Stability of Ground States in Sectors and Its Application to the Wigner-Weisskopf Model

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Abstract

We consider two kinds of stability (under a perturbation) of the ground state of a self-adjoint operator, being concerned with (i) the sector to which the ground state belongs and (ii) the uniqueness of the ground state. As an application to the Wigner-Weisskopf model which describes one mode fermion coupled to a quantum scalar field, we prove in the massive case the following: (a) For a value of the coupling constant, the Wigner-Weisskopf model has degenerate ground states ; (b) for a value of the coupling constant, the Wigner-Weisskopf model has a first excited state with energy level below the bottom of the essential spectrum.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space and H_0 a self-adjoint operator on $\mathcal{H}$, bounded from below. Let $\mathcal{I}$ be an open interval of $\mathbf{R}$ containing the origin 0 and $\{H(\alpha)\}_{\alpha\in\mathcal{I}}$ be a family of self-adjoint operators acting in $\mathcal{H}$ with $H(\alpha)$ bounded from below for every $\alpha \in \mathcal{I}$ such that

$$H(0) = H_0. \tag{1.1}$$

For a linear operator T on a Hilbert space, we denote its domain (resp. spectrum, point spectrum) by $D(T)$ (resp. $\sigma(T)$, $\sigma_{\rm p}(T)$). If T is self-adjoint and bounded from below, then

$$E_0(T) := \inf \sigma(T) > -\infty \tag{1.2}$$

is called the *ground-state energy* of T. We say that T has a ground state if $\ker(T-E_0(T)) \neq \{0\}$; a non-zero vector in $\ker(T - E_0(T))$ is called a *ground state* of T. The ground state of T is said to be unique (resp. degenerate) if $\dim\ker(T - E_0(T)) = 1$ (resp. ≥ 2).

In this paper we are concerned with stabilities of ground states of $H(\alpha)$ in the parameter $\alpha \in \mathcal{I}$. In particular we are interested in the following two kinds of stability:

(S.1) (*Stability in sectors*) Suppose that $\mathcal{H}$ has an orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \tag{1.3}$$

with $\mathcal{H}_j$ $(j = 0, 1)$ being a closed subspace of $\mathcal{H}$ such that, for all $\alpha \in \mathcal{I}$, $H(\alpha)$ is reduced by each $\mathcal{H}_j$. In the context of quantum field theory, where $\mathcal{H}$ describes the Hilbert space of state vectors for the model under consideration, each Hilbert space $\mathcal{H}_j$ is called a sector. Suppose that H_0 has a ground state in $\mathcal{H}_0$. Then a natural question is: To which sector does the ground states of $H(\alpha)$ belong ?

(S.2) Uniqueness of ground states of $H(\alpha)$.

As for (S.2), there are already fundamental results available (e.g., [Ka, Chapter VII], [RS4, §XII.2]). We apply these results in a more restricted situation to obtain a stronger result.

On the other hand, to our best knowledge, the problem (S.1) seems not to have been considered, at least, on an abstract level.

In Section 2 we prove abstract results on problem (S.1) and degeneracy of ground states. These results are applied to a special class of self-adjoint operators in Section 3. In the last section we consider the Wigner-Weisskopf model (WW model) which describes one mode fermion coupled to a quantum scalar field [WW]. We apply the results of Section 3 to this model in the massive case to establish the following properties: (a) For a value of the coupling constant, the WW model has degenerate ground states ; (b) for a value of the coupling constant, the WW model has a first excited state with energy level below the bottom of the essential spectrum.

2 Stability of Ground States in Sectors : Abstract Results

2.1 Main results

We denote the resolvent of $H(\alpha)$ $(\alpha \in \mathbf{R})$ by

$$R_z(\alpha) := (H(\alpha) - z)^{-1}, \qquad z \in \rho(H(\alpha)), \tag{2.1}$$

where $\rho(A)$ denotes the resolvent set of a closed operator A. We set

$$E_0(\alpha) := E_0(H(\alpha)), \quad \alpha \in \mathcal{I}. \tag{2.2}$$

Our basic assumptions are as follows:

(A.1) For all $z \in \mathbf{C} \setminus \mathbf{R}$, $R_z : \alpha \to R_z(\alpha)$ is continuous on $\mathcal{I}$ in operator norm.

(A.2) For each $\alpha \in \mathcal{I}$, there exists a constant $C_\alpha > 0$ such that, for all sufficiently small $|\kappa|$,

$$E_0(\alpha + \kappa) \geq C_\alpha. \tag{2.3}$$

(A.3) For all $\alpha \in \mathcal{I}$, $E_0(\alpha)$ is an isolated eigenvalue of $H(\alpha)$ (hence $H(\alpha)$ has a ground state).

A solution to the stability problem (S.1) is given in the following theorem:

Theorem 2.1 *Assume (A.1)–(A.3) and that $\mathcal{H}$ has the orthogonal decomposition (1.3) such that, for all $\alpha \in \mathcal{I}$, $H(\alpha)$ is reduced by $\mathcal{H}_0$. Suppose that, for all $\alpha \in \mathcal{I}$, the ground state $H(\alpha)$ is unique and that the ground state of H_0 is in $\mathcal{H}_0$. Then, for all $\alpha \in \mathcal{I}$, the ground state of $H(\alpha)$ is in $\mathcal{H}_0$.*

This theorem can be used to show a degeneracy of ground states:

Corollary 2.2 *Assume (A.1)–(A.3) and that $\mathcal{H}$ has the orthogonal decomposition (1.3) such that, for all $\alpha \in \mathcal{I}$, $H(\alpha)$ is reduced by $\mathcal{H}_0$. Suppose that the ground state of H_0 is unique and in $\mathcal{H}_0$. Moreover, suppose that there exists an $\alpha' \in \mathcal{I}$ such that $H(\alpha')$ has a ground state which is not in $\mathcal{H}_0$. Then, for some $\alpha_0 \in \mathcal{I} \setminus \{0\}$, the ground state of $H(\alpha_0)$ is degenerate.*

Proof. If the conclusion does not hold, then the ground state $H(\alpha)$ is unique for all $\alpha \in \mathcal{I}$. Hence, by Theorem 2.1, the ground state of $H(\alpha)$ is in $\mathcal{H}_0$ for all $\alpha \in \mathcal{I}$. But this contradicts the assumption that $H(\alpha')$ has a ground state which is not in $\mathcal{H}_0$. ■

To prove Theorem 2.1, we establish two lemmas.

Lemma 2.3 *Assume (A.1) and (A.2). Then the ground state energy $E_0(\alpha)$ is continuous in $\alpha \in \mathcal{I}$.*

Proof. Fix $\alpha \in \mathcal{I}$ arbitrarily. By (A.2), there exists a constant $\gamma_\alpha \in \mathbf{R}$ such that, for all sufficiently small $|\kappa|$, $\gamma_\alpha \in \rho\left(H\left(\alpha + \kappa\right)\right)$ and $\gamma_\alpha < E_0(\alpha + \kappa)$. Assumption (A.1) implies that $\|R_{\gamma_\alpha}(\alpha + \kappa) - R_{\gamma_\alpha}(\alpha)\| \to 0$ $(\kappa \to 0)$. Hence

$$\lim_{\kappa \to 0} \frac{1}{E_0(\alpha + \kappa) - \gamma_\alpha} = \lim_{\kappa \to 0} \|R_{\gamma_\alpha}(\alpha + \kappa)\| = \|R_{\gamma_\alpha}(\alpha)\| = \frac{1}{E_0(\alpha) - \gamma_\alpha},$$

which implies that $\lim_{\kappa \to 0} E_0(\alpha + \kappa) = E_0(\alpha)$. Thus the desired result follows. ■

Lemma 2.4 *Assume (A.1)–(A.3). Suppose that, for all $\alpha \in \mathcal{I}$, the ground state $H(\alpha)$ is unique. Let $\Psi_0(\alpha)$ be a normalized ground state of $H(\alpha)$. Then, for all $\alpha \in \mathcal{I}$,*

$$\lim_{\kappa \to 0} (\Psi_0(\alpha + \kappa), \Psi_0(\alpha))\Psi_0(\alpha + \kappa) = \Psi_0(\alpha). \tag{2.4}$$

Proof. For each $\alpha \in \mathcal{I}$, we denote by $P_\alpha(\cdot)$ the spectral measure of $H(\alpha)$. Fix $\alpha \in \mathcal{I}$ arbitrarily. By (A.3), there exists a constant $a, b \in \mathbf{R} \cap \rho(H(\alpha))$ such that $a < E_0(\alpha) < b$ and $(a, b) \cap \sigma(H(\alpha)) = \{E_0(\alpha)\}$. By (A.1) and a general fact [RS1, Theorem VIII.23(b)],

$$\|P_{\alpha+\kappa}((a, b)) - P_\alpha((a, b))\| \to 0 \ (\kappa \to 0). \tag{2.5}$$

Hence, by [RS4, p.14, Lemma], $\dim \mathrm{Ran}\, P_{\alpha+\kappa}((a, b)) = \dim \mathrm{Ran}\, P_\alpha((a, b)) = 1$ for all sufficiently small $|\kappa|$. By Lemma 2.3, $E_0(\alpha+\kappa) \in (a, b)$ for all $|\kappa| < \delta$ with some constant $\delta > 0$. Hence, for all $|\kappa| < \delta$, $P_{\alpha+\kappa}((a, b))$ is the orthogonal projection onto $\ker(H(\alpha + \kappa) - E_0(\alpha+\kappa))$, which implies that $P_{\alpha+\kappa}((a, b))\Psi_0(\alpha) = (\Psi_0(\alpha+\kappa), \Psi_0(\alpha))\Psi_0(\alpha+\kappa)$. On the other hand, (2.5) implies that $P_{\alpha+\kappa}((a, b))\Psi_0(\alpha) \to P_\alpha((a, b))\Psi_0(\alpha) = \Psi_0(\alpha)$ $(\kappa \to 0)$. Thus (2.4) follows. ■

Proof of Theorem 2.1

Let $\Psi_0(\alpha)$ be a normalized ground state of $H(\alpha)$. By the uniqueness of the ground state of $H(\alpha)$, either $\Psi_0(\alpha) \in \mathcal{H}_0$ or $\Psi_0(\alpha) \in \mathcal{H}_1$. By the present assumption, $\Psi_0(0) \in \mathcal{H}_0$.

Suppose that there existed a sequence $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_n \to 0$ $(n \to \infty)$ and $\Psi_0(\alpha_n) \in \mathcal{H}_1$. Hence $(\Psi_0(\alpha_n), \Psi_0(0)) = 0$ for all $n \geq 1$. Then, by applying Lemma 2.4 to the case $\alpha = 0$, we have $\Psi_0(0) = 0$. But this is a contradiction. Thus there exists a constant $\delta > 0$ such that, for all $|\alpha| < \delta$, we have $\alpha \in \mathcal{I}$ and $\Psi_0(\alpha) \in \mathcal{H}_0$.

Let

$$\alpha_- := \inf\{\alpha \in \mathcal{I} | \Psi_0(\alpha) \in \mathcal{H}_0\}, \quad \alpha_+ := \sup\{\alpha \in \mathcal{I} | \Psi_0(\alpha) \in \mathcal{H}_0\}.$$

Then, by the above fact, $\alpha_- < 0 < \alpha_+$. We first consider the case $\mathcal{I} = (c, d)$ with $-\infty < c < 0 < d < \infty$. We show that $\alpha_- = c, \alpha_+ = d$. Suppose that $\alpha_+ < d$. Then there exists a sequence $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_n \to \alpha_+$ $(n \to \infty)$ and $\Psi_0(\alpha_n) \in \mathcal{H}_0$. Suppose that $\Psi_0(\alpha_+) \in \mathcal{H}_1$. Then $(\Psi_0(\alpha_n), \Psi_0(\alpha_+)) = 0$. Applying Lemma 2.4 to the case $\alpha = \alpha_+$, we have $\Psi_0(\alpha_+) = 0$. But this is a contradiction. Hence $\Psi_0(\alpha_+) \in \mathcal{H}_0$. Then, in the same way as above, we can show that there exists a constant $\alpha' \in (\alpha_+, d)$ such that $\Psi_0(\alpha') \in \mathcal{H}_0$. Hence, by the definition of α_+, $\alpha' \leq \alpha_+$. But this is a contradiction. Thus $\alpha_+ = d$. Similarly we can show that $\alpha_- = c$. The same method works in the other cases of $\mathcal{I}$. ■

The proof of Theorem 2.1 shows in an obvious way that Theorem 2.1 can be generalized to the case of other eigenvectors of $H(\alpha)$:

Theorem 2.5 *Assume (A.1) and that $\mathcal{H}$ has the orthogonal decomposition (1.3) such that, for all $\alpha \in \mathcal{I}$, $H(\alpha)$ is reduced by $\mathcal{H}_0$. Suppose that, for each $\alpha \in \mathcal{I}$, $H(\alpha)$ has an isolated eigenvalue $E(\alpha)$ such that* $\dim \ker(H(\alpha) - E(\alpha)) = 1$, *$E(\cdot)$ is continuous on $\mathcal{I}$ and* $\ker(H_0 - E(0)) \subset \mathcal{H}_0$. *Then, for all $\alpha \in \mathcal{I}$,* $\ker(H(\alpha) - E(\alpha)) \subset \mathcal{H}_0$.

2.2 Uniqueness of ground states

We first prove a general fact on the stability of uniqueness of eigenvectors of $H(\alpha)$.

Proposition 2.6 *Assume (A.1). Suppose that, for each* $\alpha \in \mathcal{I}$*, there exist constants* $E(\alpha) \in \mathbf{R}$*,* $\delta_\alpha > 0$ *and* $K_\alpha > 0$ *such that*

$$[E(\alpha) - \delta_\alpha, E(\alpha) + \delta_\alpha] \cap \sigma(H(\alpha)) = \{E(\alpha)\} \tag{2.6}$$

and, for all $|\kappa| < K_\alpha$*,*

$$[E(\alpha) - \delta_\alpha, E(\alpha) + \delta_\alpha] \cap \sigma(H(\alpha + \kappa)) = \{E(\alpha + \kappa)\}, \tag{2.7}$$

so that $E(\alpha)$ *is an eigenvalue of* $H(\alpha)$*. Suppose that* $\dim\ker(H_0 - E(0)) = 1$*. Then, for all* $\alpha \in \mathcal{I}$*,* $\dim\ker(H(\alpha) - E(\alpha)) = 1$*.*

Proof. Let $a_0 := E(0)-\delta_0, b := E(0)+\delta_0$. As in the proof of Lemma 2.4, we see that, for all $|\alpha| < \delta$ with some $\delta > 0$ sufficiently small, $\dim \mathrm{Ran}P_\alpha((a_0, b_0)) = \dim \mathrm{Ran}P_0((a_0, b_0)) = 1$. By (2.7), $\mathrm{Ran}P_\alpha((a_0, b_0)) = \ker(H(\alpha)-E(\alpha)),\ |\alpha| < \delta$. Hence $\dim\ker(H(\alpha)-E(\alpha)) = 1,\ |\alpha| < \delta$. Let

$$\begin{aligned} a_- &:= \inf\{\alpha \in \mathcal{I} |\, \dim\ker(H(\alpha) - E(\alpha)) = 1\} \\ a_+ &:= \sup\{\alpha \in \mathcal{I} |\, \dim\ker(H(\alpha) - E(\alpha)) = 1\}. \end{aligned}$$

By the above fact, we have $a_- < 0 < a_+$. Consider the case $\mathcal{I} = (c, d)$ with $-\infty < c < 0 < d < \infty$. We show that $a_- = c, a_+ = d$. Suppose that $a_+ < d$. Then there exists a sequence $\{\alpha_n\}_{n=1}^\infty$ such that $\alpha_n \to a_+$ $(n \to \infty)$ and $\dim\ker(H(\alpha_n) - E(\alpha_n)) = 1$. Suppose that $\dim\ker(H(a_+) - E(a_+)) \geq 2$. We have for all $n \geq n_0$ with some $n_0 \geq 1$

$$\dim \mathrm{Ran}P_{\alpha_n}((E(a_+) - \delta_{a_+}, E(a_+) + \delta_{a_+})) = \dim \mathrm{Ran}P_{a_+}((E(a_+) - \delta_{a_+}, E(a_+) + \delta_{a_+})).$$

Hence, for all $n \geq n_0$, $\dim \mathrm{Ran}P_{\alpha_n}((E(a_+) - \delta_{a_+}, E(a_+) + \delta_{a_+})) \geq 2$. By (2.7),

$$\mathrm{Ran}P_{\alpha_n}((E(a_+) - \delta_{a_+}, E(a_+) + \delta_{a_+})) = \ker(H(\alpha_n) - E(\alpha_n)),\ n \geq n_0,$$

which implies $\dim \mathrm{Ran}P_{\alpha_n}((E(a_+) - \delta_{a_+}, E(a_+) + \delta_{a_+})) = 1$. But this is a contradiction. Thus $a_+ = d$. Similarly we can show that $a_- = c$. The same method works in the other cases of $\mathcal{I}$. ■

We consider a sufficient condition for (2.6) and (2.7) to hold in the case $E(\alpha) = E_0(\alpha)$. Let

$$E_1(\alpha) := \inf\{\sigma(H(\alpha)) \setminus \{E_0(\alpha)\}\}. \tag{2.8}$$

Proposition 2.7 *Assume (A.1) and (A.2). Suppose that, for every* $\alpha \in \mathcal{I}$*, there exists a constant* $L_\alpha > 0$ *such that*

$$\alpha \pm L_\alpha \in \mathcal{I}, \tag{2.9}$$

$$\inf_{0\leq|\kappa|\leq L_\alpha} \{E_1(\alpha + \kappa) - E_0(\alpha + \kappa)\} > E_0(\alpha) - \inf_{0\leq|\kappa|\leq L_\alpha} E_0(\alpha + \kappa). \tag{2.10}$$

Then $H(\alpha)$ *satisfies (2.6) and (2.7).*

Proof. Fix $\alpha \in \mathcal{I}$ arbitrarily. By (2.10), there is a real constant M_α such that

$$\inf_{0\leq|\kappa|\leq L_\alpha} \{E_1(\alpha+\kappa) - E_0(\alpha+\kappa)\} > M_\alpha > E_0(\alpha) - \inf_{0\leq|\kappa|\leq L_\alpha} E_0(\alpha+\kappa). \tag{2.11}$$

Hence, for every κ with $0 \leq |\kappa| \leq L_\alpha$, we have

$$M_\alpha < E_1(\alpha+\kappa) - E_0(\alpha+\kappa). \tag{2.12}$$

In particular, putting $\kappa = 0$, we have

$$E_0(\alpha) + M_\alpha < E_1(\alpha). \tag{2.13}$$

By the second inequality in (2.11), there exists a constant δ_α such that

$$0 < \delta_\alpha < M_\alpha + \inf_{0\leq|\kappa|\leq L_\alpha} E_0(\alpha+\kappa) - E_0(\alpha). \tag{2.14}$$

By (2.12) and (2.14), we have

$$\begin{aligned} E_0(\alpha) + \delta_\alpha &< M_\alpha + \inf_{0\leq|\kappa'|\leq L_\alpha} E_0(\alpha+\kappa') \\ &\leq (E_1(\alpha+\kappa) - E_0(\alpha+\kappa)) + E_0(\alpha+\kappa) \\ &= E_1(\alpha+\kappa) \end{aligned}$$

for $0 \leq |\kappa| \leq L_\alpha$, which, together with Lemma 2.3 and (2.9), implies (2.6) and (2.7). ■

Propositions 2.6 and 2.7 immediately yield the following theorem.

Theorem 2.8 *Let the assumption of Proposition 2.7 be satisfied. Suppose that the ground state of H_0 is unique. Then, for all $\alpha \in \mathcal{I}$, the ground state of $H(\alpha)$ is unique.*

A sufficient condition for (2.9) and (2.10) to hold is given in the following proposition.

Proposition 2.9 *Assume (A.1) and (A.2). Suppose that $E_0(\alpha) < E_1(\alpha)$ for all $\alpha \in \mathcal{I}$, and $E_1(\alpha)$ is continuous in $\alpha \in \mathcal{I}$. Then (2.9) and (2.10) hold.*

Proof. Fix $\alpha \in \mathcal{I}$ arbitrarily. Let ε be such that

$$0 < \varepsilon < \frac{E_1(\alpha) - E_0(\alpha)}{3}. \tag{2.15}$$

By Lemma 2.3, there exists a constant $K_{0,\alpha} > 0$ such that if $0 \leq |\kappa| \leq K_{0,\alpha}$, then $\alpha \pm K_{0,\alpha} \in \mathcal{I}$ and

$$|E_0(\alpha) - E_0(\alpha+\kappa)| < \varepsilon. \tag{2.16}$$

Since $E_1(\alpha)$ is continuous in $\alpha \in \mathcal{I}$ by the present assumption, there exists a constant $K_{1,\alpha} > 0$ such that if $0 \leq |\kappa| \leq K_{1,\alpha}$, then $\alpha \pm K_{1,\alpha} \in \mathcal{I}$ and

$$|E_1(\alpha) - E_1(\alpha + \kappa)| < \varepsilon. \tag{2.17}$$

Let

$$L_\alpha := \min\{K_{0,\alpha}\,,\ K_{1,\alpha}\}. \tag{2.18}$$

Then $\alpha \pm L_\alpha \in \mathcal{I}$, i.e., (2.9) holds. By Lemma 2.3, there exists a constant κ_0 with $0 \leq |\kappa_0| \leq L_\alpha$ such that

$$\inf_{0 \leq |\kappa| \leq L_\alpha} E_0(\alpha + \kappa) = E_0(\alpha + \kappa_0).$$

Hence we have

$$|E_0(\alpha) - \inf_{0 \leq |\kappa| \leq L_\alpha} E_0(\alpha + \kappa)| = |E_0(\alpha) - E_0(\alpha + \kappa_0)| < \varepsilon. \tag{2.19}$$

Since $E_1(\alpha) - E_0(\alpha)$ is continuous in $\alpha \in \mathcal{I}$, there exists a constant κ_1 with $0 \leq |\kappa_1| \leq L_\alpha$ such that

$$\inf_{0 \leq |\kappa| \leq L_\alpha} \{E_1\,(\alpha + \kappa) - E_0\,(\alpha + \kappa)\} = E_1(\alpha + \kappa_1) - E_0(\alpha + \kappa_1),$$

Hence we have by (2.15), (2.16), (2.17) and (2.19)

$$\begin{aligned}
&\inf_{0 \leq |\kappa| \leq L_\alpha} \{E_1\,(\alpha + \kappa) - E_0\,(\alpha + \kappa)\} \\
=\ & E_1(\alpha + \kappa_1) - E_0(\alpha + \kappa_1) \\
=\ & (E_1\,(\alpha + \kappa_1) - E_1\,(\alpha)) + (E_0\,(\alpha) - E_0\,(\alpha + \kappa_1)) + (E_1\,(\alpha) - E_0\,(\alpha)) \\
\geq\ & -2\varepsilon + (E_1\,(\alpha) - E_0\,(\alpha)) \\
>\ & \varepsilon \\
>\ & |E_0(\alpha) - \inf_{0 \leq |\kappa| \leq L_\alpha} E_0(\alpha + \kappa)|.
\end{aligned}$$

Thus (2.10) follows. ■

Theorem 2.8 and Proposition 2.9 imply the following theorem:

Theorem 2.10 *Assume (A.1), (A.2) and that* $E_0(\alpha) < E_1(\alpha)$ *for all* $\alpha \in \mathcal{I}$ *and* $E_1(\alpha)$ *is continuous in* $\alpha \in \mathcal{I}$. *Suppose that the ground state of* H_0 *is unique. Then, for all* $\alpha \in \mathcal{I}$, *the ground state of* $H(\alpha)$ *is unique.*

3 A Special Class of Self-adjoint Operators

Let H_I be a symmetric operator on $\mathcal{H}$ satisfying the following condition:

(B.1) $D(H_0) \subset D(H_I)$ and there exist constants $a, b > 0$ such that, for all $\psi \in D(H_0)$,

$$\|H_I\psi\| \le a\|H_0\psi\| + b\|\psi\|. \tag{3.1}$$

We define

$$T(\alpha) := H_0 + \alpha H_I \tag{3.2}$$

with $\alpha \in \mathbf{R}$ a coupling constant. Let $\mathcal{I}_a$ be an open interval from $-1/a$ to $1/a$:

$$\mathcal{I}_a := \left(-\frac{1}{a}, \frac{1}{a}\right). \tag{3.3}$$

By the Kato-Rellich theorem (e.g., [RS2, Theorem X.12]), for all $\alpha \in \mathcal{I}_a$, $T(\alpha)$ is self-adjoint with $D(T(\alpha)) = D(H_0)$ and bounded from below with

$$E_0(T(\alpha)) \ge E_0 - \max\left\{\frac{b|\alpha|}{1 - a|\alpha|}, |\alpha|(a|E_0| + b)\right\}, \tag{3.4}$$

where

$$E_0 := E_0(H_0). \tag{3.5}$$

We assume the following:

(B.2) For all $\alpha \in \mathcal{I}_a$, $E_0(T(\alpha))$ is an isolated eigenvalue of $T(\alpha)$.

Theorem 3.1 *Assume (B.1), (B.2) and that $\mathcal{H}$ has the orthogonal decomposition (1.3) such that, for all $\alpha \in \mathcal{I}_a$, $T(\alpha)$ is reduced by $\mathcal{H}_0$. Suppose that, for all $\alpha \in \mathcal{I}_a$, the ground state $T(\alpha)$ is unique and that the ground state of H_0 is in $\mathcal{H}_0$. Then, for all $\alpha \in \mathcal{I}_a$, the ground state of $T(\alpha)$ is in $\mathcal{H}_0$.*

Corollary 3.2 *Assume (B.1), (B.2) and that $\mathcal{H}$ has the orthogonal decomposition (1.3) such that, for all $\alpha \in \mathcal{I}_a$, $T(\alpha)$ is reduced by $\mathcal{H}_0$. Suppose that the ground state of H_0 is unique and in $\mathcal{H}_0$. Moreover, suppose that there exists an $\alpha' \in \mathcal{I}_a$ such that $T(\alpha')$ has a ground state which is not in $\mathcal{H}_0$. Then, for some $\alpha_0 \in \mathcal{I}_a \setminus \{0\}$, the ground state of $T(\alpha_0)$ is degenerate.*

We prove these results by applying Theorem 2.1 and Corollary 2.2. To do this we need a lemma.

Let

$$Q_z(\alpha) := (T(\alpha) - z)^{-1}, \quad z \in \rho(T(\alpha)). \tag{3.6}$$

Lemma 3.3 *Assume (B.1). Then, for all* $z \in \mathbf{C} \setminus \mathbf{R}$*, the operator-valued function:* $\alpha \to Q_z(\alpha)$ *is continuous on* $\mathcal{I}_a$ *in operator norm topology.*

Proof. Fix $\alpha \in \mathcal{I}_a$ and $z \in \mathbf{C} \setminus \mathbf{R}$ arbitrarily. Since $D(T(\alpha)) = D(T(\alpha + \kappa)) = D(H_0)$ for every $\kappa \in \mathbf{R}$ with $\alpha + \kappa \in \mathcal{I}$, we have

$$Q_z(\alpha + \kappa) - Q_z(\alpha) = -\kappa Q_z(\alpha + \kappa) H_I Q_z(\alpha). \tag{3.7}$$

For $\Psi \in D(H_0)$, we have by the triangle inequality and (3.1)

$$\begin{aligned} \|H_0\Psi\| &\leq \|T(\alpha)\Psi\| + |\alpha|\|H_I\Psi\| \\ &\leq \|T(\alpha)\Psi\| + a|\alpha|\|H_0\Psi\| + b|\alpha|\|\Psi\|. \end{aligned}$$

Hence

$$\|H_0\Psi\| \leq \frac{1}{1 - a|\alpha|}\|T(\alpha)\Psi\| + \frac{b|\alpha|}{1 - a|\alpha|}\|\Psi\|,$$

where $|\alpha|$ satisfies that $0 < |\alpha| < 1/a$. Putting this into (3.1), we obtain

$$\|H_I\Psi\| \leq \frac{a}{1 - |\alpha|a}\|T(\alpha)\Psi\| + \left(\frac{ab|\alpha|}{1 - a|\alpha|} + b\right)\|\Psi\|, \tag{3.8}$$

which implies that $H_I Q_z(\alpha)$ is bounded. Since $\|Q_z(\alpha + \kappa)\| \leq 1/|\Im z|$, we obtain

$$\|Q_z(\alpha + \kappa) - Q_z(\alpha)\| \leq \frac{|\kappa|}{|\Im z|}\|H_I Q_z(\alpha)\| \to 0$$

as $\kappa \to 0$. Hence the desired result follows. ∎

Proof of Theorem 3.1

By th present assumption, (3.4) and Lemma 3.3, the assumption of Theorem 2.1 with $H(\alpha) = T(\alpha)$ and $\mathcal{I} = \mathcal{I}_a$ is satisfied. Thus the assertion follows. ∎

Remark 3.1 *Assume (B.1) and fix* $\alpha \in \mathcal{I}_a$ *arbitrarily. Then* $T(\alpha + \kappa)$ *is an analytic family of type (A) near* $\kappa = 0$. This follows from (3.8) and a general fact [RS4, p.16, Lemma].

We can obtain results on uniqueness of ground states of $T(\alpha)$ by applying the results in §2.2 to the operator $T(\alpha)$. But we omit writing down them.

4 Application to the WW Model

In this section we apply the main results of Section 3 to the WW model. We first recall the definition of the WW model.

We take a Hilbert space of bosons to be

$$\mathcal{F}_{\rm b} := \mathcal{F}_{\rm b}\left(L^2\left(\mathbf{R}^d\right)\right) := \bigoplus_{n=0}^{\infty}\left[\otimes_{\rm sym}^n L^2\left(\mathbf{R}^d\right)\right] \tag{4.1}$$

($d \in \mathbf{N}$) the symmetric Fock space over $L^2(\mathbf{R}^d)$ ($\otimes_{\rm s}^n \mathcal{K}$ denotes the n-fold symmetric tensor product of a Hilbert space $\mathcal{K}$, $\otimes_{\rm s}^0 \mathcal{K} := \mathbf{C}$). In this paper, we set both of $\hbar$ (the Planck constant divided by 2π) and c (the speed of light) one, i.e., $\hbar = c = 1$.

Let $\omega : \mathbf{R}^d \to [0, \infty)$ be Borel measurable such that $0 < \omega(k) < \infty$ for almost everywhere (a.e.) $k \in \mathbf{R}^d$ with respect to the d-dimensional Lebesgue measure and

$$H_{\rm b} := d\Gamma(\omega),$$

the second quantization of the multiplication operator on $L^2(\mathbf{R}^d)$ by the function ω [RS2, §X.7].

Let λ be a function on $\mathbf{R}^d$. We assume the following (W.1) and (W.2):

(W.1) The function λ is continuous on $\mathbf{R}^d$, not identically zero with $\lambda, \lambda/\omega \in L^2(\mathbf{R}^d)$.

(W.2) The function $\omega(k)$ is continuous with

$$\lim_{|k|\to\infty} \omega(k) = \infty, \tag{4.2}$$

and there exist constants $\gamma_\omega > 0$ and $C_\omega > 0$ such that

$$|\omega(k) - \omega(k')| \leq C_\omega |k - k'|^{\gamma_\omega} \left(1 + \omega\left(k\right) + \omega\left(k'\right)\right), \qquad k, k' \in \mathbf{R}^d. \tag{4.3}$$

We define a matrix c by

$$c := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{4.4}$$

The Hamiltonian $H_{\rm ww}(\alpha)$ of the WW model is defined by

$$H_{\rm ww}(\alpha) := H_0 + \alpha H_I \tag{4.5}$$

acting in

$$\mathcal{H} = \mathbf{C}^2 \otimes \mathcal{F}_{\rm b} \tag{4.6}$$

with

$$H_0 := \mu_0 c^* c \otimes I + I \otimes H_{\rm b}, \tag{4.7}$$
$$H_I := c^* \otimes a(\lambda) + c \otimes a(\lambda)^*, \tag{4.8}$$

where $\mu_0, \alpha \in \mathbf{R} \setminus \{0\}$ are constant parameters and $a(\cdot)$ (resp. I) denotes the annihilation operator on $\mathcal{F}_{\rm b}$ (resp. identity operator). It is easy to prove the following fact:

Lemma 4.1 (i) *The operator H_I is infinitesimally small with respect to H_0.*

(ii) *For all $\alpha \in \mathbf{R}$, $H_{\rm ww}(\alpha)$ is self-adjoint with $D(H_{\rm ww}(\alpha)) = D(H_0)$ and bounded from below.*

The WW model has a conservation law for a kind of the particle number in the sense described below. Let σ_3 be the third of the Pauli matrices:

$$\sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.9}$$

and define

$$N_P := \frac{1+\sigma_3}{2} \otimes I + I \otimes N_{\rm b}, \tag{4.10}$$

where $N_{\rm b} := d\Gamma(I)$ is the boson number operator. The operator N_P was introduced in [HS95, §6]. Let $P^{(\ell)}$ be the orthogonal projection onto the ℓ-particle space of $\mathcal{F}_{\rm b}$ ($\ell \geq 0$). Then we have

$$N_{\rm b} = \sum_{\ell=0} \ell P^{(\ell)}. \tag{4.11}$$

The spectral resolution of N_P is given by

$$N_P = \sum_{\ell=0} \ell P_\ell, \tag{4.12}$$

where

$$P_\ell := \begin{cases} \dfrac{1-\sigma_3}{2} \otimes P^{(0)} & \text{if } \ell = 0, \\ \\ \dfrac{1+\sigma_3}{2} \otimes P^{(\ell-1)} + \dfrac{1-\sigma_3}{2} \otimes P^{(\ell)} & \text{if } \ell \in \mathbf{N}. \end{cases} \tag{4.13}$$

It is easy to see that, for every $\alpha \in \mathbf{R}$ and each $\ell \in \{0\} \cup \mathbf{N}$,

$$P_\ell H_{\rm ww}(\alpha) \subset H_{\rm ww}(\alpha) P_\ell. \tag{4.14}$$

Hence $H_{\rm ww}(\alpha)$ is reduced by $P_\ell \mathcal{H}$.

Let

$$\mathcal{H}_0 := (P_0 + P_1)\,\mathcal{H} \tag{4.15}$$

and

$$\mathcal{H}_1 := \mathcal{H}_0^{\perp} \text{ (the orthogonal complement of } \mathcal{H}_0). \tag{4.16}$$

Then

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1. \tag{4.17}$$

The following lemma easily follows:

Lemma 4.2 (i) *For each $\alpha \in \mathbf{R}$, $H_{\mathrm{WW}}(\alpha)$ is reduced by $\mathcal{H}_j, j = 1, 2$.*

(ii) *H_0 has a unique ground state in $\mathcal{H}_0$.*

Let

$$E_0^{\mathrm{WW}}(\alpha) := E_0\left(H_{\mathrm{WW}}\left(\alpha\right)\right) \tag{4.18}$$

and

$$\mu := \text{ess.} \inf_{k \in \mathbf{R}^d} \omega(k) \geq 0. \tag{4.19}$$

We say that the WW model is massive (resp. massless) if $\mu > 0$ (resp. $\mu = 0$).

Proposition 4.3 ([Ar99, Remark 3.1], [AH99, Proposition 6.10(i)])

$$\sigma\left(H_{\mathrm{WW}}\left(\alpha\right)\right) = [E_0^{\mathrm{WW}}\left(\alpha\right) + \mu\,,\,\infty),$$

where $\sigma_{\mathrm{ess}}(\cdot)$ denotes essential spectrum.

We define

$$D_\mu^\alpha(z) := -z + \mu_0 - \alpha^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k) - z}, \qquad z \in \mathbf{C}_\mu := \mathbf{C} \setminus [\mu\,,\,\infty) \tag{4.20}$$

The limit

$$C_\mu := \lim_{t \downarrow 0} \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k) - \mu + t} \tag{4.21}$$

exists or is infinity. In the former case, $C_\mu > 0$ by (W.1). It is easy to see that $D_\mu^\alpha(x)$ is monotone decreasing in $x < \mu$. Hence the limit

$$d_\mu^\alpha := \lim_{x \uparrow \mu} D_\mu^\alpha(x) \tag{4.22}$$

exists or is $-\infty$ and

$$d_\mu^\alpha = -\mu + \mu_0 - \alpha^2 C_\mu.$$

Let

$$\beta_0 := \begin{cases} \dfrac{\mu_0 - \mu}{C_\mu} & \text{if } 0 < C_\mu < \infty, \\ \\ 0 & \text{if } C_\mu = \infty. \end{cases} \tag{4.23}$$

and

$$\mathcal{A}_\mu := \left\{ \alpha \in \mathbf{R} \mid -\infty \le d_\mu^\alpha < 0 \right\} = \{\alpha \in \mathbf{R} | \alpha^2 > \beta_0\}. \tag{4.24}$$

For all $\alpha \in \mathcal{A}_\mu$, there exists a unique zero $E_{\mathrm{ww}}(\alpha)$ of $D_\mu^\alpha(z)$:

$$E_{\mathrm{ww}}(\alpha) = \mu_0 - \alpha^2 \int_{\mathbf{R}^d} dk \, \frac{|\lambda(k)|^2}{\omega(k) - E_{\mathrm{ww}}(\alpha)}. \tag{4.25}$$

Proposition 4.4 ([H99, Theorem 2.3 (b),(c)]) *Let $\alpha \in \mathcal{A}_\mu$. Assume either (i) $\mu > 0$ or (ii) $\mu = 0$ with $\nabla\omega \in L^\infty(\mathbf{R}^d)$. Then there exists a constant $\alpha_{\mathrm{ww}} \in \mathcal{A}_\mu \cap (0, \infty)$ such that, for all $|\alpha| > \alpha_{\mathrm{ww}}$,*

$$\{E_0^{\mathrm{ww}}(\alpha), \ E_{\mathrm{ww}}(\alpha), \ 0\} \subset \sigma_{\mathrm{p}}(H_{\mathrm{ww}}(\alpha))$$

with

$$E_0^{\mathrm{ww}}(\alpha) < \min\{E_{\mathrm{ww}}(\alpha), 0\}$$

and

$$\Psi_0(\alpha) \notin \mathcal{H}_0.$$

Let

$$E_1^{\mathrm{ww}}(\alpha) := \inf\{\sigma(H_{\mathrm{ww}}(\alpha)) \setminus \{E_0^{\mathrm{ww}}(\alpha)\}\} \tag{4.26}$$

and

$$\varepsilon_0 := \min\{0, \mu_0\}, \quad \varepsilon_1 := \max\{0, \mu_0\}. \tag{4.27}$$

Note that, if $E_1^{\mathrm{ww}}(\alpha)$ is an eigenvalue of $H_{\mathrm{ww}}(\alpha)$, then each eigenvector corresponding to it physically describes one of the first excited states of the WW model.

Theorem 4.5 *Let $\mu > 0$. Then :*

(i) *There exists a constant $\alpha_0 \in \mathcal{A}_\mu$ such that $H_{\mathrm{ww}}(\alpha_0)$ has degenerate ground states.*

(ii) *There exists a constant $\alpha_1 \in \mathcal{A}_\mu$ such that $E_1^{\mathrm{ww}}(\alpha_1)$ is an eigenvalue of $H_{\mathrm{ww}}(\alpha_1)$ and*

$$E_1^{\mathrm{ww}}(\alpha_1) < E_0^{\mathrm{ww}}(\alpha_1) + \mu = \inf\sigma(H_{\mathrm{ww}}(\alpha_1)). \tag{4.28}$$

Moreover, if $0 < \mu < |\mu_0|$, then

$$E_1^{\mathrm{ww}}(\alpha_1) < \varepsilon_1. \tag{4.29}$$

Proof. (i) Since $\mu > 0$, it follows from [AH97, Theorem 1.2] that, for all $\alpha \in \mathbf{R}$, $H_{\mathrm{WW}}(\alpha)$ has a ground state and $E_0^{\mathrm{WW}}(\alpha)$ is an isolated eigenvalue of $H_{\mathrm{WW}}(\alpha)$. These facts together with Lemmas 4.1, 4.2, Proposition 4.4 imply that the assumption of Corollary 3.2 with $T(\alpha) = H_{\mathrm{WW}}(\alpha)$ is satisfied. Hence there exists a constant $\alpha_0 \neq 0$ such that the ground state of $H_{\mathrm{WW}}(\alpha_0)$ is degenerate. If $\alpha_0 \notin \mathcal{A}_\mu$ so that $d_\mu^{\alpha_0} \geq 0$, then, by [AH99, Theorem 6.14(i)], $H_{\mathrm{WW}}(\alpha_0)$ has a unique ground state. But this is a contradiction.

(ii) By Lemma 4.3, we have for all $\alpha \in \mathbf{R}$

$$E_0^{\mathrm{WW}}(\alpha) < E_1^{\mathrm{WW}}(\alpha) \leq E_0^{\mathrm{WW}}(\alpha) + \mu.$$

Suppose that, for all $\alpha \in \mathbf{R} \setminus \{0\}$,

$$E_1^{\mathrm{WW}}(\alpha) = \inf \sigma_{\mathrm{ess}}\left(H_{\mathrm{WW}}(\alpha)\right) = E_0^{\mathrm{WW}}(\alpha) + \mu.$$

By an application of Lemma 2.4, $E_0^{\mathrm{WW}}(\alpha)$ is continuous in $\alpha \in \mathbf{R}$. Hence so is $E_1^{\mathrm{WW}}(\alpha)$. Then, by an application of Theorem 2.10, for all $\alpha \in \mathbf{R}$, the ground state of $H_{\mathrm{WW}}(\alpha)$ is unique. But this contradicts part (i). Hence there exists a constant $\alpha_1 \neq 0$ such that (4.28) holds and $E_1^{\mathrm{WW}}(\alpha_1)$ is an eigenvalue of $H_{\mathrm{WW}}(\alpha_1)$. We show that $\alpha_1 \in \mathcal{A}_\mu$. If $\mu_0 < 0$, then $d_\mu^\alpha < 0$ for all $\alpha \in \mathbf{R}$, which implies $\mathcal{A}_\mu^\alpha = \mathbf{R}$ $(\mu_0 < 0)$. Hence $\alpha_1 \in \mathcal{A}_\mu$. Let $\mu_0 > 0$. Suppose that $d_\mu^{\alpha_1} \geq 0$. Then, by [AH99, Theorem 6.14(i)] we have $E_1^{\mathrm{WW}}(\alpha_1) = E_0^{\mathrm{WW}}(\alpha_1) + \mu$, which contradicts (4.28). Hence $d_\mu^{\alpha_1} < 0$. Therefore $\alpha_1 \in \mathcal{A}_\mu$.

Finally we prove (4.29). Let $\mu < |\mu_0|$. Since $0 \in \sigma_{\mathrm{p}}\left(H_{\mathrm{WW}}(\alpha)\right)$ for all $\alpha \in \mathbf{R}$ by [AH99, Proposition 6.13], we have

$$E_1^{\mathrm{WW}}(\alpha_1) < E_0^{\mathrm{WW}}(\alpha_1) + \mu \leq 0 + \mu = \mu.$$

We first consider the case $0 < \mu_0$. In this case, $\varepsilon_0 = 0$, $\varepsilon_1 = \mu_0$. Hence $E_1^{\mathrm{WW}}(\alpha_1) < \varepsilon_1$. We next consider the case $\mu_0 < 0$. In this case, $\varepsilon_0 = \mu_0$ and $\varepsilon_1 = 0$. Since $\alpha_1 \in \mathcal{A}_\mu$(i.e., $d_\mu^{\alpha_1} < 0$), we have by [AH99, Proposition 6.13 (ii)] $0, E_{\mathrm{WW}}(\alpha_1) \in \sigma_{\mathrm{p}}\left(H_{\mathrm{WW}}(\alpha)\right)$ with $E_{\mathrm{WW}}(\alpha_1) < 0$. Since $\mu_0 < 0$, we have

$$D_\mu^{\alpha_1}(\mu_0) = -\alpha_1^2 \int_{\mathbf{R}^d} dk \frac{|\lambda(k)|^2}{\omega(k) - \mu_0} < 0.$$

This implies that $E_{\mathrm{WW}}(\alpha_1) < \mu_0$, since $D_\mu^{\alpha_1}(x)$ is monotone decreasing in $x < \mu$ and $D_\mu^{\alpha_1}\left(E_{\mathrm{WW}}\left(\alpha_1\right)\right) = 0$. Hence we have

$$E_1^{\mathrm{WW}}(\alpha_1) < E_0^{\mathrm{WW}}(\alpha_1) + \mu \leq E_{\mathrm{WW}}(\alpha_1) + \mu < \mu_0 + \mu < 0 = \varepsilon_1.$$

Thus (4.29) follows. ∎

Remark 4.1 Generally speaking, in a quantum field model, it is difficult to prove the existence of an eigenvalue corresponding to the first excited states of the model. There are many papers stating the *possibility* of the existence of the first excited states, but, to authors' best knowledge, there is few papers pointing out the real existence of those. In this sense, Theorem 4.5-(ii) has a meaning. Moreover, note that, if $0 < \mu < |\mu_0| = \varepsilon_1 - \varepsilon_0$, then ε_1 is an embedded eigenvalue of H_0. In this case too, Theorem 4.5-(ii) holds, showing that, in the WW model, the embedded eigenvalue does not necessarily disappear under the perturbation αH_I. The phenomena mentioned in Theorem 4.5 do not occur in the region of the coupling constant treated by Hübner and Spohn [HS95, §6] and ourselves in [AH99, Theorem 6.14(i)].

Remark 4.2 We may expect that, in the massless case too (i.e. $\mu = 0$), Theorem 4.5-(i) holds.

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References

[Ar99] Arai, A.: Essential Spectrum of a Self-Adjoint Operators on an Abstract Hilbert Space of Fock Type and Applications to Quantum Field Hamiltonians, Hokkaido University Preprint Series in Mathematics # 445, February 1999.

[AH97] Arai, A. and Hirokawa, M.: On the Existence and Uniqueness of Ground States of a Generalized Spin-Boson Model, *J. Funct. Anal.* **151** (1997), 455–503.

[AH99] Arai, A. and Hirokawa, M.: Ground States of a General Class of Quantum Field Hamiltonians, to appear in *Rev. Math. Phys.*. (mp_arc 99-179)

[H99] Hirokawa, M.: Remarks on the Ground State Energy of the Spin-Boson Model. An Application of the Wigner-Weisskopf Model, (preprint, 1999).

[HS95] Hübner, M. and Spohn, H.: Spectral properties of the spin-boson Hamiltonian, *Ann. Inst. Henri. Poincaré* **62** (1995), 289–323.

[Ka] Kato, T.: *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin Heidelberg New York, 1980.

[RS1] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics Vol. I*, Academic Press, New York, 1975.

[RS2] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics Vol. II*, Academic Press, New York, 1975.

[RS4] Reed, M. and Simon, B.: *Methods of Modern Mathematical Physics Vol. IV*, Academic Press, New York, 1978.

[WW] Weisskopf, V. F. and Wigner, E. P.: Berechnung der natürlichen Linienbreite auf Grund der Diracschen Lichttheorie, *Z. Phys.* **63** (1930), 54-73.