The following paper was published in *Symmetry: Culture and Science* **17**, Nos.1-2, 141-157 (2006).

FUNDAMENTAL SYMMETRY PRINCIPLE IN QUANTUM MECHANICS AND ITS PHILOSOPHICAL PHASES

Asao Arai

Mathematician, Mathematical Physicist (b. Chichibu, Saitama, Japan, 1954).
Address: Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
E-mail: arai@math.sci.hokudai.ac.jp.
Fields of interest: Mathematics, mathematical physics (also mathematical theory of music and formative arts, mathematical aesthetic, philosophy).
Publications : Arai, A. (1993) Mathematics of Symmetry : Nippon-hyoronsha, v+221 pp (in Japanese). Arai, A. (1997) Hilbert Spaces and Quantum Mechanics: Kyoritsu-shuppan, viii+276 pp (in Japanese). Arai, A. (with Ezawa, H.)(1999) Mathematical Structures of Quantum Mechanics I, II: Asakura-shoten, ix+547pp (in Japanese). Arai, A. (2005), Handbook of Modern Mathematical Physics: Asakura-shoten, xxiv+707 pp (in Japanese). Arai, A. (2006), Mathematical Principles of Quantum Phenomena: Asakura-shoten, x+535 pp (in Japanese).

Abstract

A fundamental symmetry principle in quantum mechanics is formulated in the framework of the standard axiomatic quantum mechanics and a new philosophical interpretation for quantum mechanics, which dissolves "difficulties" in the conventional interpretations for quantum mechanics, is presented. Moreover, philosophical phases of the fundamental symmetry principle are discussed in connection with Plato's philosophy and Oriental philosophies, in particular, Zen Buddhism.

Keywords: quantum mechanics, phenomenal multifoldness, symmetry, canonical commutation relation, canonical anti-commutation relation, quantum field, Hamiltonian, time operator, metaphysics, Plato, Oriental philosophy, Zen Buddhism.

1 INTRODUCTION

The purpose of this paper is threefold. The first one is to point out a symmetry principle in quantum mechanics, which we call it a fundamental symmetry principle, in the framework

of the standard axiomatic formulation of quantum mechanics which is essentially due to von Neumann (1932). The second one, which is closely related to the first one, is to present a new philosophical interpretation for quantum mechanics, which may make it possible to obtain a unified point of view to quantum phenomena, overcoming "difficulties" in the conventional interpretations for quantum mechanics, such as "reduction of wave function" and "non-locality" of quantum phenomena[1]. The third one is to show that the philosophical interpretation is in accordance with the wisdom of Plato's philosophy and Oriental philosophies, in particular, the philosophy of Zen Buddhism.

The present paper is organised as follows: In Section 2 we first recall the axioms of quantum mechanics. Then we formulate in a mathematically rigorous manner the fundamental symmetry principle mentioned above. Section 3 is devoted to philosophical considerations of the contents in Section 2. These considerations lead us in a natural way to a new interpretation on quantum states, physical quantities and "time-development" of a quantum state. In the last section we show how the fundamental symmetry principle is harmonically positioned in the philosophies mentioned above.

2 GENERAL STRUCTURE OF QUANTUM MECHANICS AND FUNDAMENTAL SYMMETRY PRINCIPLE

2.1 Axioms of quantum mechanics

In the present paper , we mean by quantum mechanics not only the one with finite degrees of freedom, but also the one with infinite degrees of freedom, including quantum field theory, unless otherwise stated. The general structure of quantum mechanics is given by a set of axioms which is essentially due to von Neumann (1932). We first recall it[2].

(QM.1) (Quantum state)

(a) For each quantum system S, there is a complex Hilbert space $\mathcal{H}$, and a state of S is represented by a non-zero vector in $\mathcal{H}$, called a *state vector*. The Hilbert space $\mathcal{H}$ is called a Hilbert space of state vectors of S.

(b) (The identity principle of quantum states) Two non-zero vectors Ψ and Φ in $\mathcal{H}$ represent an identical state if and only if $\Psi = \alpha\Phi$ for some complex constant $\alpha \neq 0$.

[1] We do not review the conventional interpretations for quantum mechanics and controversies on them here. See, e.g., Peres (1993).

[2] For mathematical terminology , see textbooks on Hilbert space theory, e.g., Reed and Simon (1972), Weidmann (1980), Arai (1997), Arai and Ezawa (1999), Arai (2005b).

(QM.2) (Physical quantities) A physical quantity (observable) of the quantum system S is represented by a self-adjoint operator acting in $\mathcal{H}$. In particular, a self-adjoint operator which represents the total energy of the system S is called the *Hamiltonian* of S.

(QM.3) (Measurement and probability interpretation) In a state $\Psi \in \mathcal{H}$, the probability that the result of a measurement of the physical quantity A lies in a Borel set $J \subset \mathbf{R}$ (the field of real numbers) is given by $\|E_A(J)\Psi\|^2/\|\Psi\|^2$, where is E_A is the spectral measure of the self-adjoint operator A and $\|\cdot\|$ the norm of $\mathcal{H}$.

(QM.4) ("Time-development") Given a state vector $\Psi \in \mathcal{H}$ at time $t = 0$, the state vector $\Psi(t) \in \mathcal{H}$ at time $t \in \mathbf{R}$ is given by

$$\Psi(t) = e^{-iHt/\hbar}\Psi,$$

provided that no measurement is done on the system S in the meantime, where i is the imaginary unit, H is the Hamiltonian of S, and $\hbar = h/2\pi$ with h being the Planck constant.

(QM.5) If a measurement of the physical quantity A yields a result in a Borel set $J \subset \mathbf{R}$, then the state of the system "immediately after" the measurement is in the range of $E_A(J)$.

Here we give some remarks on the axioms described above. As is seen later, these remarks are essential for our new philosophical interpretation for quantum mechanics.

(R.1) The first remark is about Axioms (QM.1) and (QM.2).

(1) The Hilbert space $\mathcal{H}$ of state vectors is not uniquely chosen. In fact, if $\mathcal{H}$ is a Hilbert space of state vectors of the quantum system S, then every Hilbert space which is a unitary transformation of $\mathcal{H}$ can be a Hilbert space of state vectors of S. Physically this is interpreted as a reflection of an essential nature of a *quantum particle*[3]. As is well-known, a quantum particle can appear in different manners depending on the measurement apparatuses used. A typical example is the so-called *wave-particle duality*: a quantum particle behaves like a classical wave under a certain measurement condition, while it moves like a classical particle under another measurement condition. In this case it is important to note that the two appearances are mutually exclusive. This is a

[3]We mean by a *quantum particle* a microscopic object which does not follow the laws of classical physics, but does those of quantum mechanics : atoms, nucleuses and elementary particles (electron, nucleon, pion, photon etc.) are typical examples.

basic example of the *complementary principle* advocated by N. Bohr (1948). There are other examples of complementary quantum phenomena. Thus we are led to the following insight: a quantum particle can take, in principle, infinitely many appearances according to the measurement apparatuses used, including mutually exclusive ones. We call this property of a quantum particle the *phenomenal multifoldness* (Arai, 2003). Each appearance of a quantum particle is connected with a physical picture. This exactly corresponds to non-uniqueness of the Hilbert space of state vectors of a quantum system : there corresponds to a choice of the Hilbert space of state vectors according to each physical picture of the quantum system under consideration. The bridge between two different physical pictures of a quantum particle can be made by a unitary transformation (unitary operator) between the corresponding Hilbert spaces of state vectors[4].

(2) It is well-known that the identity principle of quantum states (QM.1)-(b) together with the principle of indistinguishability of a quantum particle plays an important role in deriving statistics for a quantum many body system consisting of identical quantum particles (Arai, 2002). But , on the axiomatic level too, it implies some very important consequence. We say that two non-zero vectors $\mathcal{H}$ are equivalent if there exists a non-zero complex constant α such that $\Psi = \alpha\Phi$. In this case we write $\Psi \sim \Phi$. Then it is easy to show that this relation $\sim$ is an equivalence relation in the set $\mathcal{H} \setminus \{0\}$. The equivalence class of Ψ is given by

$$[\Psi] = \{\beta\Psi | \beta \in \mathbf{C} \setminus \{0\}\},$$

where $\mathbf{C}$ is the field of complex numbers. This object is called a *ray* and the quotient space

$$P(\mathcal{H}) = (\mathcal{H} \setminus \{0\})/\sim$$

is called the *projective space* of $\mathcal{H}$. Therefore the identity principle of quantum states tells us that a quantum state is a ray in $\mathcal{H}$. Hence a quantum state is a very abstract object and has no physical correspondent. Thus we have to conclude that a quantum state is not physical[5]. In relation to this, it is

[4]An elementary example of a Hilbert space of state vectors is the Hilbert space$L^2(\mathbf{R}^n_{\mathbf{x}})$ of square integrable functions on the n-dimensional space $\mathbf{R}^n_{\mathbf{x}} = \{\mathbf{x} = (x_1, ..., x_n) | x_j \in \mathbf{R}, j = 1, ..., n\}$ (n is a natural number). In the physics literature, this Hilbert space is called the state space in the coordinate representation or q-representation of "wave functions". If an n-dimensional vector $\mathbf{p} \in \mathbf{R}^n_{\mathbf{p}}$ denotes the momentum of a quantum particle , then $L^2(\mathbf{R}^n_{\mathbf{p}})$ is called the state space in the momentum representation or p-representation of "wave functions". The Fourier transform $\mathcal{F}$ from $L^2(\mathbf{R}^n_{\mathbf{x}})$ to $L^2(\mathbf{R}^n_{\mathbf{p}})$ is a unitary operator which acts as a bridge between the two Hilbert spaces (two physical pictures: position and momentum which are mutually exclusive). See also Example 1 below.

[5]This implies the following: even if a Hilbert space of state vectors of a quantum system is given by a

also difficult to interpret physically (QM.2) which tells us that physical quantities are obviously unphysical in the usual sense of "physical". But, from the philosophical point of view we take in the present paper, these structures are natural as we shall see later.

(R.2) The second remark is concerned with Axiom (QM.4). It follows from (QM.4) that, for all $t, t_0 \in \mathbf{R}$,

$$\Psi(t) = e^{-iH(t-t_0)/\hbar}\Psi(t_0),$$

which gives the relation between the state at time t_0 and that at time t under the condition that no measurement is made on the system between time t_0 and time t. The strongly continuous one-parameter unitary group

$$U(t) = e^{-iHt/\hbar} \quad (t \in \mathbf{R})$$

is a symmetry transformation on the Hilbert space $\mathcal{H}$ in the sense that it is bijective, preserving the inner product $\langle \cdot, \cdot \rangle$ of $\mathcal{H}$ and hence the transition probabilities $|\langle \Psi, \Phi \rangle|^2$ between arbitrary state vectors Ψ and Φ in $\mathcal{H}$. Hence (QM.4) can be regarded as an expression of a symmetry principle in quantum mechanics. From a group theoretical point of view, $\{U(t)|t \in \mathbf{R}\}$ is a strongly continuous unitary representation of $\mathbf{R}$ as the time translation group [6]. One of the important things on (QM.4) is that, for each state vector $\Psi \in \mathcal{H}$, a mapping $\Psi_H : \mathbf{R} \to \mathcal{H}$ is defined by

$$\Psi_H(t) = U(t)\Psi \quad (t \in \mathbf{R}).$$

This mapping is strongly continuous in $t \in \mathbf{R}$. Hence it is a curve in $\mathcal{H}$. We call the curve the *state curve* with initial state Ψ. The following theorem tells us an interesting and important aspect of state curves in $\mathcal{H}$:

Theorem 1

(1) *There is no intersection between two state curves with different initial states, i.e.,* $\Psi_H(t) \neq \Phi_H(t)$ *for all* $t \in \mathbf{R}$ *if* $\Psi \neq \Phi$.

(2) *The Hilbert space* $\mathcal{H}$ *is filled with state curves, i.e.,* $\mathcal{H} = \cup_{\Psi \in \mathcal{H}\setminus\{0\}}\{\Psi_H(t)|t \in \mathbf{R}\}$.

function space such as $L^2(\mathbf{R}^n_{\mathbf{x}})$, each element of it can not be interpreted as a wave function in the sense of classical field theory (note also that, strictly speaking, $L^2(\mathbf{R}^n_{\mathbf{x}})$ is not a function space, but a set of equivalence classes of functions). In this sene it is not suitable to call such an element a "wave function" as done in many textbooks of quantum mechanics in physics. In the context of quantum mechanics, it should be called a *state function* if one wants to give a name to it, using "function"(Arai, 2002).

[6]For more details, see, e.g., Arai (2006a), Chapter 4.

The proof of this theorem is not so difficult, but, we omit it [see Arai (2006a), Proposition 4.39].

As is shown in Section 3, this theorem is philosophically significant too.

(R.3) The last remark on the above axioms goes to (QM.5). This axiom is needed to identify the initial state or the final state in a measurement process. But, to identify a state uniquely by a measurement, one has to observe a set of physical quantities, called a maximal set of strongly commuting physical quantities [Arai (2006a), Chapter 1]. In what follows we mean by "measurement" such a measurement[7]

2.2 Fundamental Symmetry Principle

Axioms (QM.1)–(QM.5) do not tell us in what way a Hilbert space of state vectors of a quantum system is chosen and physical quantities are determined. At this point a symmetry principle acts. We call it a *fundamental symmetry principle* in quantum mechanics. To describe it, however, we need a preliminary.

Definition 1 Let n be a natural number. The Lie algebra of Heisenberg type with degree n is a complex Lie algebra, denoted $\mathrm{HL}(n)$, with a basis $\{X_j, Y_j, Z | j = 1, \cdots, n\}$ satisfying

$$[X_j, Y_k] = i\hbar Z, \quad [X_j, X_k] = 0, \quad [Y_j, Y_k] = 0,$$
$$[X_j, Z] = 0, \quad [Y_j, Z] = 0 \quad (j, k = 1, ..., n),$$

where $[\cdot, \cdot]$ is the Lie bracket of $\mathrm{HL}(n)$ and δ_{jk} is the Kronecker $\delta_{jj} = 1$; $\delta_{jk} = 0, j \neq k$.

Remark 1 The case where n is the countable infinity ∞ is also defined. The Lie algebra $\mathrm{HL}(\infty)$ is called the *Lie algebra of Heisenberg type with infinite degrees.*

For a vector space $\mathcal{V}$, we denote by $\mathrm{L}(\mathcal{V})$ the space of linear operators on $\mathcal{V}$.

Definition 2 A representation of the Lie algebra $\mathrm{HL}(n)$ is a pair $(\mathcal{V}, \rho)$ consisting of a complex vector space $\mathcal{V}$ and a linear mapping $\rho : \mathrm{HL}(n) \to \mathrm{L}(\mathcal{V})$ such that $\rho(Z) = I$ (the identity on $\mathcal{V}$) and , for all $X, Y \in \mathrm{HL}(n)$,

$$[\rho(X), \rho(Y)] = \rho([X, Y]),$$

where $[\ ,\ ]$ on the left hand side is the commutator on $\mathrm{L}(\mathcal{V})$.

[7]The uniqueness in the strict sense is only valid for a maximal set of strongly commuting physical quantities whose spectra are respectively purely discrete. In the case where the maximal set contains a physical quantity whose spectrum is not purely discrete, the notion of the uniqueness in this context has to be weakend a little bit [Arai (2006a), Chapter 1].

Let $(\mathcal{V}, \rho)$ be a representation of HL(n). Then, putting

$$Q_j = \rho(X_j), \quad P_j = \rho(Y_j)$$

we have

$$[Q_j, P_k] = i\hbar\delta_{jk}, \quad [Q_j, Q_k] = 0, \quad [P_j, P_k] = 0,$$

where we omit the identity on the right hand side of the first equation. In the context of quantum mechanics, these relations are called the *canonical commutation relations* (CCR) with n degrees of freedom [abbreviated to CCR(n)].

A notion of Hilbert space representation of HL(n) is defined as follows (the case where $\mathcal{V}$ is a subspace of a complex Hilbert space):

Definition 3 A triple $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j | j = 1, ..., n\})$ consisting of a complex Hilbert space $\mathcal{H}$, a dense subspace $\mathcal{D}$ of $\mathcal{H}$ and a set $\{Q_j, P_j | j = 1, ..., n\}$ of symmetric operators acting in $\mathcal{H}$ is called a *representation of* CCR(n) if the following conditions are satisfied:

(i)

$$\mathcal{D} \subset D(Q_j) \cap D(P_j), \quad Q_j\mathcal{D} \subset \mathcal{D}, P_j\mathcal{D} \subset \mathcal{D} \ (j = 1, ..., n),$$

where, for a linear operator A, $D(A)$ denotes its domain.

(ii)

$$[Q_j, P_k] = i\hbar\delta_{jk}, \quad [Q_j, Q_k] = 0, \quad [P_j, P_k] = 0$$

on $\mathcal{D}$ $(j, k = 1, ..., n)$.

The Hilbert space $\mathcal{H}$ is called a representation space of CCR(n).

If Q_j and P_j $(j = 1, ..., n)$ are self-adjoint , then $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j | j = 1, ..., n\})$ is called a *self-adjoint representation* of CCR(n).

Remark 2 It follows that, for each j, at least, one of Q_j and P_j is unbounded (Arai, 1997).

Remark 3 There exist non-self-adjoint representaitons of CCR(n).

Remark 4 The first relation in (ii) implies the following inequality, called the *Heisenberg uncertainty relation* (von Neumann, 1932):

$$(\Delta Q_j)_\Psi (\Delta P_j)_\Psi \geq \frac{\hbar}{2}$$

for all $\Psi \in \mathcal{D} \setminus \{0\}$, where, for a linear operator A acting in $\mathcal{H}$,

$$(\Delta A)_\Psi = \frac{\left\| \left(A - \frac{\langle \Psi, A\Psi \rangle}{\|\Psi\|^2} \right) \Psi \right\|}{\|\Psi\|}, \quad \Psi \in D(A) \setminus \{0\}$$

called the *uncertainty* of A in the state vector Ψ.

Example 1 Let $\mathcal{H} = L^2(\mathbf{R}^n_{\mathbf{x}})$, $\mathcal{D} = C_0^\infty(\mathbf{R}^n_{\mathbf{x}})$ (the space of infinitely many differentiable functions on $\mathbf{R}_{\mathbf{x}}$ with compact support),

$$q_j = x_j(\text{the multiplication operator by the } j\text{-th coordinate variable in } \mathbf{R}^n_{\mathbf{x}}),$$
$$p_j = -i\hbar D_j,$$

where D_j is the generalised partial differential operator in the variable x_j. Then

$$\Pi_{\mathrm{S}}(n) = (L^2(\mathbf{R}^n_{\mathbf{x}}), C_0^\infty(\mathbf{R}_{\mathbf{x}}), \{q_j, p_j | j = 1, \cdots, n\})$$

is a self-adjoint representation of CCR(n). This representation of CCR(n) is called the *Schrödinger representation* with n degrees of freedom. In the physics literature, it is called the q-representation or the coordinate representation.

Let $\mathcal{F}$ be the Fourier transform from $L^2(\mathbf{R}^n_{\mathbf{x}})$ onto $L^2(\mathbf{R}^n_{\mathbf{p}})$ and put

$$q'_j = \mathcal{F} q_j \mathcal{F}^{-1}, \quad p'_j = \mathcal{F} p_j \mathcal{F}^{-1}, \mathcal{D}' = \mathcal{F}(\mathcal{D}).$$

Then

$$\Pi_{\mathrm{S}}(n)' = (L^2(\mathbf{R}^n_{\mathbf{p}}), \mathcal{D}', \{q'_j, p'_j | j = 1, \cdots, n\})$$

is a self-adjoint representation of CCR(n). In the physics literature this representation is called the momentum representation or the p-representation .

Example 2 Let $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j | j = 1, \cdots, n\})$ be an arbitrary representation of CCR(n) and $\mathcal{K}$ be a complex Hilbert space . Let U be a unitary operator from $\mathcal{H}$ to $\mathcal{K}$ and put

$$Q_j(U) = U Q_j U^{-1}, P_j(U) = U P_j U^{-1}.$$

Then $(\mathcal{K}, U\mathcal{D}, \{Q_j(U), P_j(U) | j = 1, \cdots, n\})$ is a representation of CCR(n). Therefore, for each natural number n, infinitely many representations of CCR(n) can be constructed from a representation of CCR(n).

Taking the fact shown in Example 2 into account, one introduces a notion of equivalence for representations of CCR(n):

Definition 4 Two representations $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j | j = 1, \cdots, n\})$ and $(\mathcal{H}', \mathcal{D}', \{Q'_j, P'_j | j = 1, \cdots, n\})$ are said to be *equivalent* if there exists a unitary operator W from $\mathcal{H}$ to $\mathcal{K}$ such that $Q'_j = W Q_j W^{-1}, P'_j = W P_j W^{-1}$.

Example 3 The Schrödinger representation $\Pi_{\mathrm{S}}(n)$(Example 1) is equivalent to $\Pi_{\mathrm{S}}(n)'$. It is also equivalent to a representation called the *Born-Heisenberg-Jordan representation* or the *Fock representation* with n degrees of freedom (Arai and Ezawa , 1999).

With the notion of equivalence for representations of CCR(n), the set of representations of CCR(n) is classified into equivalence classes. This classification is non-trivial, because there exist representations inequivalent each other [8]. Physically important examples of such representations appear, e.g., in gauge quantum mechanics in two space dimensions (Arai, 1992, 1995a, 1995b, 1996, 1998) .

We are now ready to formulate the fundamental symmetry principle in quantum mechanics.

Fundamental Symmetry Principle

(F.1) A Hilbert space of state vectors of a quantum system with n external degrees of freedom is a representation space of a self-adjoint representation of CCR(n). Two representations with different representation spaces correspond to different frameworks of physical picture.

(F.2) If the system has internal degrees of freedom (e.g., spin) in addition to the the external ones, the Hilbert space must be also a representation space of the algebra describing the internal degrees of freedom. We call such an algebra an *internal algebra*[9].

(F.3) Let $(\mathcal{H}, \mathcal{D}, \{Q_j, P_j | j = 1, ..., n\})$ be the representation in (F.1). Then physical quantities of the system under consideration are constructed from the self-adjoint operators Q_j, P_j $(j = 1, \cdots, n)$ and the operators in the representation of the internal algebra if the system has internal degrees of freedom.

(F.4) Two equivalent representations of CCR(n) and an internal algebra give a physically equivalent description of a quantum system.

This general principle can be derived from structural analyses of models in quantum mechanics. But some remarks may be in order.

The word "fundamental" in the principle stated above is used to emphasise the fundamentality of CCR and any other internal algebras as the theoretical starting point from which quantum mechanics is developed, while "symmetry" indicates (abstract) symmetries associated with CCR (Lie algebra of Heisenberg type) and internal algebras . This kind of symmetry is not a "visible" one like translation symmetry and rotation symmetry in the n-dimensional space $\mathbf{R}^n$. In this sense too, symmetries associated with CCR and

[8]The celebrated von Neumann uniqueness theorem (von Neumann, 1931) holds only for a stronger form of representation of CCR , called the Weyl representation (Reed and Simon, 1972; Arai, 2006). Unfortunately misunderstanding on this respect can be seen in the physics literature.

[9]The notion of equivalence of representations of an internal algebra is defined similarly to the case of representations of CCR(n).

internal algebras may be more fundamental. In relation to symmetry in the most general sense, we consider algebras like Lie algebras as more fundamental than objects like Lie groups.

A quantum system can have symmetries such as translation symmetry, rotation symmetry and reflection symmetry. We call those symmetries *secondary symmetries* in comparison with the fundamental symmetry principle. One can also consider CCR with infinite degrees of freedom, denoted CCR(∞). A representation of it gives a framework of a *bosonic quantum field theory* (Arai, 2000). On the other hand, a representation of the *canonical anti-commutation relation* (CAR) with infinite degrees of freedom, which defines an internal algebra, describes a framework of a *fermionic quantum field theory* (Arai, 2000). From these viewpoints, *supersymmetric quantum mechanics* and *supersymmetric quantum field theory* can be regarded as representations of CCR and CAR with additional secondary symmetries (Arai, 2000; Arai, 2006a). Thus *the fundamental symmetry principle applies to quantum field theory too, including supersymmetric one.*

Symbolically one can say : *quantum phenomena are representations of CCR(n) ($n = 1, \cdots, n$ or $n = \infty$) and internal algebras.*

From the viewpoint of the fundamental symmetry principle, the phenomenal multifoldness of a quantum particle mentioned in Section 2.1 can be understood as a natural consequence of the infinite variety of representation spaces of CCR and internal algebras.

2.3 Secondary representations of CCR

Given a self-adjoint representation $\Pi = (\mathcal{H}, \mathcal{D}, \{Q_j, P_j | j = 1, \cdots, n\})$ of CCR(n), a different representation of CCR(m) may be constructed from Π, where m is not necessarily equal to n. We call such a representation a *secondary representation* of CCR (if it exists) with respect to Π. This class of representations is also classified into equivalence classes. It is possible for a secondary representation of CCR to be inequivalent to the Schrödinger representation, giving a framework for a description of physically important quantum phenomena. Indeed, examples of such secondary representations of CCR appear in gauge quantum mechanics in two space dimensions in connection with the so-called *Aharonov-Bohm effect* (Arai, 1992). Properties of these representations have been studied in detail in a series of papers (Arai, 1995a, 1995b, 1996, 1998).

Examples of secondary representation of CCR(∞) in quantum field theory, which are inequivalent to the Fock space representation of CCR(∞), are discussed in Arai (1995c, 2001).

Another interesting example of secondary representation of CCR is a (not necessarily self-adjoint) representation $(\mathcal{H}, \mathcal{D}, (T, H))$ of CCR(1) with H being the Hamiltonian of the system under consideration so that

$$[T, H] = i\hbar$$

on $\mathcal{D}$. Such an operator T is called a *time operator* with respect to the Hamiltonian H (Miyamoto, 2001; Arai, 2005a). If H has a time operator T, then a *time-energy uncertainty relation* follows in the sense that

$$(\Delta T)_\Psi (\Delta H)_\Psi \geq \frac{\hbar}{2}, \quad \Psi \in \mathcal{D} \setminus \{0\}$$

(see Remark 4). This gives a rigorous formulation of the time-energy uncertainty relation[10]. Time operators are related to the usual notion of time in terms of transition probabilities in the "time-development" of the quantum system. For the details, see Miyamoto (2001) and Arai (2005a).

3 NEW PHILOSOPHICAL INTERPRETATION

Based on the physical and mathematical considerations given in Section 2, we now go to reading philosophical contents implied by quantum mechanics[11]. As is pointed out in the remarks on Axioms (QM.1) and (QM.2), even on the level of physical interpretations for quantum mechanics, one encounters with some difficulties. But we show that these difficulties are not real ones and naturally dissolve if one takes the metaphysical dimension of existence into account. As is well known, great Greek philosophers (Pythagoras, Plato, Aristotle, ...) and Oriental philosophers (Lao-tzu, Confucius, Buddha, ...) suggested profound worldviews which take not only the material-sensorial dimension of existence, but also the metaphysical dimension of existence, although these two dimensions are inseparable[12]. With these preliminary philosophical remarks, we present a new philosophical

[10]In many textbooks of quantum mechanics, the time-energy uncertainty relation is treated ambiguously.

[11]Needless to say, no philosophical interpretations may be needed from pragmatic or practical viewpoints. But sciences without philosophy are spiritually blind and even dangerous in connection with their applications in the material dimension as some histories have already shown.

[12]For example, Plato talks about IDEAs as the origins of all phenomenal (material-sensorial) things (we write "idea" in the sense of Plato as IDEA to distnguish it from the usual, every day concept of idea) . Obviously IDEAs are metaphysical. The mathematical IDEA world forms part of the metaphysical dimension. For Lao-tzu, a great saint in ancient China, the WAY (Tao) is the absolute metaphysical origin (Lao-tzu, 2001). But we should remark that it is extremely difficult to talk about beings or "things" in the metaphysical dimension in terms of the usual, ordinary words, because the ordinary words are primarily organised to be used in the material-sensorial, daily life dimension and, in this sense, the metaphysical dimension, which is non-spatial and non-temporal, is beyond the words (this is a reason why many people do not realise the reality of the metaphysical dimenison). But it may be possible, to some extent, to describe states in the metaphysical dimension by using metaphor, symbolic or poetic expressions. The description concerning the metaphysical dimension in the present paper should be understood in this sense.

interpretation for quantum mechanics as follows[13]. First we notice some insight which is obtained from an empirical fact on quantum particles, i.e., the phenomenal multifoldness of a quantum particle. This shows that a quantum particle is "something" which is not uniquely characterised in the physical dimension. For example, the following way of expression is possible: a quantum particle is neither a classical particle nor a classical wave . More generally, one can say that a quantum particle is neither A nor B, where A and B are arbitrary properties the quantum particle has in the physical dimension, according to the phenomenal multifoldness. Hence one is led to the viewpoint that a quantum particle must be a metaphysical being, at least, if one assumes an eternal "substance" from which quantum phenomena appear following the phenomenal multifoldness. As is realised immediately, this is exactly in accordance with unphysicality of quantum states and physical quantities pointed out in remark (R.1) on (QM.1) and (QM.2). Thus we conclude that quantum states as well as physical quantities are also metaphysical beings. Conversely speaking, empirical facts together with the axioms of quantum mechanics suggest the metaphysical dimension of existence. If one accepts the philosophical interpretations for (QM.1) and (QM.2) as stated in the preceding paragraph, then one is naturally led to interpret (QM.4) as follows: the "time-development" of a quantum state is *not* a *real* time-development, i.e., not a change in the physical-sensory dimension, because a quantum state is not a physical one (note that the metaphysical dimension is *non-spatial* and *non-temporal*, and hence the notion of the usual time-development loses meaning). Hence the so-called "measurement problem" with "reduction of wave function (quantum state)", which is one of the main controversies in the conventional interpretations of quantum mechanics, has no meaning. This "problem" just comes from a misunderstanding on the nature of quantum states and is out of the question in our interpretation. Then, what is the true meaning of (QM.4) ?

To answer this important question, we notice Theorem 1. It tells us that the Hamiltonian H generates, in terms of state curves , a metaphysical order in the Hilbert space of state vectors. Taking this order into account, we can assume that the real process of a quantum phenomenon proceeds as follows: a measurement of the system S at time t_0 chooses a state vector in the Hilbert space $\mathcal{H}$, say, Ψ. Then the state curve $\Psi(t) = e^{-iH(t-t_0)/\hbar}\Psi$ is assigned in the metaphysical dimension and, at the same time, this curve becomes a *reference curve* for future measurements made at time $t > t_0$ in such a way that the probability of finding (by a measurement) the system in a state Φ at time $t > t_0$ is

$$\frac{|\langle \Phi, \Psi(t) \rangle|^2}{\|\Phi\|^2 \|\Psi\|^2}.$$

The states do not change, which always exist "unmovedly" in the metaphysical dimen-

[13]The present author discussed, in Arai (2006b), some of the philosophical contents below in connection with the natural philosophy of Schelling and that of Goethe.

sion with the order mentioned above, but only the time of measurement changes in the physical dimension. This is also in accordance with impossibility of causal description of a quantum particle in the sense of classical mechanics. Moreover it is obvious that the non-locality (long range correlation of quantum phenomena) is a natural consequence of the metaphysicality of quantum states[14]. In this way one can arrive at a complete understanding on the nature of quantum phenomena.

4 PHILOSOPHICAL PHASES

Finally we briefly sketch philosophical phases of the interpretation presented in the preceding section.

In a previous paper (Arai, 2006b), we pointed out that the mathematical IDEA world has "graded" and "multiple" structures, being organised in such a way that all the constituent elements are mutually related in various manners. And these structures are in accordance with the ontology of Buddhism suggested in the Avatamsaka-sutra (Kegon-kyou in Japanese) and that of Ibnu'l-'Arabi (1165-1240), a great Islamic philosopher, where, as for the ontology of Buddhism and that of Ibnu'l-'Arabi, we follow Izutsu (1989) [15]. An important feature of the structures is the existence of IDEAs, called fundamental IDEAs (Arai, 2003), each of which is characterised by a set of axioms (e.g., semi-group, group, ring, field, vector space, topological space, algebra). In the philosophy of Zen Buddhism, the metaphysical origin is called the *Absolute Nothing* or the *Absolute Non-segment*, from which metaphysical beings appear first in the ontological structural sense (Izutsu, 1991). From this viewpoint, the fundamental mathematical IDEAs are some of the "nearest" ones to the Absolute Nothing, in other words, they each belong to some "rank" to which the Absolute Nothing "acts" most "directly". Then the mathematical IDEA world can be viewed as developments of the Absolute Nothing through the fundamental mathematical IDEAs. This is a "shape" of the mathematical IDEA world from a purely ideal or spiritual viewpoint. On the other hand, one can view the mathematical IDEA world in another way, namely, in relation to the generation of physical phenomena. Then the mathematical IDEA world shows different structures and orders[16]. In the latter

[14]The metaphysical dimension is "beyond" or "over" the physical dimenison and hence *non-spacial* and *non-temporal*.

[15]Toshihiko Izutsu (1914-1993) is a distinguished scholar of Islamic thought and Oriental philosophy. The memorial book (al-Din Ashtiyani et al., 1998) may be a good reference to know him as well as his great works.

[16]It is well known that, in the esoteric Buddhism, the structure of existence or the universe is represented by pictures called *mandalas*. Among them, there are distinguished two kinds of mandala, called *Kongokai-mandala* and *Taizokai-mandala* in Japanese, the both forming one. We remark that the two kinds of views for the mathematical IDEA world presented here exactly correspond to these two kinds of mandala: the Kongokai-mandala corresponds to the first view of the mathematical IDEA world and the Taizokai-

viewpoint, it is easily seen from the description made so far in the present paper that the fundamental IDEAs for quantum phenomena are CCR's and internal algebras. In this way one can see how the fundamental principles of quantum mechanics are harmonically positioned in the metaphysical dimension and get a unified philosophical viewpoint for quantum phenomena and their IDEAs.

REFERENCES

al-Din Ashtiyani, S.J., Matsubara, H., Iwami, T. and Matsumoto, A. (ed.) (1998) *Conciousness and Reality*: Iwanami-shoten, xv+472 pp.

Arai, A. (1992) Momentum operators with gauge potentials, local quantization of magnetic flux, and representation of canonical commutation relations, J. Math. Phys. 33, 3374-3378.

Arai, A. (1995a) Gauge theory on a non-simply connected domain and representations of canonical commutation relations, J. Math. Phys. 36, 2569-2580.

Arai, A. (1995b), Representation of canonical commutation relations in a gauge theory, the Aharonov-Bohm effect, and the Dirac-Weyl operator, J. Nonlinear Math. Phys. 2 , 247-262.

Arai, A. (1995c) A class of representations of the $*$-algebra of the canonical commutation relations over a Hilbert space and instability of embedded eigenvalues in quantum field models, J. Nonlinear Math. Phys. 2, 247-262.

Arai, A. (1996) Canonical commutation relations, the Weierstrass Zeta function and infinite dimensional Hilbert space representations of the quantum group $\mathrm{Uq}(\mathrm{sl}_2)$, J. Math. Phys. 37, 4203-4218.

Arai, A. (1997) *Hilbert Spaces and Quantum Mechanics*: Kyoritsu-shuppan, viii+276 pp (in Japanese).

Arai, A. (1998) Representation-theoretic aspects of two-dimensional quantum systems in singular vector potentials: canonical commutation relations, quantum algebras, and reduction to lattice quantum systems, J . Math. Phys. 39, 2476-2498.

Arai, A.(2000) *Fock Spaces and Quantum Fields*: Nippon-hyoronsha , xiv+636 pp (in Japanese).

mandala the second one. In this sense too, the philosophy in the present paper is in accordance with the ontological images of the esoteric Buddhism.

Arai, A.(2001) Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation, Rev. Math. Phys. 13, 1075-1094.

Arai, A. (2002) *Many Body Systems and Quantum Fields*: Iwanami-shoten, x+80 pp (in Japanese).

Arai, A. (2003), *Mathematical Principles of Physical Phenomena*: Kyoritsu-shuppan, vi+557 pp (in Japanese).

Arai, A. (2005a) Generalized weak Weyl relation and decay of quantum dynamics, Rev. Math. Phys. 17, 1071-1109.

Arai, A. (2005b), *Handbook of Modern Mathematical Physics*: Asakura-shoten, xxiv+707 pp (in Japanese).

Arai, A. (2006a), *Mathematical Principles of Quantum Phenomena*: Asakura-shoten, x+535 pp (in Japanese).

Arai, A. (2006b) On the ontological structures revealed by quantum phenomena (in Japanese with a German abstract), to be published in Schelling-Jahrbuch Bd.14, Schelling-Gesellschaft Japan.

Arai, A. and Ezawa, H.(1999) *Mathematical Structures of Quantum Mechanics I, II*: Asakura-shoten, ix+476 pp+71 pp (in Japanese).

Bohr, N.(1948) On the notions of causality and complementarity: Dialectica, Vol 2., p.312; Science, Vol. 111, 1950, p.51.

Izutsu, T. (1989) *Cosmos and Anti-Cosmos — For Oriental Philosophy*: Iwanami-shoten (in Japanese).

Izutsu, T. (1991) *Consciousness and Essence—Searching for a Structural Coincidence of Oriental Philosophies*: Iwanami-shoten (in Japanese).

Lao-tsu (2001) *The Way and Its Virtue* (translated and annotated by T. Izutsu): The Izutsu Library Series on Oriental Philosophy 1, Keiou University Press.

Miyamoto, M. (2001) A generalized Weyl relation approach to the time operator and its connection to survival probability, J. Math. Phys. 42, 1038-1052.

Von Neumann, J. (1932) *Die Mathematische Grundlagen der Quantenmechanik*: Springer.

Von Neumann, J.(1931) Die Eindeutigkeit der Schrödingerschen Operatoren , Math. Ann. 104, 570-578.

Peres, A.(1993) *Quantum Theory: Concepts and Methods*: Kluwer Academic Publishers.

Reed, M. and Simon, B.(1972) *Methods of Modern Mathematical Physics Vol.I: Functional Analysis*: Academic Press.

Weidmann, J.(1980) *Linear Operators in Hilbert Spaces*: Springer.