Spectral Analysis of a Dirac Operator with a Meromorphic Potential

Asao Arai*and Kunimitsu Hayashi
Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

Abstract

We consider an operator $Q(V)$ of Dirac type with a meromorphic potential given in terms of a function V of the form $V(z) = \lambda V_1(z) + \mu V_2(z),\ z \in \mathbb{C} \setminus \{0\}$, where V_1 is a complex polynomial of $1/z$, V_2 is a polynomial of z, and λ and μ are non-zero complex parameters. The operator $Q(V)$ acts in the Hilbert space $L^2(\mathbb{R}^2; \mathbb{C}^4) = \oplus^4 L^2(\mathbb{R}^2)$. The main results we prove include: (i) the (essential) self-adjointness of $Q(V)$; (ii) the pure discreteness of the spectrum of $Q(V)$; (iii) if $V_1(z) = z^{-p}$ and $4 \le \deg V_2 \le p+2$, then $\ker Q(V) \ne \{0\}$ and $\dim\ker Q(V)$ is independent of (λ, μ) and lower order terms of $\partial V_2/\partial z$; (iv) a trace formula for $\dim\ker Q(V)$.

1 Introduction and Main Results

In this paper we consider an operator of Dirac type with a meromorphic potential and analyze spectral properties of it. To define the operator and state the main results, we first need to introduce some basic objects.

As usual, we denote by $\sigma_1, \sigma_2, \sigma_3$ the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.1}$$

which give a self-adjoint representation of a Clifford algebra:

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}, \quad j,k = 1,2,3, \tag{1.2}$$

with $\{X, Y\} := XY + YX$. We also define the following 2×2 matrices:

$$p_+ := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1+\sigma_3), \quad p_- := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(1-\sigma_3). \tag{1.3}$$

*Corresponding author: arai@math.sci.hokudai.ac.jp

We write $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$. We denote by D_x and D_y the generalized partial differential operators in the variables x and y respectively acting in $L^2(\mathbb{R}^2) = L^2(\mathbb{C})$, the Hilbert space of square integrable functions on $\mathbb{R}^2$ with respect to the two-dimensional Lebesgue measure, and introduce

$$\partial := \frac{1}{2}(D_x - iD_y), \quad \bar{\partial} := \frac{1}{2}(D_x + iD_y). \tag{1.4}$$

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. For two linear operators S and T on a Hilbert space, $D(T + S) := D(T) \cap D(S)$ unless otherwise stated.

The meromorphic potential of the Dirac operator we consider is given in terms of a function of the following form:

$$V(z) = \lambda V_1(z) + \mu V_2(z), \quad z \in \mathbb{C}^\times := \mathbb{C} \setminus \{0\}, \tag{1.5}$$

where V_1 is a complex polynomial of $1/z$, V_2 is a polynomial of z, and λ and μ are non-zero complex parameters, called the *coupling constants*.

In general, for a densely defined linear operator T on a Hilbert space, T^* denotes the adjoint of T.

We introduce two Dirac-Weyl type operators acting in $L^2(\mathbb{R}^2; \mathbb{C}^2) = \oplus^2 L^2(\mathbb{R}^2)$ by

$$\begin{aligned} Q_-(V) &:= -\frac{1}{2}\sigma_1 D_x - \frac{1}{2}\sigma_2 D_y + ip_+\partial V - ip_-(\partial V)^* \\ &= -\frac{1}{2}(\sigma_1 + i\sigma_2)\partial - \frac{1}{2}(\sigma_1 - i\sigma_2)\bar{\partial} + ip_+(\partial V) - ip_-(\partial V)^*, \end{aligned} \tag{1.6}$$

$$Q_+(V) := \frac{1}{2}(\sigma_1 - i\sigma_2)\bar{\partial} + \frac{1}{2}(\sigma_1 + i\sigma_2)\partial - ip_+(\partial V)^* + ip_-(\partial V). \tag{1.7}$$

Note that

$$D(Q_-(V)) = D(Q_+(V)) = D(D_x) \cap D(D_y) \cap D(\partial V) \tag{1.8}$$

and hence

$$D(Q_\pm(V)) \supset C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2), \tag{1.9}$$

the space of $\mathbb{C}^2$-valued infinitely differentiable functions on $\mathbb{R}^2 \setminus \{0\}$ with compact support in $\mathbb{R}^2 \setminus \{0\}$. Therefore $Q_+(V)$ and $Q_-(V)$ are densely defined. Moreover it is easy to see that

$$Q_-(V) \subset Q_+(V)^*, \quad Q_+(V) \subset Q_-(V)^*. \tag{1.10}$$

Hence $Q_\pm(V)$ are closable.

The Dirac operator we consider in the present paper is defined by

$$Q(V) := \begin{pmatrix} 0 & Q_-(V) \\ Q_+(V) & 0 \end{pmatrix} \tag{1.11}$$

$$= i\psi_2\bar{\partial} + i\psi_2^*\partial + i\psi_1(\partial V) - i\psi_1^*(\partial V)^*, \tag{1.12}$$

acting in the Hilbert space

$$L^2(\mathbb{R}^2; \mathbb{C}^2) \oplus L^2(\mathbb{R}^2; \mathbb{C}^2) = L^2(\mathbb{R}^2; \mathbb{C}^4) = \oplus^4 L^2(\mathbb{R}^2), \tag{1.13}$$

where

$$\psi_1 := \frac{1}{2}\begin{pmatrix} 0 & 1+\sigma_3 \\ 1-\sigma_3 & 0 \end{pmatrix}, \quad \psi_2 = \frac{1}{2}\begin{pmatrix} 0 & i\sigma_1+\sigma_2 \\ -i\sigma_1-\sigma_2 & 0 \end{pmatrix}. \tag{1.14}$$

Note that

$$\{\psi_j, \psi_k\} = 0, \quad j,k = 1,2. \tag{1.15}$$

It follows from (1.9) and (1.10) that $Q(V)$ is a symmetric operator with

$$D(Q(V)) \supset C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4). \tag{1.16}$$

Remark 1.1 In practice, in analysis of $Q_\pm(V)$ and $Q(V)$, one works with only ∂V. Hence, from a purely mathematical point of view, it may be more suitable to define $Q_\pm(V)$ and $Q(V)$ with ∂V replaced by a meromorphic function W. But, in this respect, we follow the notations used in early works [6, 1, 2, 3].

In what follows, for a closable operator T, $\overline{T}$ denotes the closure of T. For a polynomial P, we denote the degree of P by $\deg P$. For a Fredholm operator T, we set

$$\mathrm{index}(T) := \dim\ker T - \dim\ker T^*, \tag{1.17}$$

the index of F, where, for a linear operator A, $\ker A := \{\psi \in D(A) | A\psi = 0\}$.

The Dirac operator $Q(V)$ originally comes from supersymmetric quantum mechanics (SSQM) — it is called a supercharge of the $N = 2$ Wess-Zumino model — and the following facts are known:

(i) The case $V_1 = 0$ and $\deg V_2 \geq 2$ [6, 2]. In this case, $Q(V)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2; \mathbb{C}^4)$, $\overline{Q_\pm(V)}$ are Fredholm and $Q_-(V)^* = \overline{Q_+(V)}$ with

$$\ker \overline{Q_-(V)} = \{0\},$$
$$\mathrm{index}(\overline{Q_+(V)}) = \dim\ker \overline{Q_+(V)} = \deg V_2 - 1.$$

Moreover the spectrum of $\overline{Q(V)}$ is purely discrete.

(ii) The case $V_1(z) = z^{-p}$ ($p \in \mathbb{N}$) and $V_2 = 0$ [3]. In this case, $Q(V)$ is essentially self-adjoint on on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4)$ and $Q_-(V)^* = \overline{Q_+(V)}$ with

$$\ker \overline{Q_-(V)} = \{0\}, \quad \dim\ker \overline{Q_+(V)} = p - 1.$$

But $\overline{Q_\pm(V)}$ are *not* Fredholm.

As a next step of mathematical research, it is interesting to investigate the case where $V_1 \neq 0$ *and* $V_2 \neq 0$. This is one of the basic motivations of this work.

For a self-adjoint operator S, we denote by $\sigma(S)$ and $\sigma_{\mathrm{d}}(S)$ the spectrum and the discrete spectrum of S respectively ($\sigma_{\mathrm{d}}(S)$ is the set of isolated eigenvalues of S with finite multiplicity). We say that the spectrum of S is purely discrete if $\sigma(S) = \sigma_{\mathrm{d}}(S)$.

For the function V_1, we define $\deg V_1$ by

$$\deg V_1 := p \tag{1.18}$$

if $V_1(z) = \sum_{j=0}^p a_j z^{-j}$ with $a_p \neq 0$ ($a_j \in \mathbb{C},\ j = 0, \cdots, p$).

We now state the main results. We work with the following hypothesis:

Hypothesis (A) $\deg V_1 \geq 1$ and $\deg V_2 \geq 2$.

Note that, under this hypothesis, the matrix valued potential $i\psi_1(\partial V) - i\psi_1^*(\partial V)^*$ of the Dirac operator $Q(V)$ has a pole at $z = 0$ with order $\deg V_1 + 1 \geq 2$. This makes the mathematical analysis of $Q(V)$ more difficult than the regular case $V_1 = 0$.

Theorem 1.1 *Assume (A). Then:*

(i) *The operator $Q(V)$ is self-adjoint and $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4)$ is a core of $Q(V)$.*

(ii) *The operators $Q_\pm(V)$ are closed and*

$$Q_+(V)^* = Q_-(V). \tag{1.19}$$

Theorem 1.2 *Assume (A). Then:*

(i) *The spectrum of $Q(V)$ is symmetric with respect to the origin, i.e., $\lambda \in \sigma(Q(V)) \iff -\lambda \in \sigma(Q(V))$.*

(ii)

$$\ker Q_-(V) = \{0\}. \tag{1.20}$$

(iii) *The spectrum of $Q(V)$ is purely discrete.*

(iv) *The operators $Q_\pm(V)$ are Fredholm.*

For the kernel of $Q_+(V)$, we have the following theorem, which we regard as one of the most important results in the present paper.

Theorem 1.3 *Let $P(z)$ be a complex polynomial of $z \in \mathbb{C}$ with $\deg P \leq q - 1$ ($q \in \mathbb{N}$) and*

$$U(z) := \frac{\lambda}{z^p} + \mu z^q + P(z), \quad z \in \mathbb{C}^\times. \tag{1.21}$$

Let $4 \leq q \leq p + 2$. Then $\ker Q(U) \neq \{0\}$ and $\dim \ker Q(U)$ is independent of (λ, μ) and P.

Remark 1.2 This theorem implies that $\dim \ker Q(U)$ is determined by p and q only. But we have been unable to find an explicit formula for $\dim \ker Q(U)$.

Remark 1.3 By Theorem 1.2-(ii), we have

$$\ker Q(V) = \ker Q_+(V) \oplus \{0\}. \tag{1.22}$$

Hence

$$\dim \ker Q(V) = \dim \ker Q_+(V). \tag{1.23}$$

The basic methods we use to analyze the operator $Q(V)$ is to consider its square $Q(V)^2$ as was done in [6, 2, 3] (see also [10, Chapter 5]). This leads us to introduce the following operators:

$$H_-(V) := -\frac{1}{4}\Delta + |\partial V|^2, \tag{1.24}$$
$$H_+(V) := H_-(V) + H_I(V), \tag{1.25}$$

both acting in $L^2(\mathbb{R}^2; \mathbb{C}^2)$, where

$$\Delta := D_x^2 + D_y^2, \tag{1.26}$$

the two-dimensional generalized Laplacian, and

$$H_I(V) := \begin{pmatrix} 0 & -i\partial^2 V \\ i(\partial^2 V)^* & 0 \end{pmatrix}. \tag{1.27}$$

We remark that the definition of $H_+(V)$ (resp. $H_-(V)$) here is different form that in [3], although it turns out that they coincide (see Theorem 2.5-(ii) and Theorem 2.6-(ii) in Section 2). We also note that the potential $|\partial V|^2$ (resp. $H_I(V)$) has a pole at $z = 0$ with order $2(\deg V_1 + 1) \geq 4$ (resp. $\deg V_1 + 2$). Hence $H_\pm(V)$ are two-dimensional Schrödinger operators with strongly singular (matrix-valued) potentials.

We set

$$H(V) := H_+(V) \oplus H_-(V) = \begin{pmatrix} H_+(V) & 0 \\ 0 & H_-(V) \end{pmatrix}. \tag{1.28}$$

Remark 1.4 In the context of SSQM, $H(V)$ turns out to be the supersymmetric Hamiltonian of the $N = 2$ Wess-Zumino model (Theorem 1.4-(iii) below) and $H_+(V)$ (resp. $H_-(V)$) is called the bosonic (resp. fermionic) part of $H(V)$.

Theorem 1.4 *Assume (A). Then:*

(i) *The operators $H_-(V)$ and $H_+(V)$ are nonnegative and self-adjoint. Moreover they are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$.*

(ii) *The operator $H(V)$ is nonnegative and self-adjoint. Moreover it is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4)$.*

(iii) *The following operator equality holds:*

$$Q(V)^2 = H(V). \tag{1.29}$$

Theorem 1.5 *Assume (A). Then:*

(i) $\ker H_-(V) = \{0\}$.

(ii) *The operators $H_\pm(V)$ have compact resolvent. In particular, the spectra of $H_\pm(V)$ are purely discrete and*

$$\sigma(H_+(V)) \setminus \{0\} = \sigma(H_-(V)) \setminus \{0\}. \tag{1.30}$$

Theorem 1.6 *Let U be defined by (1.21) and $4 \le q \le p+2$. Then* $\ker H_+(U) \ne \{0\}$ *and* $\dim\ker H_+(U)$ *is independent of* (λ,μ) *and* P.

In addition to the theorems stated above, we prove a trace formula for $\dim\ker Q(V)$ in terms of $H_\pm(V)$ (see Theorem 6.1 in Section 6).

The present paper is organized as follows. In Section 2 we discuss the (essential) self-adjointness problem of $H_\pm(V)$ and $Q(V)$. We prove Theorem 1.1 and Theorem 1.4. Section 3 is concerned with spectral properties of $H_\pm(V)$ and $Q(V)$. We prove Theorem 1.2 and Theorem 1.5. In Section 4 we prove the continuity of the ground state energy of $H_\pm(V)$ (the infimum of the spectrum of $H_\pm(V)$) in the counpling constants λ and μ (Theorem 4.2). As a result, it is shown that $\dim\ker H_+(V)$ $(=\dim\ker Q_+(V))$ is a constant independent of (λ,μ) (Theorem 4.3). These results also may be regarded as part of the main results of the present paper. Section 5 is devoted to proofs of Theorem 1.3 and Theorem 1.6. In the last section we establish the trace formula mentioned above.

2 Self-adjointness

In this section we prove Theorems 1.1 and 1.4.

Throughout this section, we assume (A) and write simply

$$Q = Q(V),\ Q_\pm = Q_\pm(V),\ H = H(V),\ H_\pm = H_\pm(V),\ H_I = H_I(V).$$

We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm of $L^2(\mathbb{R}^2;\mathbb{C}^r)$ respectively $(r\in\mathbb{N})$:

$$\langle\Psi,\Phi\rangle := \int_{\mathbb{R}^2}\langle\Psi(\boldsymbol{r}),\Phi(\boldsymbol{r})\rangle_{\mathbb{C}^r}d\boldsymbol{r},\quad (\boldsymbol{r}=(x,y)\in\mathbb{R}^2) \tag{2.1}$$

$$\|\Psi\| := \sqrt{\langle\Psi,\Psi\rangle},\quad \Psi,\Phi\in L^2(\mathbb{R}^2;\mathbb{C}^r), \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{C}^r}$ is the standard inner product of $\mathbb{C}^r$.

Lemma 2.1 *For all* $\Psi\in C_0^\infty(\mathbb{R}^2\setminus\{0\};\mathbb{C}^2)$,

$$\frac{1}{4}\|\Delta\Psi\|^2 + \|(\partial V)^2\Psi\|^2 \le \|H_-\Psi\|^2 + \|(\partial^2 V)\Psi\|^2. \tag{2.3}$$

Proof. This fact is stated in [3, Lemma 2.7]. For the reader's convenience, we give a proof (cf. the proof of [1, Lemma 2.7]). It is obvious that

$$C_0^\infty(\mathbb{R}^2\setminus\{0\};\mathbb{C}^2)\subset D(H_\pm). \tag{2.4}$$

We have

$$\partial\bar\partial\subset\frac{1}{4}\Delta,\quad \bar\partial\partial\subset\frac{1}{4}\Delta$$

and

$$\partial\bar\partial\Psi = \bar\partial\partial\Psi = \frac{1}{4}\Delta\Psi,\quad \Psi\in C_0^\infty(\mathbb{R}^2\setminus\{0\};\mathbb{C}^2).$$

Let $\Psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$. Then

$$\|H_-\Psi\|^2 = \frac{1}{4}\|\Delta\Psi\|^2 + \|(\partial V)^2\Psi\|^2 - C_V$$

with

$$C_V := \Re\langle\bar{\partial}\partial\Psi, (\partial V)^*(\partial V)\Psi\rangle + \Re\langle\partial\bar{\partial}\Psi, (\partial V)^*(\partial V)\Psi\rangle.$$

By integration by parts, we see that

$$C_V = \|(\partial^2 V)\Psi\|^2 - \|(\partial V)\partial\Psi\|^2 - \|(\partial V)\bar{\partial}\Psi\|^2.$$

Hence

$$\|H_-\Psi\|^2 = \frac{1}{4}\|\Delta\Psi\|^2 + \|(\partial V)^2\Psi\|^2 - \|(\partial^2 V)\Psi\|^2 + \|(\partial V)\partial\Psi\|^2 + \|(\partial V)\bar{\partial}\Psi\|^2. \qquad (2.5)$$

Hence (2.3) follows. ■

Lemma 2.2 *For every $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that*

$$|\partial^2 V(z)|^2 \le \varepsilon|\partial V(z)|^4 + b_\varepsilon, \quad z \in \mathbb{C} \setminus \{0\}. \qquad (2.6)$$

Proof. See [3, Lemma 2.8]. ■

Lemma 2.3 *Let A be a densely defined closable linear operator from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$. Suppose that there exists a dense subspace D of $\mathcal{H}$ such that $D \subset D(A^*A)$ and A^*A is essentially self-adjoint on D. Then*

$$\overline{A^*A} = \bar{A}^*\bar{A}. \qquad (2.7)$$

Proof. Let $S = \overline{A^*A}$. For all $\psi \in D(S)$, there exists a sequence $\{\psi_n\}_n$ with $\psi_n \in D$ $(n \in \mathbb{N})$ such that $\psi_n \to \psi$, $S\psi_n \to S\psi$ $(n \to \infty)$. Since $S\psi_n = A^*A\psi_n$, it follows that $\{A\psi_n\}_n$ is a Cauchy sequence in $\mathcal{K}$. Hence $\psi \in D(\bar{A})$ and $A\psi_n \to \bar{A}\psi$ $(n \to \infty)$. Since A^* is closed, it follows that $\bar{A}\psi \in D(A^*)$ and $A^*\bar{A}\psi = S\psi$. This means that $S \subset A^*\bar{A} = \bar{A}^*\bar{A}$. By the von Neumann theorem ([8, Theorem X.25]), $\bar{A}^*\bar{A}$ is self-adjoint. Thus (2.7) follows. ■

Lemma 2.4 *Let A and B be closed linear operators from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ such that $A + B$ is closable. Suppose that there exist a core D of $\overline{A+B}$ and closed linear operators A', B' from $\mathcal{H}$ to $\mathcal{K}$ such that $D \subset D(A) \cap D(B)$, $D(A) = D(A')$, $D(B) = D(B')$ and*

$$\|A'\psi\| + \|B'\psi\| \le C(\|(A+B)\psi\| + \|\psi\|), \quad \psi \in D, \qquad (2.8)$$

where $C > 0$ is a constant. Then $A + B$ is closed and

$$\|A'\psi\| + \|B'\psi\| \le C(\|(A+B)\psi\| + \|\psi\|), \quad \psi \in D(A+B). \qquad (2.9)$$

Proof. Let $\psi \in D(\overline{A+B})$. Then there exists a sequence $\{\psi_n\}_n$ with $\psi_n \in D$ such that $\psi_n \to \psi$ and $(A+B)\psi_n \to (\overline{A+B})\psi$ $(n \to \infty)$. By (2.8), $\{A'\psi_n\}_n$ and $\{B'\psi_n\}_n$ are Cauchy sequences. Since A' and B' are closed, it follows that $\psi \in D(A') \cap D(B') = D(A) \cap D(B) = D(A+B)$ and $A'\psi_n \to A'\psi$, $B'\psi_n \to B'\psi$ $(n \to \infty)$. Hence $D(\overline{A+B}) \subset D(A+B)$ and (2.9) holds. In particular, $\overline{A+B} = A+B$. Thus $A+B$ is closed. ■

Theorem 2.5

(i) *The operator H_- is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$ and H_- is nonnegative, self-adjoint with*

$$(1-\varepsilon)\|(\partial V)^2\Psi\|^2 + \|\frac{1}{4}\Delta\Psi\|^2 \le \|H_-\Psi\|^2 + b_\varepsilon\|\Psi\|^2, \quad \Psi \in D(H_-), \tag{2.10}$$

where $\varepsilon \in (0,1)$ is an arbitrary constant and $b_\varepsilon > 0$ is a constant depending on ε.

(ii) *Operator equality*

$$H_- = \overline{Q}_-{}^* \overline{Q}_- \tag{2.11}$$

holds.

(iii) $\overline{Q}_- = Q_-$*, i.e., Q_- is closed.*

(iv) *For all $\Psi \in D(Q_-)$,*

$$\|Q_-\Psi\|^2 = \frac{1}{4}\|(-\Delta)^{1/2}\Psi\|^2 + \|(\partial V)\Psi\|^2. \tag{2.12}$$

Proof. (i) The essential self-adjointness of H_- on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$ is already proved in [3, Theorem 2.12]. The nonnegativity of $\overline{H}_-$ is obvious. Thus we need only to show the self-adjointness of H_- with $D(H_-) = D(\Delta) \cap D(|\partial V|^2)$ (this was not proven in [3]). By (2.3) and (2.6), we see that (2.10) holds for all $\Psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$. Hence, by applying Lemma 2.4 with $A = -\Delta/4$ and $B = |\partial V|^2$, we conclude that H_- is closed and (2.10) holds. Hence H_- is self-adjoint.

(ii) By direct computations, we have

$$H_-\Psi = Q_-^* Q_-\Psi, \quad \Psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2). \tag{2.13}$$

Hence, applying Lemma 2.3, we obtain (2.11).

(iii) Since we have established the self-adjointness and the nonnegativity of H_-, the operator $H_-^{1/2}$ exists. By definition (1.24) of H_- and a limiting argument, one can show that $D(H_-^{1/2}) \subset D((-\Delta)^{1/2}) \cap D(\partial V)$ and

$$\|H_-^{1/2}\Psi\|^2 = \frac{1}{4}\|(-\Delta)^{1/2}\Psi\|^2 + \|(\partial V)\Psi\|^2, \quad \Psi \in D(H_-^{1/2}). \tag{2.14}$$

On the other hand, (2.11) implies that $D(H_-^{1/2}) = D(\overline{Q}_-)$ and

$$\|H_-^{1/2}\Psi\|^2 = \|\overline{Q}_-\Psi\|^2, \quad \Psi \in D(H_-^{1/2}). \tag{2.15}$$

Hence, in particular, it follows that $D(\overline{Q}_-) \subset D((-\Delta)^{1/2}) \cap D(\partial V)$. Note that the right hand side is equal to $D(Q_-)$. Thus $D(\overline{Q}_-) \subset D(Q_-)$, which means that $\overline{Q}_- = Q_-$.

(iv) This follows from (2.14), (2.15) and part (iii). ■

Theorem 2.6

(i) *The operator H_+ is self-adjoint and essentially self-adjoint on every core of H_-. In particuler, H_+ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$.*

(ii) *Operator equality*

$$H_+ = \overline{Q}_+^* \overline{Q}_+ \tag{2.16}$$

holds. In particular, H_+ is nonnegative.

(iii) $\overline{Q}_+ = Q_+$, *i.e., Q_+ is closed.*

(iv) *For all $\Psi = (\Psi_1, \Psi_2) \in D(Q_+)$,*

$$\|Q_+\Psi\|^2 = \frac{1}{4}\|(-\Delta)^{1/2}\Psi\|^2 + \|(\partial V)\Psi\|^2 + q_{H_I}(\Psi), \tag{2.17}$$

where

$$q_{H_I}(\Psi) := \int_{\mathbb{R}^2} \left\{\Psi_1(\boldsymbol{r})^*(-i)(\partial^2 V)(z)\Psi_2(\boldsymbol{r}) + \Psi_2(\boldsymbol{r})^* i(\partial^2 V)(z)^*\Psi_1(\boldsymbol{r})\right\} d\boldsymbol{r}. \tag{2.18}$$

Proof. (i) Let H_I be defined by (1.27). Then we have by (2.6)

$$\|H_I\Psi\|^2 \le \varepsilon\|(\partial V)^2\Psi\|^2 + b_\varepsilon\|\Psi\|^2, \quad \Psi \in D((\partial V)^2),$$

where $\varepsilon \in (0,1)$ is arbitrary and $b_\varepsilon > 0$ is a constant. This inequality and (2.10) imply that

$$\|H_I\Psi\|^2 \le \frac{\varepsilon}{1-\varepsilon}\|H_-\Psi\|^2 + c_\varepsilon\|\Psi\|^2, \quad \Psi \in D(H_-), \tag{2.19}$$

where $c_\varepsilon > 0$ is a constant. Hence H_I is infinitesimally small with respect to H_-. Therefore, by the Kato-Rellich theorem [8, Theorem X.12], $H_+ = H_- + H_I$ is self-adjoint with $D(H_+) = D(H_-)$ and bounded from below. Moreover it is essentially self-adjoint on every core of H_-.

(ii) It is straightforward to see that

$$H_+\Psi = Q_+^* Q_+ \Psi, \quad \Psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2).$$

Hence a simple application of Lemma 2.3 yields (2.16).

(iii) By a limiting argument and definition (1.25) of H_+, we can show that $D(H_+^{1/2}) = D(H_-^{1/2})$ and

$$\|H_+^{1/2}\Psi\|^2 = \|H_-^{1/2}\Psi\|^2 + q_{H_I}(\Psi), \quad \Psi \in D(H_+^{1/2}) = D(H_-^{1/2}). \tag{2.20}$$

By (2.16), we have $D(H_+^{1/2}) = D(\overline{Q}_+)$. Since $D(H_-^{1/2}) = D((-\Delta)^{1/2}) \cap D(\partial V) = D(Q_+)$, it follows that $D(\overline{Q}_+) = D(Q_+)$. This means that $\overline{Q}_+ = Q_+$.

(iv) This follows from (2.14), (2.20) and part (iii). ■

Corollary 2.7 *The operator H defined by (1.28) is a nonnegative self-adjoint operator. It is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4)$.*

Lemma 2.8 *Let S be a symmeric operator on a Hilbert space $\mathcal{K}$ and D be a dense subspace of $\mathcal{K}$ such that $D \subset D(S^2)$ and S^2 is essentially self-adjoint on D. Then S is essentially self-adjoint on D and its closure is essentially self-adjoint on all other cores of $\overline{S^2}$.*

Proof. Let $N := \overline{S^2} + 1$. Then $N \geq 1$ and $\|S\psi\| \leq \|N\psi\|$, $\psi \in D$. Obviously $|\langle S\psi, N\psi\rangle - \langle N\psi, S\psi\rangle| = 0$, $\psi \in D$. Hence, by a simple application of Glimm-Jaffe-Nelson type commutator theorem [8, Theorem X.37], we see that S is essentially self-adjoint on D ans its closure is essentially self-adjoint on all other cores for N. ■

Theorem 2.9

(i) *The operator Q is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4)$ and every core of H is a core of $\bar{Q}$.*

(ii) *The operator Q is self-adjoint.*

(iii) *Operator equalities (1.19) and (1.29) hold.*

Proof. (i) It is obvious that $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4) \subset D(Q^2)$. By direct computations, we have

$$H\Psi = Q^2\Psi, \ \Psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^4). \tag{2.21}$$

Hence, by Lemma 2.8, the assertion of part (i) follows.

(ii) By the closedness of $Q_\pm$ (Theorem 2.5-(iii) and Theorem 2.6-(iii)), Q is closed. This fact and part (i) imply that Q is self-adjoint.

(iii) Operator equality (1.19) follows from the self-adjointness of Q. Since we have established the self-adjointness of Q, a simple application of Lemma 2.3 gives (1.29). ■

In summary we have proved Theorem 1.1 and Theorem 1.4.

3 Spectral Properties

Throughout this section, we write simply $Q = Q(V)$, $Q_\pm = Q_\pm(V)$, $H_\pm = H_\pm(V)$.

3.1 Proof of Theorem 1.2-(i)

We have the following orthogonal decomposition:

$$L^2(\mathbb{R}^2; \mathbb{C}^4) = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{3.1}$$

with

$$\mathcal{H}_+ := \left\{ \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix} \middle| f, g \in L^2(\mathbb{R}^2) \right\} \cong L^2(\mathbb{R}^2; \mathbb{C}^2), \tag{3.2}$$

$$\mathcal{H}_- := \left\{ \begin{pmatrix} 0 \\ 0 \\ f \\ g \end{pmatrix} \middle| f, g \in L^2(\mathbb{R}^2) \right\} \cong L^2(\mathbb{R}^2; \mathbb{C}^2). \tag{3.3}$$

We denote the orthogonal projections onto $\mathcal{H}_\pm$ by $\Gamma_\pm$ respectively. Let

$$\Gamma := \Gamma_+ - \Gamma_-. \tag{3.4}$$

Then Γ is self-adjoint and unitary:

$$\Gamma^* = \Gamma, \quad \Gamma^2 = I, \tag{3.5}$$

where I denotes identity.

Lemma 3.1 *For all $\Psi \in D(Q)$, $\Gamma\Psi \in D(Q)$ and*

$$Q\Gamma\Psi + \Gamma Q\Psi = 0. \tag{3.6}$$

Proof. For all $\Psi = (\Psi_+, \Psi_-) \in \mathcal{H}_+ \oplus \mathcal{H}_-$, we have $\Gamma\Psi = (\Psi_+, -\Psi_-)$. Hence, if $\Psi \in D(Q) = D(Q_+) \oplus D(Q_-)$, then $\Gamma\Psi \in D(Q)$. By direct computations, we have $Q\Gamma\Psi = -\Gamma Q\Psi$, $\Psi \in D(Q)$. ■

By Lemma 3.1, we have operator equality $\Gamma Q \Gamma^{-1} = -Q$. It follows from the unitary invariance of spectrum that $\sigma(Q) = \sigma(-Q)$, which implies Theorem 1.2-(i).

3.2 Proofs of Theorem 1.2-(ii) and Theorem 1.5-(i)

Let $\Psi \in \ker H_-$. Then $H_-\Psi = 0$. Then, by (2.10), $\Psi \in \ker \Delta = \{0\}$. Hence $\Psi = 0$. Thus Theorem 1.5-(i) follows. This fact and (2.11) imply that $\ker Q_- = \{0\}$. Hence Theorem 1.2-(ii) is proved.

3.3 Proof of Theorem 1.5-(ii)

Lemma 3.2 *Let W be a Lebesgue measurable function on $\mathbb{R}^n$ ($n \in \mathbb{N}$). Suppose that W is bounded from below with $W(x) \to \infty$ as $|x| \to \infty$ ($x \in \mathbb{R}^n$) and $D(|W|^{1/2}) \cap D((-\Delta)^{1/2})$ is dense in $L^2(\mathbb{R}^n)$ (Δ is the n-dimensional generalized Laplacian). Then $H_W := -\Delta + W$ defined as a sum of quadratic forms is a self-adjoint operator with compact resolvent. In particular, H_W has purely discrete spectrum and a complete set of eigenfunctions.*

Proof. Similar to the proof of [9, Theorem XIII] (In this theorem, it is assumed that $W \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. But this condition is not necessary. In this respect, it is sufficient to assume the denseness of $D(|W|^{1/2}) \cap D((-\Delta)^{1/2})$ which ensures the existence of the self-adjoint operator H_W defined as a sum of quadratic forms). ■

We write

$$H_- = L \oplus L \tag{3.7}$$

with

$$L = -\frac{1}{4}\Delta + |\partial V|^2 \tag{3.8}$$

acting in $L^2(\mathbb{R}^2)$.

Lemma 3.3 *The operator L has compact resolvent. In particular, L has purely discrete spectrum and a complete set of eigenfunctions.*

Proof. It is easy to see that $|\partial V|^2 \to \infty$ as $|z| \to \infty$. Hence we can apply Lemma 3.2 to obtain the desired result. ■

By Lemma 3.3 and (3.7), H_- has compact resolvent. In particular it has purely discrete spectrum. Since H_I is infinitesimally small with respect to H_- as already shown, it follows from a general theorem [8, Theorem XIII.68] that H_+ has compact resolovent. Thus the pure discreteness of spectra of $H_\pm$ is proven.

The following fact (*spectral supersymmetry*) is well known ([4] and [10, Corollary 5.6]):

Lemma 3.4 *For every densely defined closed linear operator C from a Hilbert space to a Hilbert space,*

$$\sigma(C^*C) \setminus \{0\} = \sigma(CC^*) \setminus \{0\}, \tag{3.9}$$

$$\sigma_{\rm p}(C^*C) \setminus \{0\} = \sigma_{\rm p}(CC^*) \setminus \{0\}, \tag{3.10}$$

where $\sigma_{\rm p}(T)$ denotes the point spectrum of an operator T (the set of all eigenvalues of T) and the multiplicity of $\lambda \in \sigma_{\rm p}(C^*C) \setminus \{0\}$ is equal to that of $\lambda \in \sigma_{\rm p}(CC^*) \setminus \{0\}$.

The spectral property (1.30) follows from an application of Lemma 3.4 with (1.19), (2.11) and (2.16).

3.4 Proof of Theorem 1.2-(iii)

This follows from Therem 1.4-(iii) and the functional calculus.

3.5 Proof of Theorem 1.2-(iv)

We have already seen that $Q_\pm$ are densely defined and closed. From the definition of Q, we have

$$\ker Q = \ker Q_+ \oplus \ker Q_-.$$

By Theorem 1.2-(iii), $\dim\ker Q < \infty$. Hence $\dim\ker Q_-^* = \dim\ker Q_+ < \infty$ and $\dim\ker Q_+^* = \dim\ker Q_- < \infty$. The pure discreteness of the spectrum of Q implies also that $\mathrm{Ran}(Q_\pm)$ (the range of $Q_\pm$) are closed. Thus $Q_\pm$ are Fredholm.

4 Continuity of Ground State Energies and Constancy of $\dim\ker H_+(V)$ in Coupling Constants

For a self-adjoint operator S on a Hilbert space, we definne

$$E_0(S) := \inf \sigma(S), \tag{4.1}$$

provided that S is bounded from below. By abuse of nomenclature, we call $E_0(S)$ the *ground state energy* of S.

In this section, as a preliminary to the next section, we prove the continuiuty of the ground state energies of $H_\pm(V)$ on the coupling constants λ and μ.

To make explicit the dependence of $H_\pm(V)$ on λ and μ, we write

$$H_\pm(\lambda,\mu) := H_\pm(V). \tag{4.2}$$

For a self-adjoint operator S on a Hilbert space, we denote by $\rho(S)$ the resolvent set of S: $\rho(S) = \mathbb{C} \setminus \sigma(S)$.

Lemma 4.1 *Let $(\lambda_0, \mu_0) \in (\mathbb{C}^\times)^2$ be arbitrarily fixed. Let $z \in \rho(H_\pm(\lambda_0,\mu_0))$. Then $z \in \rho(H_\pm(\lambda,\mu))$ for all $(\lambda,\mu) \in (\mathbb{C}^\times)^2$ in a neighborhood $\mathcal{N}_0 \subset (\mathbb{C}^\times)^2$ of (λ_0,μ_0) and*

$$\|(H_\pm(\lambda,\mu) - z)^{-1} - (H_\pm(\lambda_0,\mu_0) - z)^{-1}\| \le C(|\lambda - \lambda_0| + |\mu - \mu_0|) \tag{4.3}$$

for all $(\lambda,\mu) \in \mathcal{N}_0$, where $C > 0$ is a constant independent of $(\lambda,\mu) \in \mathcal{N}_0$.

Proof. It is sufficient to consider the case where $|\lambda-\lambda_0|+|\mu-\mu_0| \le \delta_1$ with a constant $\delta_1 > 0$. Throughout the proof, C_j's denote constants independent of λ and μ. Let

$$\mathcal{D} := D(-\Delta) \cap D(|\partial V_1|^2) \cap D(|\partial V_2|^2). \tag{4.4}$$

Then

$$D(H_\pm(\lambda,\mu)) = \mathcal{D} \oplus \mathcal{D}, \quad (\lambda,\mu) \in (\mathbb{C}^\times)^2. \tag{4.5}$$

Let $\Psi \in \mathcal{D} \oplus \mathcal{D}$. Then

$$\begin{aligned}[H_+(\lambda,\mu) - H_+(\lambda_0,\mu_0)]\Psi &= (|\lambda|^2 - |\lambda_0|^2)|\partial V_1|^2\Psi + (|\mu|^2 - |\mu_0|^2)|\partial V_2|^2\Psi \\ &\quad +2\Re\{(\lambda^*\mu - \lambda_0^*\mu_0)(\partial V_1)^*(\partial V_2)\}\Psi \\ &\quad +(\lambda-\lambda_0)H_I(V_1)\Psi + (\mu-\mu_0)H_I(V_2)\Psi.\end{aligned}$$

Hence

$$\begin{aligned}&\|[H_+(\lambda,\mu) - H_+(\lambda_0,\mu_0)]\Psi\| \\ \le\ & C_1(|\lambda-\lambda_0| + |\mu-\mu_0|) \\ & \times(\|(\lambda_0\partial V_1)^2\Psi\| + \|H_I(\lambda_0 V_1)\Psi\| + \|(\mu_0\partial V_2)^2\Psi\| + \|H_I(\mu_0 V_2)\Psi\|).\end{aligned}$$

By (2.10) and (2.19), we have

$$\begin{aligned}&\|[H_+(\lambda,\mu) - H_+(\lambda_0,\mu_0)]\Psi\| \\ \le\ & C_2(|\lambda-\lambda_0| + |\mu-\mu_0|)(\|H_-(\lambda_0\partial V_1)\Psi\| + \|H_-(\mu_0 V_2)\Psi\| + \|\Psi\|).\end{aligned}$$

Note that

$$\begin{aligned}H_+(\lambda_0,\mu_0) &= H_-(\lambda_0 V_1) + 2\Re[(\lambda_0\partial V_1)^*(\mu_0\partial V_2)] + |\mu_0\partial V_2|^2 + H_I(\lambda_0 V_1 + \mu_0 V_2), \\ &= H_-(\mu_0 V_2) + 2\Re[(\lambda_0\partial V_1)^*(\mu_0\partial V_2)] + |\lambda_0\partial V_1|^2 + H_I(\lambda_0 V_1 + \mu_0 V_2)\end{aligned}$$

By this fact and the closedness of $H_+(\lambda_0,\mu_0)$, there exists a constant $C_3 > 0$ such that

$$\|H_-(\lambda_0 V_1)\Psi\| \le C_3(\|H_+(\lambda_0,\mu_0)\Psi\| + \|\Psi\|),$$
$$\|H_-(\mu_0 V_2)\Psi\| \le C_3(\|H_+(\lambda_0,\mu_0)\Psi\| + \|\Psi\|).$$

Hence

$$\|[H_+(\lambda,\mu) - H_+(\lambda_0,\mu_0)]\Psi\| \le C_4(|\lambda-\lambda_0| + |\mu-\mu_0|)(\|H_+(\lambda_0,\mu_0)\Psi\| + \|\Psi\|). \tag{4.6}$$

Let $z \in \rho(H_+(\lambda_0,\mu_0))$. Then we have

$$\begin{aligned} H_+(\lambda,\mu) - z &= \left\{1 + [H_+(\lambda,\mu) - H_+(\lambda_0,\mu_0)](H_+(\lambda_0,\mu_0) - z)^{-1}\right\} \\ &\quad \times (H_+(\lambda_0,\mu_0) - z). \end{aligned}$$

This formula and (4.6) imply the assertion for $H_+(\lambda,\mu)$ to be proved. Similarly we can prove the properties of $H_-(\lambda,\mu)$ as stated in Lemma 4.1. ■

Let

$$E_\pm(\lambda,\mu) := E_0(H_\pm(\lambda,\mu)). \tag{4.7}$$

Theorem 4.2 *The functions* $E_\pm : (\lambda,\mu) \mapsto E_\pm(\lambda,\mu)$ *are continuous on* $(\mathbb{C}^\times)^2$.

Proof. By the nonnegativity of $H_\pm(\lambda,\mu)$ and the functional calculus, we have

$$\frac{1}{E_\pm(\lambda,\mu)+1} = \|(H_\pm(\lambda,\mu)+1)^{-1}\|.$$

By Lemma 4.1, the right hand side is continuous in (λ,μ). ■

Theorem 4.3 *Suppose that, for some* $(\lambda_0,\mu_0) \in (\mathbb{C}^\times)^2$, $E_+(\lambda_0,\mu_0) = 0$. *Then:*

(i) *For all* $(\lambda,\mu) \in (\mathbb{C}^\times)^2$, $E_+(\lambda,\mu) = 0$ *and hence* $\ker H_+(\lambda,\mu) \ne \{0\}$.

(ii) *The quantity* $\dim\ker H_+(\lambda,\mu)$ *is a constant independent of* $(\lambda,\mu) \in (\mathbb{C}^\times)^2$.

Proof. (i) By Theorem 1.5-(i), we have

$$E_-(\lambda,\mu) > 0, \quad \forall(\lambda,\mu) \in (\mathbb{C}^\times)^2. \tag{4.8}$$

In particular $\varepsilon_0 := E_-(\lambda_0,\mu_0) > 0$. By the continuity of $E_\pm(\lambda,\mu)$ in (λ,μ) (Theorem 4.2), for every $\varepsilon \in (0,\varepsilon_0/2)$, there exists a constant $r_\varepsilon > 0$ such that, if $|\lambda-\lambda_0| + |\mu-\mu_0| < r_\varepsilon$, then

$$0 \le E_+(\lambda,\mu) < \varepsilon, \quad |E_-(\lambda,\mu) - \varepsilon_0| < \varepsilon.$$

Hence, if $|\lambda-\lambda_0| + |\mu-\mu_0| < r_\varepsilon$, then

$$E_-(\lambda,\mu) > \varepsilon_0 - \varepsilon > \varepsilon > E_+(\lambda,\mu).$$

Therefore $E_-(\lambda,\mu) \ne E_+(\lambda,\mu)$ for $|\lambda-\lambda_0| + |\mu-\mu_0| < r_\varepsilon$, which, togehter with (1.30), implies that $E_+(\lambda,\mu) = 0$ for $|\lambda-\lambda_0| + |\mu-\mu_0| < r_\varepsilon$. By (4.8), we can repeat this process and see that $E_+(\lambda,\mu) = 0$ for all $(\lambda,\mu) \in (\mathbb{C}^\times)^2$. Thus the facts of part (i) follow.

(ii) We denote by $P(\lambda,\mu)$ the orthogonal projection onto $\ker H_+(\lambda,\mu)$. By the assumption, $0 \in \sigma(H_+(\lambda_0,\mu_0))$ and $a := \inf \sigma(H_+(\lambda_0,\mu_0)) \setminus \{0\} > 0$. Let $0 < \varepsilon < a$ and $C_\varepsilon := \{z \in \mathbb{C} \,|\, |z| = \varepsilon\}$. Then $C_\varepsilon \subset \rho(H_+(\lambda_0,\mu_0))$. By Theorem 4.2, there exists a constant $\delta > 0$ such that, if $|\lambda - \lambda_0| + |\mu - \mu_0| < \delta$, then $E_0(H_+(\lambda, \mu)) = 0$ (hence $\ker H_+(\lambda, \mu) \neq \{0\}$) and $C_\varepsilon \subset \rho(H_+(\lambda, \mu))$. Therefore, for such (λ, μ), we have

$$P(\lambda, \mu) = -\frac{1}{2\pi i} \int_{C_\varepsilon} (H_+(\lambda, \mu) - z)^{-1} dz.$$

Using Lemma 4.1, one can show that $P(\lambda, \mu)$ is continuous in (λ, μ) in operator norm. Hence, by a general theorem (e.g., [9, p.14, Lemma]), $\dim \operatorname{Ran}(P(\lambda, \mu))$ is a constant on the region $\{(\lambda, \mu) \in (\mathbb{C}^\times)^2 | |\lambda - \lambda_0| + |\mu - \mu_0| < \delta\}$. Repeating this process, we see that $\dim \operatorname{Ran}(P(\lambda, \mu))$ is a constant independent of $(\lambda, \mu) \in (\mathbb{C}^\times)^2$. On the other hand, $\dim \operatorname{Ran}(P(\lambda, \mu)) = \dim \ker H_+(\lambda, \mu)$. Thus the assertion of part (ii) follows. ■

5 Kernels of $H_+(V)$ and $Q_+(V)$

In this section we prove Theorem 1.3 and Theorem 1.6.

5.1 A special case

We first prove Theorem 1.6 and Theorem 1.3 for the function U with $P = 0$:

Theorem 5.1 *Let $\lambda, \mu \in \mathbb{C}^\times$ and*

$$W_{\lambda,\mu}(z) := \frac{\lambda}{z^p} + \mu z^q, \quad z \in \mathbb{C}^\times. \tag{5.1}$$

Let $4 \leq q \leq p + 2$. Then

$$\ker H_+(W_{\lambda,\mu}) \neq \{0\}. \tag{5.2}$$

Moreover, $\dim \ker H_+(W_{\lambda,\mu})$ *is independent of λ and μ.*

Corollary 5.2 *Let $4 \leq q \leq p + 2$. Then*

$$\ker Q_+(W_{\lambda,\mu}) \neq \{0\}. \tag{5.3}$$

Moreover, $\dim \ker Q_+(W_{\lambda,\mu})$ *is independent of λ and μ.*

Proof. By Theorem 2.6-(ii) and (iii), we have

$$\ker Q_+(W_{\lambda,\mu}) = \ker H_+(W_{\lambda,\mu}). \tag{5.4}$$

By this fact and Theorem 5.1, we obtain the desired result. ■

To prove Theorem 5.1, we need some preliminaries. We first recall a known fact:

Lemma 5.3 [6, 2] *Suppose that* $\deg V_2 \geq 2$. *Then:*

(i) *The operators $H_\pm(V_2)$ are nonnegative and self-adjoint with $D(H_\pm(V_2)) = D(\Delta) \cap D(|\partial V_2|^2)$. Moreover they are essentially self-adjoint on $C_0^\infty(\mathbb{R}^2; \mathbb{C}^2)$.*

(ii) *The spectra of $H_\pm(V_2)$ are purely discrete and*

$$\dim\ker H_+(V_2) = \deg V_2 - 1, \tag{5.5}$$
$$\dim\ker H_-(V_2) = 0. \tag{5.6}$$

.

The following fact may be interesting to keep in mind, although it is not necessary in the arguments below.

Proposition 5.4 *For all complex polynomials Y of $z \in \mathbb{C}$, $H_\pm(Y)$ are not essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$.*

Proof. For a constant $\mu \in \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{N}$, we define

$$M_q(z) := \mu z^q, \quad z \in \mathbb{C}. \tag{5.7}$$

Applying [8, Theorem X.7, Theorem X.10, Theorem X.11], one can prove that $H_-(M_q)$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$.

For a general polynomial Y with $\deg Y = q$, we write $Y(z) = M_q(z) + P_1(z)$ with $\deg P_1 \le q-1$. Then $|\partial P_1|^2 + 2\Re\{(\partial M_q)^*(\partial P_1)\}$ is infinitesimally small with respect to $|\partial M_q|^2$. Using this fact and the Kato-Rellich theorem, one can show that $H_-(Y)$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$.

By (2.10), $H_I(Y)$ is infinitesimally small with respect to $H_-(Y)$. Hence, by a perturbation argument again, we can show that $H_+(Y)$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$. ■

Because of the fact stated in Proposition 5.4, the following lemma, which is a key to the proof of Theorem 5.1, is non-trivial.

Lemma 5.5 *Let M_q be defined by (5.7) and $q \ge 4$. Then*

$$\inf_{\Psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2), \|\Psi\|=1} \langle \Psi, H_+(M_q)\Psi \rangle = 0. \tag{5.8}$$

Proof. Let $K_\nu(z)$ be the modified Bessel function of the third kind (e.g., [5, §7.2.2]) and

$$\begin{aligned}\Phi_m &= \mu r^{q-1} K_{m/q}(2\mu r^q)(2\pi)^{-1/2} e^{i(q-1-m)\theta}, \\ &\quad (r = |z|,\ \theta = \arg z,\ m = 1, \ldots, q-1).\end{aligned}$$

Put

$$\Omega_m = \begin{pmatrix} \Phi_m \\ -i\Phi_{q-m}^* \end{pmatrix}.$$

Then it was shown in [2, Proposition 3.2] that, for $m = 1, \cdots, q-1$, $\Omega_m \in \ker H_+(M_q) \setminus \{0\}$:

$$H_+(M_q)\Omega_m = 0.$$

Let $q \geq 4$. Then, by the regularity of Ω_m (cf. [2, p.2975, Remark (1)]) and (2.16), we see that $\Omega_m \in D(Q_+(M_q))$ and

$$Q_+(M_q)\Omega_m = 0. \tag{5.9}$$

Let f be a C_0^∞-function on $\mathbb{R}^2$ such that $f(\boldsymbol{r}) = 1$ for $|\boldsymbol{r}| \leq 1$, $f(\boldsymbol{r}) = 0$ for $|\boldsymbol{r}| \geq 2$ and $|f(\boldsymbol{r})| \leq 1$ for $1 \leq |\boldsymbol{r}| \leq 2$ and $g \in C_0^\infty(\mathbb{R}^2)$ be such that $g(0) = 1$. For $n \in \mathbb{N}$, we define

$$h_n(\boldsymbol{r}) = g\left(\frac{\boldsymbol{r}}{n}\right)[1 - f(n\boldsymbol{r})].$$

Then $h_n \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ and

$$\lim_{n\to\infty} h_n(\boldsymbol{r}) = 1, \quad \forall \boldsymbol{r} \in \mathbb{R}^2 \setminus \{0\}.$$

Let

$$\Psi_n^{(m)}(\boldsymbol{r}) = h_n(\boldsymbol{r})\Omega_m(\boldsymbol{r}).$$

Then $\Psi_n^{(m)} \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$ and

$$\lim_{n\to\infty} \|\Psi_n^{(m)}\| = \|\Omega_m\| \neq 0,$$

which implies that $\|\Psi_n^{(m)}\| \neq 0$ for all $n \geq n_0$ with some $n_0 \in \mathbb{N}$. Hence we can define a sequence $\{\widetilde{\Psi}_n^{(m)}\}_{n\geq n_0}$ of unit vectors in $C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2)$ by

$$\widetilde{\Psi}_n^{(m)} := \frac{\Psi_n^{(m)}}{\|\Psi_n^{(m)}\|}.$$

Using (5.9), we see that

$$Q_+(M_q)\Psi_n^{(m)} = \frac{1}{2}\{(\sigma_1 - i\sigma_2)(\bar{\partial}h_n) + (\sigma_1 + i\sigma_2)(\partial h_n)\}\Omega_m.$$

By the fact that $\|\sigma_j\| = 1$ $(j = 1, 2)$, we have

$$\|Q_+(M_q)\Psi_n^{(m)}\| \leq \|(D_x h_n)\Omega_m\| + \|(D_y h_n)\Omega_m\|.$$

By direct computations, we have

$$(D_\# h_n)(\boldsymbol{r}) = \frac{1}{n}(D_\# g)\left(\frac{\boldsymbol{r}}{n}\right)[1 - f(n\boldsymbol{r})] - ng\left(\frac{\boldsymbol{r}}{n}\right)(D_\# f)(n\boldsymbol{r}),$$

where $\# = x, y$. Hence

$$\|(D_\# h_n)\Omega_m\| \leq I_n^{1/2} + J_n^{1/2}$$

with

$$I_n := \frac{1}{n^2}\int_{\mathbb{R}^2}\left|(D_\# g)\left(\frac{\boldsymbol{r}}{n}\right)[1 - f(n\boldsymbol{r})]\right|^2 \|\Omega_m(\boldsymbol{r})\|^2 d\boldsymbol{r},$$

$$J_n := n^2 \int_{\mathbb{R}^2}\left|g\left(\frac{\boldsymbol{r}}{n}\right)(D_\# f)(n\boldsymbol{r})\right|^2 \|\Omega_m(\boldsymbol{r})\|^2 d\boldsymbol{r}.$$

We have

$$0 \le I_n \le \frac{1}{n^2}\|D_\# g\|_\infty^2 \|\Omega_m\|^2,$$

where, for $u : \mathbb{R}^2 \to \mathbb{C}$, we define $\|u\|_\infty := \sup_{\boldsymbol{r}\in\mathbb{R}^2} |u(\boldsymbol{r})|$. Hence

$$\lim_{n\to\infty} I_n = 0.$$

On the other hand, by change of variables, we have

$$0 \le J_n \le \|g\|_\infty^2 \int_{\mathbb{R}^2} |(D_\# f)(\boldsymbol{r})|^2 \left\| \Omega_m\left(\frac{\boldsymbol{r}}{n}\right)\right\|^2 d\boldsymbol{r}.$$

Under the present assumption $q \ge 4$, we can take $m \ge 2$. Then it follows from the asymptotic properties of the Bessel function K_ν

$$K_\nu(r) \sim \sqrt{\pi/(2r)}e^{-r} \quad (r \to \infty),$$
$$K_\nu(r) \sim \frac{\text{const.}}{r^\nu} \quad (r \to 0)\ (\nu \ge 1)$$

that Ω_m is bounded on $\mathbb{R}^2$ and

$$\lim_{\boldsymbol{r}\to 0} \Omega_m(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \mathbb{R}^2.$$

Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty} J_n = 0.$$

Thus we have for $m = 2, \cdots, q-1$

$$\lim_{n\to\infty} \|Q_+(M_q)\Psi_n^{(m)}\| = 0.$$

This implies that

$$\lim_{n\to\infty} \langle \widetilde{\Psi}_n^{(m)}, H_+(M_q)\widetilde{\Psi}_n^{(m)}\rangle = 0 \quad (m = 2, \cdots, q-1).$$

Thus (5.8) is proved. ∎

Proof of Theorem 5.1

By Theorem 4.3, we need only to prove (5.2) for sufficiently small $|\lambda|$. Throughout the proof, we write $W_\lambda = W_{\lambda,\mu}$. Let $f(z) = z^{-p}$. Then we have

$$H_+(W_\lambda) = H_+(M_q) + F_\lambda + \lambda H_I(f),$$

where

$$F_\lambda(z) := |\lambda|^2 |(\partial f)(z)|^2 + 2\Re\{(\lambda \partial f)^* \partial M_q(z)\}.$$

By the variational principle, we have

$$E_0(H_+(W_\lambda)) \le \langle \Psi, H_+(M_q)\Psi\rangle + \langle \Psi, F_\lambda \Psi\rangle + \lambda\langle \Psi, H_I(f)\Psi\rangle$$

for all $\Psi \in D(H_+(W_\lambda))$ with $\|\Psi\| = 1$. Hence

$$\limsup_{\lambda \to 0} E_0(H_+(W_\lambda)) \le \langle \Psi, H_+(M_q)\Psi \rangle.$$

We have

$$\inf_{\Psi \in D(H_+(W_\lambda)), \|\Psi\|=1} \langle \Psi, H_+(M_q)\Psi \rangle \le \inf_{\Psi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}; \mathbb{C}^2), \|\Psi\|=1} \langle \Psi, H_+(M_q)\Psi \rangle.$$

By Lemma 5.5, the right hand side is equal to zero. Hence

$$\lim_{\lambda \to 0} E_0(H_+(W_\lambda)) = 0.$$

Therefore, for every $\varepsilon > 0$, there exists a constant $\delta_\varepsilon > 0$ such that

$$0 \le E_0(H_+(W_\lambda)) < \varepsilon, \quad \forall |\lambda| < \delta_\varepsilon. \tag{5.10}$$

On the other hand, we have

$$H_-(W_\lambda) = H_-(M_q) + F_\lambda.$$

Under the condition that $p + 2 \ge q$, we can show that

$$F_\lambda(z) \ge -C_{p,q}|\lambda|^{2(q-1)/(p+q)}, \quad \forall z \in \mathbb{C}^\times,$$

with a constant $C_{p,q} > 0$. Hence, by the variational principle again, we have

$$E_0(H_-(W_\lambda)) \ge E_0(H_-(M_q)) - C_{p,q}|\lambda|^{2(q-1)/(p+q)}.$$

By Lemma 5.3-(ii), we have $E_0(H_-(M_q)) > 0$. Hence one can take a constant $\eta > 0$ such that $E_0(H_-(M_q)) - \eta > 0$. Then, for all $\lambda \in \mathbb{C}^\times$ satisfying

$$C_{p,q}|\lambda|^{2(q-1)/(p+q)} < E_0(H_-(M_q)) - \eta,$$

we have $E_0(H_-(W_\lambda)) > \eta > 0$. By (1.30), we have

$$\sigma(H_-(W_\lambda)) \setminus \{0\} = \sigma(H_+(W_\lambda)) \setminus \{0\}. \tag{5.11}$$

It follows from this property and (5.10) that, for all sufficiently small $|\lambda|$, $E_0(H_+(W_\lambda)) = 0$. Thus (5.2) follows for all sufficiently small $|\lambda|$. ■

5.2 A more general case

Let U be given by (1.21). and

$$R := \begin{pmatrix} i\partial P & 0 \\ 0 & -i(\partial P)^* \end{pmatrix}.$$

Then we have

$$Q_-(U) = Q_-(W_{\lambda,\mu}) + R. \tag{5.12}$$

Lemma 5.6 *The operator R is relatively compact with respect to $Q_-(W_{\lambda,\mu})$.*

Proof. Let $\{\Psi_n\}_n$ be a sequence with $\Psi_n \in D(Q_-(W_{\lambda,\mu}))$ such that

$$\|\Psi_n\| + \|Q_-(W_{\lambda,\mu})\Psi_n\| \leq 1.$$

Then $\|\Psi_n\| \leq 1$ and, by (2.12),

$$\|(-\Delta)^{1/2}\Psi_n\| \leq 2, \quad \|(\partial W_{\lambda,\mu})\Psi_n\| \leq 1.$$

Let $B_\ell := \{\boldsymbol{r} \in \mathbb{R}^2 | \, |\boldsymbol{r}| \leq \ell\}$ $(\ell > 0)$ and χ_ℓ be the characteristic function of B_ℓ. It is well known (or easy to see) that, for all $\ell > 0$, $\chi_\ell(-\Delta + 1)^{-1/2}$ is a compact operator on $L^2(\mathbb{R}^2)$. Hence there is a subsequence $\{\Xi_m\}_m$ of $\{\Psi_n\}$ such that, for all $\ell \in \mathbb{N}$, $\{\chi_\ell \Xi_m\}_m$ is strongly convergent in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Let $a \in \mathbb{N}$. Then

$$\begin{aligned} \|(1-\chi_a)R\Xi_m\| &= \|(1-\chi_a)R(\partial W_{\lambda,\mu})^{-1}(\partial W_{\lambda,\mu})\Xi_m\| \\ &\leq \|(1-\chi_a)R(\partial W_{\lambda,\mu})^{-1}\|_\infty. \end{aligned}$$

It is easy to see that, for all sufficiently large $a \in \mathbb{N}$

$$\|(1-\chi_a)R(\partial W_{\lambda,\mu})^{-1}\|_\infty \leq \frac{C}{a}$$

with a constant $C > 0$. Hence, for every $\varepsilon > 0$, there is a constant a such that

$$\|(1-\chi_a)R(\Xi_m - \Xi_n)\| < \varepsilon, \quad m, n \in \mathbb{N}.$$

We fix such an a. Since $|\partial P|$ is bounded on every bounded set of $\mathbb{R}^2$, it follows from the strong convergence of $\{\chi_\ell \Xi_m\}_m$ that there is a constant $n_0 \in \mathbb{N}$ such that

$$\|\chi_a R(\Xi_m - \Xi_n)\| < \varepsilon, \quad m, n \geq n_0.$$

Hence we obtain

$$\|R(\Xi_m - \Xi_n)\| < 2\varepsilon, \quad m, n \geq n_0.$$

This means that $\{R\Xi_m\}_m$ is a strongly convergent sequence in $L^2(\mathbb{R}^2; \mathbb{C}^2)$. Thus R is relatively compact with respect to $Q_-(W_{\lambda,\mu})$. ■

Lemma 5.7 *Suppose that $4 \leq q \leq p+2$. Then*

$$\dim\ker Q_+(U) = \dim\ker Q_+(W_{\lambda,\mu}). \tag{5.13}$$

Proof. By Lemma 5.6 and the Fredholmness of $Q_-(W_{\lambda,\mu})$, we can apply a stability theorem ([7, p.238, Theorem 5.26]) for Fredholm index to obtain

$$\text{index}(Q_-(W_{\lambda,\mu})) = \text{index}(Q_-(U)).$$

Using Theorem 1.1-(ii) and Theorem 1.2-(ii), we obtain (5.13). ■

Theorem 1.3 now follows from Corollary 5.2, Lemma 5.7 and (1.23). This fact together with Theorem 2.6-(ii) and (iii) yields Theorem 1.6.

6 A Trace Formula

Finally, we present a formula for $\dim\ker Q_+(V)$ in terms of $H_\pm(V)$.

Theorem 6.1 *Assume (A). Then, for all $t > 0$, $e^{-tH_\pm(V)}$ are trace class and*

$$\dim\ker Q(V) = \dim\ker Q_+(V) = \mathrm{tr}\, e^{-tH_+(V)} - \mathrm{tr}\, e^{-tH_-(V)} \tag{6.1}$$

independently of $t > 0$, where tr *means trace.*

Proof. It is easy to see that

$$|\partial V(z)|^2 \geq \omega^2|z|^2 - c,$$

where $\omega > 0$ and c are positive constants depending on λ and μ. Hence

$$H_-(V) \geq H_{\mathrm{os}} \oplus H_{\mathrm{os}} - c,$$

where

$$H_{\mathrm{os}} := -\frac{1}{4}\Delta + \omega^2|z|^2$$

acting in $L^2(\mathbb{R}^2)$. It is well known that H_{os} is the Hamiltonian of a two-dimensional quantum harmonic oscillator and

$$\sigma(H_{\mathrm{os}}) = \sigma_{\mathrm{d}}(H_{\mathrm{os}}) = \left\{\omega(n+m+1)\middle| n, m \in \{0\} \cup \mathbb{N}\right\},$$

where each eigenvalue $\omega(n+m+1)$ is simple. In particular, for all $t > 0$, $e^{-tH_{\mathrm{os}}}$ is trace class on $L^2(\mathbb{R}^2)$. Therefore, via the min-max principle (Theorem XIII.1 and Problem 1 (p.364) in [9]), one can show that, for all $t > 0$, $e^{-tH_-(V)}$ is trace class and

$$\mathrm{tr}\, e^{-tH_-(V)} \leq e^{ct}\mathrm{tr}\, e^{-tH_{\mathrm{os}}}.$$

It follows from (2.19) and a general theorem [8, Theorem X.18] that $H_I(V)$ is infinitesimally form-bounded with respect to $H_-(V)$. Hence we have

$$H_+(V) \geq (1-\varepsilon)H_-(V) - b_\varepsilon,$$

where $\varepsilon \in (0, 1)$ is arbitrary and $b_\varepsilon > 0$ is a constant. Therefore, by the min-max principle again, $e^{-tH_+(V)}$ is trace class for all $t > 0$. It follows from these facts and Theorem 1.4-(ii) that $e^{-tQ(V)^2}$ is trace class for all $t > 0$. Hence, by applying a well known formula on Fredholm index (e.g., [10, Theorem 5.19]), we have

$$\mathrm{index}(Q_+(V)) = \mathrm{tr}\left[\Gamma e^{-tQ(V)^2}\right],$$

where Γ is defined by (3.4). By Theorem 1.1-(ii), Theorem 1.2-(ii) and (1.23), the left hand side is equal to $\dim\ker Q_+(V) = \dim\ker Q(V)$, while the right hand side is equal to

$$\mathrm{tr}\, e^{-tH_+(V)} - \mathrm{tr}\, e^{-tH_-(V)}$$

by Theorem 1.4-(iii). Thus (6.1) follows. ■

Remark 6.1 The right hand side of (6.1) has a functional integral representation. But here we do not go into the details. It will be discussed in a separate paper.

References

[1] A. Arai, Existence of infinitely many zero-energy states in a model of supersymmetric quantum mechanics, *J. Math. Phys.* **30** (1989), 1164–1170.

[2] A. Arai, On the degeneracy in the ground state of the $N = 2$ Wess-Zumino supersymmetric quantum mechanics, *J. Math. Phys.* **30** (1989), 2973–2977.

[3] A. Arai and O. Ogurisu, Meromorphic $N = 2$ Wess-Zumino supersymmetric quantum mechanics, *J. Math. Phys.* **32** (1991), 2427–2434.

[4] P. A. Deift, Application of a commutation formula, *Duke Math. J.* **45** (1978), 267–310.

[5] A. Erdélyi, *Higher Transcendental Functions, Vol.II*, McGraw-Hill, New York, 1953.

[6] A. Jaffe, A. Lesniewski, and M. Lewenstein, Ground state structure in supersymmetric quantum mechanics, *Ann. Phys. (NY)* **178** (1987), 313–329.

[7] T. Kato, *Perturbation Theory for Linear Operators*, Second Edition, Springer, Berlin, Heidelberg, 1976.

[8] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol.II: Fourier Analysis, Self-Adjointness*, Academic Press, 1975, New York.

[9] M. Reed and B. Simon, *Methods of Modern Mathematical Physics Vol.IV: Analysis of Operators*, Academic Press, 1978, New York.

[10] B. Thaller, *The Dirac Equation*, Springer, 1992, Berlin, Heidelberg.