need to investigate conditions for K(t) to be bounded.

Proposition 7.5 Suppose that (4.12) holds with $\alpha > 1$. Then, for all $t \in \mathbb{R}$, K(t) is a bounded self-adjoint operator with $D(K(t)) = \mathcal{H}$.

Proof. This follows from an application of Theorem 4.5 to the case where $b_{nm} = e^{it(E_n - E_m)} - 1$, $n, m \in \mathbb{N}$.

Proposition 7.6 Suppose that (6.1) holds. Then, for all $t \in \mathbb{R}$, K(t) is Hilbert-Schmidt and self-adjoint with

$$||K(t)||_{2}^{2} = \sum_{k=1}^{\infty} \sum_{n \neq k}^{\infty} \left| \frac{e^{it(E_{n} - E_{k})} - 1}{E_{n} - E_{k}} \right|^{2}.$$
(7.6)

Proof. Similar to the proof of Proposition 6.1.

7.2 Non-differentiability of K

From the view-point of the theory of generalized time operators [2], it is interesting to examine differentiability of the operator-valued function K.

Proposition 7.7 For all $k \in \mathbb{N}$, the \mathcal{H} -valued function : $\mathbb{R} \ni t \mapsto K(t)e_k$ is not strongly differentiable on \mathbb{R} .

Proof. We first show that $K(t)e_k$ is not strongly differentiable at t = 0. Since $K(0)e_k = 0$, we have for all $t \in \mathbb{R} \setminus \{0\}$ and N > k

$$\left\|\frac{K(t)e_k - K(0)e_k}{t}\right\|^2 = \sum_{\substack{n \neq k}}^{\infty} \frac{|e^{it(E_n - E_k)} - 1|^2}{t^2 |E_n - E_k|^2}$$
$$\geq \sum_{\substack{n \neq k}}^{N+1} \frac{|e^{it(E_n - E_k)} - 1|^2}{t^2 |E_n - E_k|^2}$$

Hence

$$\liminf_{t \to 0} \left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 \ge \sum_{n \neq k}^{N+1} 1 = N.$$

Since N > k is arbitrary, it follows that

$$\lim_{t \to 0} \left\| \frac{K(t)e_k - K(0)e_k}{t} \right\|^2 = +\infty.$$

This implies that $K(t)e_k$ is not strongly differentiable at t = 0.

We next show that $K(t)e_k$ is not strongly differentiable at each $t \neq 0$. By (7.5), we have for all $s \in \mathbb{R} \setminus \{0\}$

$$\frac{K(t+s)e_k - K(t)e_k}{s} = e^{it(H-E_k)}\frac{K(s)e_k}{s}.$$

Hence

$$\left\|\frac{K(t+s)e_k - K(t)e_k}{s}\right\| = \left\|\frac{K(s)e_k}{s}\right\|.$$

By the preceding result, the right hand side diverges to $+\infty$ as $s \to 0$. Therefore $K(t)e_k$ is not strongly differentiable at t.

Remark 7.2 We have

$$\langle e_{\ell}, K(t)e_{k} \rangle = \begin{cases} i \frac{e^{it(E_{\ell} - E_{k})} - 1}{E_{\ell} - E_{k}} & ; \ \ell \neq k \\ 0 & ; \ \ell = k \end{cases}$$
(7.7)

Hence, for all $k, \ell \in \mathbb{N}, \langle e_{\ell}, K(t)e_k \rangle$ is differentiable in $t \in \mathbb{R}$ and

$$\frac{d}{dt} \langle e_{\ell}, K(t)e_k \rangle = (\delta_{\ell k} - 1)e^{it(E_{\ell} - E_k)}.$$
(7.8)

Proposition 7.7 tells us some singularity of K(t) acting on \mathcal{D}_0 . But, as shown in the next proposition, K(t) restricted to \mathcal{D}_c is strongly differentiable at t = 0.

Proposition 7.8 For all $\psi \in \mathcal{D}_c$, the \mathcal{H} -valued function $K(t)\psi$ is strongly differentiable at t = 0 with

$$\left. \frac{d}{dt} K(t) \psi \right|_{t=0} = \psi.$$
(7.9)

Proof. We need only to prove the statement for ψ of the form $\psi = e_k - e_\ell$ $(k, \ell \in \mathbb{N}, k \neq \ell)$. For all $t \in \mathbb{R} \setminus \{0\}$, we have

$$\frac{K(t)(e_k - e_\ell)}{t} = A(t) + B(t),$$

where

$$A(t) := i \frac{e^{it(E_{\ell} - E_{k})} - 1}{t(E_{\ell} - E_{k})} e_{\ell} - i \frac{e^{it(E_{k} - E_{\ell})} - 1}{t(E_{k} - E_{\ell})} e_{k},$$

$$B(t) := i \sum_{n \neq k, \ell}^{\infty} \left(\frac{e^{it(E_{n} - E_{k})} - 1}{t(E_{n} - E_{k})} - \frac{e^{it(E_{n} - E_{\ell})} - 1}{t(E_{n} - E_{\ell})} \right) e_{n}.$$

It is easy to see that

$$\lim_{t \to 0} A(t) = e_k - e_\ell.$$

As for B(t), we have

$$||B(t)||^2 = \sum_{n \neq k, \ell}^{\infty} |F_n(t)|^2,$$

where

$$F_n(t) := \frac{e^{it(E_n - E_k)} - 1}{t(E_n - E_k)} - \frac{e^{it(E_n - E_\ell)} - 1}{t(E_n - E_\ell)}.$$

It is easy to see that

$$\lim_{t \to 0} F_n(t) = 0.$$

Moreover, one can show that

$$|F_n(t)| \le \frac{C}{|E_n - E_k|}, \quad n \ne k, \ell,$$

where C > 0 is a constant independent of n and t. Since $\sum_{n \neq k}^{\infty} 1/|E_n - E_k|^2 < \infty$, one can apply the dominated convergence theorem to conclude that $\lim_{t\to 0} ||B(t)||^2 = 0$. Thus $K(t)(e_k - e_\ell)$ is strongly differentiable at t = 0 and (7.9) with $\psi = e_k - e_\ell$ holds.

Proposition 7.9 For all $k, \ell \in \mathbb{N}$ with $k \neq \ell$, the \mathfrak{H} -valued function $K(t)(e_k - e_\ell)$ is not strongly differentiable at $t \notin \{2\pi n/(E_k - E_\ell) | n \in \mathbb{Z}\}$.

Proof. Let $t \neq 2\pi n/(E_k - E_\ell)$ $(n \in \mathbb{Z})$ and $s \in \mathbb{R} \setminus \{0\}$. Then, by (7.5), we have

$$\frac{(K(t+s) - K(t))(e_k - e_\ell)}{s} = e^{itH} \frac{K(s)}{s} e^{-itH} (e_k - e_\ell).$$

Hence

$$\left\|\frac{(K(t+s) - K(t))(e_k - e_\ell)}{s}\right\| = \|u(s)\|$$

with

$$u(s) := \frac{K(s)}{s} (e^{-itE_k} e_k - e^{-itE_\ell} e_\ell).$$

We write

$$u(s) = u_1(s) + u_2(s)$$

with

$$u_1(s) := e^{-itE_k} \frac{K(s)}{s} (e_k - e_\ell), \quad u_2(s) := (e^{-itE_k} - e^{-itE_\ell}) \frac{K(s)}{s} e_\ell.$$

By Proposition 7.8, we have $\lim_{s\to 0} u_1(s) = e^{-itE_k}(e_k - e_\ell)$. On the other hand, we have from the proof of Proposition 7.7 and the assumed condition for t

$$\lim_{s \to 0} \|u_2(s)\| = +\infty$$

Hence $\lim_{s\to 0} ||u(s)|| = +\infty$. Thus the desired result follows.

8 Possible Connections with Regular Perturbation Theory

We consider a perturbation of H by a symmetric operator H_I on \mathcal{H} :

$$H(\lambda) := H + \lambda H_I, \tag{8.1}$$

where $\lambda \in \mathbb{R}$ is a perturbation parameter. For simplicity, we assume that H_I is *H*-bounded: $D(H) \subset D(H_I)$ and there exist constants $a, b \geq 0$ such that

$$||H_I\psi|| \le a||H\psi|| + b||\psi||, \quad \psi \in D(H).$$

Then, by the Kato-Rellich theorem (e.g., [15, Theorem X.12]), for all $\lambda \in \mathbb{R}$ satisfying

$$a|\lambda| < 1, \tag{8.2}$$

 $H(\lambda)$ is self-adjoint and bounded below. In what follows, we assume (8.2).

8.1 Eigenvalues of $H(\lambda)$

We fix $n \in \mathbb{N}$ arbitrarily. By a general theorem in regular perturbation theory (e.g., [16, Theorem XII.9]), there exists a constant $c_n > 0$ such that, for all $|\lambda| < c_n$, H has a unique, isolated non-degenerate eigenvalue $E_n(\lambda)$ near E_n . Moreover, $E_n(\lambda)$ is analytic in λ with Taylor expansion

$$E_n(\lambda) = E_n + E_n^{(1)}\lambda + E_n^{(2)}\lambda^2 + \cdots,$$
 (8.3)

where

$$E_n^{(1)} := \langle e_n, H_I e_n \rangle, \quad E_n^{(2)} := \sum_{m \neq n}^{\infty} \frac{|\langle e_n, H_I e_m \rangle|^2}{E_n - E_m}.$$
 (8.4)

As an eigenvector of $H(\lambda)$ with eigenvalue $E_n(\lambda)$, one can take a vector $\psi_n(\lambda)$ analytic in λ with Taylor expansion

$$\psi_n(\lambda) = e_n + e_n^{(1)}\lambda + \cdots, \qquad (8.5)$$

where

$$e_n^{(1)} := \sum_{m \neq n}^{\infty} \frac{\langle e_m, H_I e_n \rangle}{E_n - E_m} e_m.$$
(8.6)

By Lemma 2.2, we have

$$\langle e_n, Te_m \rangle = \begin{cases} \frac{i}{E_n - E_m} & ; n \neq m \\ 0 & ; n = m \end{cases}$$

Hence $E_n^{(2)}$ can be written

$$E_n^{(2)} = (-i) \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 \langle e_n, T e_m \rangle.$$
(8.7)

To rewrite the right hand side only in terms of e_n and linear operators on \mathcal{H} , we note that

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 = ||H_I e_n||^2 < \infty$$

by the Parseval equality. Hence

$$\sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^4 < \infty$$

Therefore the infinite series

$$f_n := \sum_{m=1}^{\infty} |\langle e_n, H_I e_m \rangle|^2 e_m$$
(8.8)

strongly converges and defines a vector in \mathcal{H} . Thus we can define a linear operator V on \mathcal{H} as follows:

$$D(V) := \mathcal{D}_0, \tag{8.9}$$

$$V\psi := -i\sum_{n=1}^{\infty} \langle e_n, \psi \rangle f_n, \quad \psi \in \mathcal{D}_0$$
(8.10)

where the right hand side of (8.10) is a sum over a finite term. It is easy to see that V is a symmetric operator.

Proposition 8.1 For all $n \in \mathbb{N}$,

$$E_n^{(2)} = \langle Te_n, Ve_n \rangle.$$
(8.11)

Proof. We have $Ve_n = -if_n$. Hence $\langle Te_n, Ve_n \rangle = -i \langle Te_n, f_n \rangle$, which is equal to the right hand side of (8.7).

This proposition suggests some role of the time operator $T_1 = T | \mathcal{D}_0$ in the perturbation expansions of the eigenvalues of H.

As for the first order term $e_n^{(1)}\lambda$ of the eigenvector $\psi_n(\lambda)$, we have

$$e_n^{(1)} = (-i) \sum_{m=1}^{\infty} \langle e_m, H_I e_n \rangle \langle e_n, T e_m \rangle e_m.$$
(8.12)

8.2 Transition probability amplitudes

In the context of quantum mechanics where $H(\lambda)$ is the Hamiltonian of a quantum system, the complex number $\langle \phi, e^{-itH(\lambda)}\psi \rangle$ with unit vectors $\phi, \psi \in \mathcal{H}$ is called the *transition* probability amplitude for the probability such that the state of the quantum system at time t is found in the state ϕ under the condition that the state of the quantum system at time zero is ψ . **Lemma 8.2** Let $\phi, \psi \in D(H)$. Then, for all $t \in \mathbb{R}$,

$$\left\langle \phi, e^{-itH(\lambda)}\psi \right\rangle = \left\langle \phi, e^{-itH}\psi \right\rangle - i\lambda \int_0^t \left\langle e^{i(t-s)H}\phi, H_I e^{-isH}\psi \right\rangle ds + O(\lambda^2).$$
(8.13)

Proof. By a simple application of a general formula for the unitary group generated by a self-adjoint operator ([5, Lemma 5.9]), we have

$$e^{-itH(\lambda)}\psi = e^{-itH}\psi - i\lambda \int_0^t e^{-i(t-s)H(\lambda)}H_I e^{-isH}\psi ds,$$
(8.14)

where the integral is taken in the strong sense. Hence

$$\begin{split} \left\langle \phi, e^{-itH(\lambda)}\psi \right\rangle &= \left\langle \phi, e^{-itH}\psi \right\rangle - i\lambda \int_0^t \left\langle e^{i(t-s)H(\lambda)}\phi, H_I e^{-isH}\psi \right\rangle ds \\ &= \left\langle \phi, e^{-itH}\psi \right\rangle - i\lambda \int_0^t \left\langle e^{i(t-s)H}\phi, H_I e^{-isH}\psi \right\rangle ds + R(\lambda), \end{split}$$

where

$$R(\lambda) := -i\lambda \int_0^t \left\langle \left(e^{i(t-s)H(\lambda)} - e^{i(t-s)H} \right) \phi, H_I e^{-isH} \psi \right\rangle ds.$$

Using (8.14) again, we have

$$R(\lambda) = -\lambda^2 \int_0^t ds \int_0^{-(t-s)} ds' \left\langle e^{i(t-s+s')H(\lambda)} H_I e^{-is'H} \phi, H_I e^{-isH} \psi \right\rangle.$$

Hence

$$|R(\lambda)| \le \lambda^2 \int_0^{|t|} ds \int_0^{|t-s|} ds' ||H_I e^{-is'H} \phi|| ||H_I e^{-isH} \psi|$$

Therefore $R(\lambda) = O(\lambda^2)$. Thus (8.13) holds.

Applying (8.13) with $\phi = e_m$ and $\psi = e_n$ $(n \neq m)$, we have

$$\left\langle e_m, e^{-itH(\lambda)} e_n \right\rangle = -\lambda \frac{e^{-itE_n} - e^{-itE_m}}{E_m - E_n} \left\langle e_m, H_I e_n \right\rangle + O(\lambda^2), \tag{8.15}$$

which, combined with (7.7), gives

$$\langle e_m, e^{-itH(\lambda)}e_n \rangle = i\lambda \langle e_n, e^{-itH}K(t)e_m \rangle \langle e_m, H_Ie_n \rangle + O(\lambda^2). \quad m \neq n.$$
 (8.16)

This suggests a physical meaning of the commutation factor K.

By Theorem 7.3, one can rewrite the first term on the right hand side in terms of T_1 and e^{-itH} , obtaining

$$\langle e_m, e^{-itH(\lambda)}e_n \rangle = i\lambda \langle e_n, [T_1, e^{-itH}]e_m \rangle \langle e_m, H_Ie_n \rangle + O(\lambda^2), \quad m \neq n.$$
 (8.17)

This also is suggestive on physical meaning of the time operator T_1 .

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