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# Vafa-Witten Theory on $N$=2 and $N$=4 Twisted Superspace in Four Dimensions

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**Abstract**

We construct a new off-shell twisted real form of the hypermultiplet with a scalar and an anti-self-dual tensor superfields. Using the $N = 2$ twisted superspace formalism, we construct a Donaldson-Witten theory coupled to the real form of the hypermultiplet. We show that this action possesses the Vafa-Witten type $N = 4$ twisted supersymmetry at the on-shell level. We also reconstruct the action using a $N = 4$ twisted superconnection formalism.

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# 1 Introduction

One of the important characteristic of a topological twist [1, 2] is that the BRST charge of a quantized topological field theory relates to a supercharge. One can construct a new algebra which is called twisted superalgebra by using the twisting procedure. These algebras consist of scalar, vector and tensor generators. The twisted supersymmetry is investigated in various models and various dimensions [3–16]. We are especially interested in Dirac-Kähler twist [17] which is connected with lattice fermion. The advantage of this twist is that ghost fields in the quantized topological field theory are directly related to the gaugino or matter fields in supersymmetric theory and that an untwisted theory is easily constructed from the twisted supersymmetric theory. A twisted superspace formalism are then constructed from the twisted superalgebra [18, 19]. Twisted supersymmetric Yang-Mills theories on the superspace formulation based on the Dirac-Kähler twist was derived in two and four dimensions [20, 21]. Three dimensional twisted supersymmetric Yang-Mills theory was investigated in [22]. A recent development of the twisted superspace is that a path-integral quantization procedure with respect to the subsuperspace which consists of scalar and vector fermionic coordinates is proposed in four dimensional $N$=2 super Yang-Mills case [23, 24].

In previous paper [25] we constructed a $N$=2 twisted superspace formalism with a central charge using the Dirac-Kähler twist. Our formulation was based on a tensor formulation coming from the Dirac-Kähler twisting procedure. We proposed a $N$=2 twisted multiplet with a central charge. This multiplet includes a bosonic vector field. We then proposed a new off-shell twisted supersymmetric action and gave a gauge covariant version of this action. It turned out that this action plus Donaldson-Witten action has the on-shell $N$=4 TSUSY and the four-dimensional Dirac-Kähler twist is equivalent to the Marcus's twist [26].

In this paper we propose a new $N$=2 twisted real form of the hypermultiplet with a central charge [27–29]. This multiplet consist of a bosonic scalar and a bosonic self-dual antisymmetric tensor fields. In this case a covariantized on-shell action similarly possesses a $N$=4 twisted supersymmetry. An algebra of this symmetry is different from the Marcus's one. An assignment of ghost number is especially different in these theories. We insist that this theory is equivalent to the Vafa-Witten theory [30, 31].

Another important motivation of this work comes from the recent study of lattice SUSY. It is well known that the Dirac-Kähler fermion mechanism is well-defined on the lattice [32–34]. Supersymmetric models with modified Leibniz rule on the lattice was studied in [35–38]. The characteristic of these models is that all of the twisted supersymmetries are exactly defined on the lattice. Other models by using the Dirac-Kähler fermion without modified Leibniz rule possess the partial twisted supersymmetries. These models was investigated in [39–49]. In the recent development of lattice SUSY, there are matrix formulations on which we impose $Z_N$ orbifold conditions [50–66]. These models are constructed under the orbifold condition coming from some global symmetries of some mother theories. The lattice

SUSY models based on the twisted SUSY may be related to the one based on the matrix formulation.

This paper is organized as follows. In Sec. 2 we give a brief introduction of the $N$=2 twisted superspace formulation with a central charge based on the Dirac-Kähler twist. In Sec. 3 we propose a twisted real form of the hypermultiplet which is constructed by the scalar and tensor superfields. we summarize a correspondence between the theories of the $N$=2 hypermultiplets and the twisted theories. In Sec. 4 we derive a Donaldson-Witten theory coupling to the hypermultiplet. We show that the supersymmetries of the theory become the $N$=4 twisted supersymmetry which is correspondence with Vafa-Witten type one. In Sec. 5 we reconstruct the $N$=4 twisted SYM theory with respect to the $N$=4 twisted superspace formulation. We summarize the results in section 5. We provide several appendices to summarize the notations and show the full transformation of on-shell $N$=4 TSUSY.

# 2 $N = 2$ twisted SUSY with central charge

In a previous paper [25] we derived the N=2 twisted SUSY algebra and superspace formalism with a central charge with respect to the Dirac-Kähler fermion [67, 68]. This algebra is a twisted version of the ordinary $N$=2 SUSY algebra with a central charge [69, 70]. $N = 2$ twisted SUSY generators consist of a scalar, vector, anti-self-dual tensor and central charge $\{s^+, s^+_\mu, s^+_{\mu\nu}$ and $Z\}$. The algebra of the twisted supercharges are

$$\begin{aligned}
&\{s^+, s^+_\mu\} = P_\mu, \qquad &&\{s^+_A, s^+_\mu\} = -\delta^+_{A,\mu\nu}P^\nu, \qquad &&\{s^+, s^+_A\} = 0,\\
&\{s^+, s^+\} = Z, \qquad &&\{s^+_\mu, s^+_\nu\} = \delta_{\mu\nu}Z, \qquad &&\{s^+_A, s^+_B\} = \delta^+_{A,B}Z,
\end{aligned} \tag{2.1}$$

where the others (anti)commute, the capital $\{A\}$ denotes the second rank tensor indices $\mu\nu$ and $\delta^+_{\mu\nu,\rho\sigma}$ ($\delta^+_{A,B}$) is defined as $\delta^+_{\mu\nu,\rho\sigma} \equiv \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \epsilon_{\mu\nu\rho\sigma}$. Through this paper we consider the Euclidean flat spacetime.

We now introduce the $N = 2$ twisted superspace based on the algebras. The superspace consist of the bosonic coordinates $x_\mu$, $z$ and fermionic ones $\theta^+$, $\theta^+_\mu$, $\theta^+_{\mu\nu}$, where $z$ is a bosonic parameter corresponding to the central charge $Z$ and $\theta^+_{\mu\nu}$ are fermionic anti-self-dual tensor parameters with $\theta^+_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\theta^{+\rho\sigma}$. We define differential operators $\{\mathcal{Q}^+, \mathcal{Q}^+_\mu, \mathcal{Q}^+_A, \mathcal{Z}\}$ corresponding with the twisted supercharges $\{s^+, s^+_\mu, s^+_A, Z\}$ as follows:

$$\begin{aligned}
\mathcal{Q}^+ &= \frac{\partial}{\partial\theta^+} + \frac{i}{2}\theta^{+\mu}\partial_\mu + \frac{i}{2}\theta^+\frac{\partial}{\partial z},\\
\mathcal{Q}^+_\mu &= \frac{\partial}{\partial\theta^{+\mu}} + \frac{i}{2}\theta^+\partial_\mu - \frac{i}{2}\theta^+_{\mu\nu}\partial^\nu + \frac{i}{2}\theta^+_\mu\frac{\partial}{\partial z},\\
\mathcal{Q}^+_A &= \frac{\partial}{\partial\theta^{+A}} - \frac{i}{2}\delta^+_{A,\mu\nu}\theta^{+\mu}\partial^\nu + \frac{i}{2}\theta^+_A\frac{\partial}{\partial z},\\
\mathcal{Z} &= -i\frac{\partial}{\partial z}.
\end{aligned} \tag{2.2}$$

These differential operators satisfy the following anticommuting relations:

$$\begin{aligned} &\{\mathcal{Q}^+, \mathcal{Q}^+_\mu\} = i\partial_\mu, \qquad \{\mathcal{Q}^+_A, \mathcal{Q}^+_\mu\} = -i\delta^+_{A,\mu\nu}\partial^\nu, \qquad \{\mathcal{Q}^+, \mathcal{Q}^+_A\} = 0, \\ &\{\mathcal{Q}^+, \mathcal{Q}^+\} = -\mathcal{Z}, \qquad \{\mathcal{Q}^+_\mu, \mathcal{Q}^+_\nu\} = -\delta_{\mu\nu}\mathcal{Z}, \qquad \{\mathcal{Q}^+_A, \mathcal{Q}^+_B\} = -\delta^+_{A,B}\mathcal{Z}, \end{aligned} \tag{2.3}$$

where $\frac{\partial}{\partial\theta^{+A}}\theta^+_B \equiv \delta^+_{A,B}$ and $\mathcal{Z}$ commutes all the differential operators. The sign of the spacetime derivative is reversed with respect to the algebra (2.1). We then introduce differential operators $\{\mathcal{D}^+, \mathcal{D}^+_\mu, \mathcal{D}^+_A\}$ which anticommute with the differential operators $\{\mathcal{Q}^+, \mathcal{Q}^+_\mu, \mathcal{Q}^+_A\}$,

$$\begin{aligned} \mathcal{D}^+ &= \frac{\partial}{\partial\theta^+} - \frac{i}{2}\theta^{+\mu}\partial_\mu - \frac{i}{2}\theta^+\frac{\partial}{\partial z}, \\ \mathcal{D}^+_\mu &= \frac{\partial}{\partial\theta^{+\mu}} - \frac{i}{2}\theta^+\partial_\mu + \frac{i}{2}\theta^+_{\mu\nu}\partial^\nu - \frac{i}{2}\theta^+_\mu\frac{\partial}{\partial z}, \\ \mathcal{D}^+_A &= \frac{\partial}{\partial\theta^{+A}} + \frac{i}{2}\delta^+_{A,\rho\sigma}\theta^{+\rho}\partial^\sigma - \frac{i}{2}\theta^+_A\frac{\partial}{\partial z}. \end{aligned} \tag{2.4}$$

These operators $\{\mathcal{D}^+, \mathcal{D}^+_\mu, \mathcal{D}^+_A\}$ satisfy the following relations:

$$\begin{aligned} &\{\mathcal{D}^+, \mathcal{D}^+_\mu\} = -i\partial_\mu, \qquad \{\mathcal{D}^+_A, \mathcal{D}^+_\mu\} = i\delta^+_{A,\mu\nu}\partial^\nu, \qquad \{\mathcal{D}^+, \mathcal{D}^+_A\} = 0, \\ &\{\mathcal{D}^+, \mathcal{D}^+\} = \mathcal{Z}, \qquad \{\mathcal{D}^+_\mu, \mathcal{D}^+_\nu\} = \delta_{\mu\nu}\mathcal{Z}, \qquad \{\mathcal{D}^+_A, \mathcal{D}^+_B\} = \delta^+_{A,B}\mathcal{Z}, \end{aligned} \tag{2.5}$$

where $\mathcal{Z}$ commutes these operators $\{\mathcal{D}^+, \mathcal{D}^+_\mu, \mathcal{D}^+_A\}$.

# 3 A new twisted hypermultiplet

In the previous paper we constructed the twisted multiplet introducing the bosonic superfield with vector index. We now introduce bosonic superfields with a scalar and an anti-self-dual tensor indices, $\mathcal{V}^+$ and $\mathcal{V}^+_A$ respectively. Since a general superfield has many component fields, we need to eliminate superfluous fields. We then use the R-symmetry in order to impose a condition on the superfield. We also introduce R-transformations for the superfield $\mathcal{V}^+, \mathcal{V}^+_A$ and supercharges:

$$\begin{aligned} R^+_A\mathcal{V}^+ &= -\frac{i}{2}\mathcal{V}^+_A, \\ R^+_A\mathcal{V}^+_B &= \frac{i}{2}\delta^+_{A,B}\ \mathcal{V}^+ - \frac{i}{8}\Gamma^+_{ABC}\ \mathcal{V}^{+C}, \\ R^+_A s^+_\mu &= -\frac{i}{2}\delta^+_{A,\mu\nu}s^{+\nu}, \\ R^+_A s^+ &= -\frac{i}{2}s^+_A, \\ R^+_A s^+_B &= \frac{i}{2}\delta^+_{A,B}\ s^+ - \frac{i}{8}\Gamma^+_{ABC}\ s^{+C}, \\ R^+_A s^+_\mu &= -\frac{i}{2}\delta^+_{A,\mu\nu}s^{+\nu}, \end{aligned} \tag{3.6}$$

where $\Gamma^+_{ABC}$ is an anti-symmetric tensor defined in the Appendix. The $\{\mathcal{D}^+\}$ operators transform in the same manner with respect to the supercharges. We can find the following R-invariant terms,

$$
\begin{aligned}
&R^+_A(\mathcal{D}^+_\mu\mathcal{V}^+ + \mathcal{D}^{+\nu}\mathcal{V}^+_{\mu\nu}) = 0,\\
&R^+_A(\mathcal{D}^+\mathcal{V}^+ + \frac{1}{4}\mathcal{D}^{+B}\mathcal{V}^+_B) = 0,\\
&R^+_A(\mathcal{D}^+_B\mathcal{V}^+ - \mathcal{D}^+\mathcal{V}^+_B + \frac{1}{16}\Gamma^+_{BCD}\mathcal{D}^{+C}\mathcal{V}^{+D}) = 0.
\end{aligned}
\tag{3.7}
$$

We may impose the following conditions:

$$
\begin{aligned}
R^+_A(\mathcal{D}^+\mathcal{V}^+) = R^+_A(\mathcal{D}^+_\mu\mathcal{V}^+) = R^+_A(\mathcal{D}^+_B\mathcal{V}^+) = 0,\\
R^+_A(\mathcal{D}^+\mathcal{V}^+_B) = R^+_A(\mathcal{D}^+_\mu\mathcal{V}^+_B) = R^+_A(\mathcal{D}^+_B\mathcal{V}^+_C) = 0.
\end{aligned}
\tag{3.8}
$$

The constraints (3.8) mean that $\mathcal{D}^+_I\mathcal{V}^+$ and $\mathcal{D}^+_I\mathcal{V}^+_A$ should be the R-invariant, where $\{\mathcal{D}^+_I\} = \{\mathcal{D}^+, \mathcal{D}^+_\mu, \mathcal{D}^+_A\}$. We can then find the following relations between $\mathcal{V}^+$ and $\mathcal{V}^+_A$ by using eqs. (3.7) and (3.8):

$$
\begin{aligned}
&\delta^+_{A,\mu\nu}\mathcal{D}^{+\nu}\mathcal{V}^+ + \mathcal{D}^+_\mu\mathcal{V}^+_A = 0,\\
&\mathcal{D}^+_A\mathcal{V}^+ + \mathcal{D}^+\mathcal{V}^+_A = 0,\\
&\mathcal{D}^+_A\mathcal{V}^+_B + \mathcal{D}^+_B\mathcal{V}^+_A - 2\delta^+_{A,B}\mathcal{D}^+\mathcal{V}^+ = 0,\\
&\mathcal{D}^+_A\mathcal{V}^+_B - \mathcal{D}^+_B\mathcal{V}^+_A - \frac{1}{4}\Gamma^+_{ABC}(\mathcal{D}^{+C}\mathcal{V}^+ - \mathcal{D}^+\mathcal{V}^{+C}) = 0.
\end{aligned}
\tag{3.9}
$$

We define the component fields as follows:

$$
\begin{aligned}
\mathcal{V}^+| = v, \qquad & \mathcal{V}^+_A| = v^+_A, \qquad & \mathcal{Z}\mathcal{V}^+| = K, \qquad & \mathcal{Z}\mathcal{V}^+_A| = K^+_A,\\
\mathcal{D}^+_A\mathcal{V}^+| = \lambda^+_A, \qquad & \mathcal{D}^+\mathcal{V}^+| = \lambda, \qquad & \mathcal{D}^+_\mu\mathcal{V}^+| = \psi_\mu, &
\end{aligned}
\tag{3.10}
$$

where $|$ means to take the lowest components of the $\theta$'s. Higher components of $\theta$'s in the superfields $\mathcal{V}^+$ and $\mathcal{V}^+_A$ are expressed by the derivative of the fields (3.10). From eqs. (3.9), we show some twisted supertransformations:

$$
\begin{aligned}
s^+v^+_A &\equiv \mathcal{Q}^+\mathcal{V}^+_A| = \mathcal{D}^+\mathcal{V}^+_A| = -\lambda^+_A,\\
s^+\lambda &\equiv \mathcal{Q}^+\mathcal{D}^+\mathcal{V}^+| = \mathcal{D}^+\mathcal{D}^+\mathcal{V}^+| = \frac{1}{2}\mathcal{Z}\mathcal{V}^+| = \frac{1}{2}K.
\end{aligned}
\tag{3.11}
$$

We can also find the other transformation laws of components fields.

$N$=2 TSUSY with a central charge and R-invariant action is the following form,

$$
\begin{aligned}
\mathcal{S}_H &= \frac{1}{2}\int d^4x \mathrm{Tr}\Big( -\partial^\mu v\partial_\mu v - \frac{1}{4}\partial^\mu v^A\partial_\mu v_A + 4i\psi^\mu(\partial_\mu\lambda - \partial^\nu\lambda_{\mu\nu})\\
&\qquad + K^2 + \frac{1}{4}K^{+A}K^+_A\Big).
\end{aligned}
\tag{3.12}
$$

This action can be represented by the superfields. We omit the explicit form of the superfield action because of a complicated one. We will show a covariantized action with respect to the superfields in the next section.

An untwisted theory of the twisted multiplet corresponds to the real form of the hypermultiplet [29]. The spacetime tensor fields $v^+_{\mu\nu}$ and $K^+_{\mu\nu}$ in the action (3.12) are not spacetime tensor in the untwisted theory any more. We will discuss the detail of the untwist of the theory. We apply the Dirac-Kähler formalism to the super transformations,

$$\begin{aligned}(QP_+)_{\alpha I}(P_+\overline{\Psi})_{J\beta} &= -\frac{i}{2}\partial_\mu v\gamma^\mu_{\alpha\beta}(P_+)_{IJ} - \frac{i}{8}\partial_\mu v^+_A\gamma^\mu_{\alpha\beta}(\gamma^A P_+)_{JI} \\ &\quad +\frac{1}{2}\delta_{\alpha\beta}(P_+)_{JI}K + \frac{1}{8}\delta_{\alpha\beta}(\gamma^A P_+)_{JI}K^+_A, \qquad (3.13) \\ (\Psi P_+)_{\alpha I} &\equiv \frac{1}{2}\{(\lambda + \psi_\mu\gamma^\mu + \frac{1}{4}\lambda^+_A\gamma^A)P_+\}_{\alpha I}, \\ (QP_+)_{\alpha I} &\equiv \{(s + s_\mu\gamma^\mu + \frac{1}{4}s^+_A\gamma^A)P_+\}_{\alpha I}, \qquad (3.14)\end{aligned}$$

where $\alpha, \beta, I, J \in \{1,2,3,4\}$ are still the same spinor indices. In the above transformation we may ignore $I = 3$ and $I = 4$ components because of the projection matrix $P_+$. The indices $I, J \in \{1,2\}$ are identified with indices of the $SU(2)_i$ R-symmetry group of the $N$=2 supersymmetry. We exchange $(\gamma^\mu)_{IJ}$ to $(\Gamma^\mu)_{JI}$, where $\Gamma^\mu$ is the gamma matrix of the internal $SO(4)$ symmetry. In this case the gamma matrices appears with the projection $P_+$, so that the internal symmetry reduce to the $SU(2)$ symmetry. The transformation is

$$Q_\alpha{}^i\overline{\Psi}^{N=2}_{j\beta} = -\frac{i}{2}\delta^i{}_j\gamma^\mu_{\alpha\beta}\partial_\mu v - \frac{i}{8}\gamma^\mu_{\alpha\beta}\partial_\mu v^+_A(\sigma^A)_i{}^j + \frac{1}{2}\delta^i{}_j\delta_{\alpha\beta}K + \frac{1}{8}\delta_{\alpha\beta}K^+_A(\sigma^A)_i{}^j, \qquad (3.15)$$

where $i \equiv I$ and $j \equiv J$, $\{i,j\} \in \{1,2\}$. In this section $\Psi^{N=2}{}_\alpha{}^i$ denotes $\Psi_\alpha{}^i$ without confusing. Thus the $v^+_A$ and $K^+_A$ are not spacetime tensor fields but spacetime scalar fields. We redefine the bosonic fields as follows,

$$\omega \equiv v, \qquad (\omega_i{}^j)^* \equiv \frac{1}{4}v^+_A(\sigma^A)_i{}^j, \qquad (K_i{}^j)^* \equiv \frac{1}{4}K^+_A(\sigma^A)_i{}^j. \qquad (3.16)$$

The $\omega_{ij}$ satisfies the following relations,

$$\omega_{ij} = \omega_{ji}, \qquad (\omega^{ij})^* = \epsilon_{ik}\epsilon_{jl}\omega^{kl} \qquad (3.17)$$

where $\omega_{ij} \equiv -\epsilon_{jk}\omega_i{}^k$. The scalar fields $\omega$ and $K$ are $\mathbf{1}$ and $\omega_{ij}$ and $K_{ij}$ are $\mathbf{3}$ of $SU(2)_i$ R-symmetry group, respectively. An action of the untwisted theory is

$$S_\omega = \int d^4x(-\frac{1}{2}\partial^\mu\omega\partial_\mu\omega - \frac{1}{4}\partial^\mu\omega^{ij}\partial_\mu\omega_{ij} + 2i\overline{\Psi}_i\gamma^\mu\partial_\mu\Psi^i + \frac{1}{2}K^2 + \frac{1}{4}K^{ij}K_{ij}). \qquad (3.18)$$

This action is known as the real form of the hypermultiplet.

The theory possesses an additional $SU(2)$ symmetry. Redefine the fields we can see clearly the $SU(2)$ symmetry.

$$\tilde{\omega}_i{}^j \equiv \omega_i{}^j + \delta_i{}^j\omega, \qquad \tilde{K}_i{}^j \equiv K_i{}^j + \delta_i{}^jK, \qquad (3.19)$$

where $\tilde{\omega}_i{}^j$ and $\tilde{K}_i{}^j$ are $\mathbf{2}\otimes\mathbf{2}$ of $SU(2)_i$ R-symmetry. One of the $SU(2)$ indices can be regarded as a independent $SU(2)$ index. For example,

$$\tilde{\omega}_i{}^j \to \tilde{\omega}_i{}^a, \qquad \tilde{K}_i{}^j \to \tilde{K}_i{}^a, \qquad \Psi_\alpha{}^i \to \Psi_\alpha{}^a \tag{3.20}$$

where index $a$ is $SU(2)_{PG}$ which is known as Pauli-Gürsey group. An action and transformations are

$$S_{\tilde{\omega}} = \int d^4x\Big(-\frac{1}{4}\partial^\mu\tilde{\omega}^{ia}\partial_\mu\tilde{\omega}_{ia} + 2i\overline{\Psi}_a\gamma^\mu\partial_\mu\Psi^a + \frac{1}{4}\tilde{K}^{ia}\tilde{K}_{ia}\Big), \tag{3.21}$$

$$\begin{aligned}
Q_\alpha{}^i\overline{\Psi}_{a\beta} &= -\frac{i}{2}(\gamma^\mu)_{\alpha\beta}\partial_\mu\epsilon_{ab}\epsilon^{ij}\tilde{\omega}_j{}^b + \frac{1}{2}\epsilon_{ab}\epsilon^{ij}\tilde{K}_j{}^b,\\
Q_\alpha{}^i\tilde{\omega}_j{}^a &= 4\Psi_\alpha{}^a\delta_j{}^i,\\
Q_\alpha{}^i\tilde{K}_j{}^a &= 4i(\gamma_\mu\partial^\mu\Psi^a)_\alpha\delta_j{}^i.
\end{aligned} \tag{3.22}$$

When we take a diagonal subgroup of $SU(2)_i\otimes SU(2)_{PG}$ in the action (3.21), the action corresponds to the action (3.18).

We here summarize twisting procedure of this theory. The global symmetry group is $SU(2)_L\otimes SU(2)_R\otimes SU(2)_i\otimes SU(2)_{PG}$, where $SU(2)_L\otimes SU(2)_R$ is the rotation group $SO(4)$. Since the theory has two independent automorphism groups, we can define three different twisted theories. We can identify $SU(2)_i$ with $SU(2)_R$ and $SU(2)_{PG}$ with $SU(2)_R$, respectively.

$$\begin{aligned}
\tilde{\omega}_i{}^a &\to \tilde{\omega}_\alpha{}^\beta = v\delta_\alpha{}^\beta + \frac{1}{4}v^+_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta,\\
\xi_\alpha{}^a &\to \xi_\alpha{}^\beta = \frac{1}{2}(\lambda\delta_\alpha{}^\beta - \frac{1}{4}\lambda^+_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta),\\
\overline{\eta}^{\dot{\alpha}a} &\to \eta^{\dot{\alpha}\beta} = \frac{1}{2}\psi_\mu(\overline{\sigma}^\mu)^{\dot{\alpha}\beta},
\end{aligned} \tag{3.23}$$

where $\Psi^T = (\xi_\alpha{}^a, i\overline{\eta}^{\dot{\alpha}a})$, and a twist of $\tilde{K}_i{}^a$ is similar to that of $\tilde{\omega}_i{}^a$ . The twisted multiplet consists of the fields $\{v, v^+_{\mu\nu}, K, K^+_{\mu\nu}, \chi, \chi^+_A$ and $\psi_\mu\}$. This multiplet is the twisted version of the real form of the hypermultiplets.

We can identify $SU(2)_i$ with $SU(2)_R$ and $SU(2)_{PG}$ with $SU(2)_L$, respectively.

$$\begin{aligned}
\tilde{\omega}_{ia} &\to \tilde{\omega}_{\alpha\dot{\alpha}} = iV_\mu(\sigma^\mu)_{\alpha\dot{\alpha}},\\
\tilde{K}_{ia} &\to \tilde{\omega}_{\alpha\dot{\alpha}} = -iK_\mu(\sigma^\mu)_{\alpha\dot{\alpha}},\\
\xi_{\alpha a} &\to \xi_{\alpha\dot{\alpha}} = \frac{1}{2}\tilde{\psi}_\mu(\sigma^\mu)_{\alpha\dot{\alpha}},\\
\overline{\eta}^{\dot{\alpha}}{}_a &\to \eta^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{2}(\tilde{\eta}\delta^{\dot{\alpha}}{}_{\dot{\beta}} + \frac{1}{4}\chi^-_{\mu\nu}(\overline{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}).
\end{aligned} \tag{3.24}$$

This multiplet consists of the fields $\{V_\mu, K_\mu, \tilde{\eta}, \chi^-_A$ and $\tilde{\psi}_\mu\}$. The twisted multiplet was derived in [25].

The action (3.21) corresponds to the Fayet-Sohnius hypermultiplet action, when we adopts the following relations,

$$A_i = \tilde{\omega}_i{}^1, \qquad A^{\dagger i} = \epsilon^{ij}\tilde{\omega}_j{}^2, \qquad \Psi_\alpha{}^1 = \Psi_\alpha, \qquad \Psi_\alpha{}^2 = -C\overline{\Psi}^T_\alpha,$$

$$K_i = \tilde{K}_i{}^1, \qquad K^{\dagger i} = \epsilon^{ij} \tilde{K}_j{}^2, \tag{3.25}$$

where $\Psi_\alpha$ is Dirac spinor. The last twist is given by taking a diagonal subgroup of $SU(2)_i \otimes SU(2)_R$ only. This twisted model was given in [19].

# 4 Connection to the Vafa-Witten theory

We pointed out that the twisted multiplet action coupling to the gauge multiplet possess the twisted $N$=4 supersymmetry at on-shell level [25]. We will investigate the $N$=4 twisted supersymmetry in this case. We define covariantized operators in order to introduce the gauge multiplet.

$$\begin{aligned} \nabla^+ &\equiv \mathcal{D}^+ - i\Gamma, \\ \nabla^+_\mu &\equiv \mathcal{D}^+_\mu - i\Gamma_\mu, \\ \nabla^+_A &\equiv \mathcal{D}^+_A - i\Gamma_A, \\ \nabla_{\underline{\mu}} &\equiv \partial_\mu - i\Gamma_{\underline{\mu}}, \end{aligned} \tag{4.26}$$

where $\{\Gamma, \Gamma_\mu, \Gamma_A\}$ and $\Gamma_{\underline{\mu}}$ are fermionic and bosonic connection superfields, respectively. We then use the superconnection formalism [19, 71, 72]. The supercurvatures are defined as the following commutation relations,

$$\begin{aligned} &\{\nabla^+, \nabla^+_\mu\} = -i\nabla_{\underline{\mu}}, \quad \{\nabla^+_A, \nabla^+_\mu\} = i\delta^+_{A,\mu\nu}\nabla^{\underline{\nu}}, \quad \{\nabla^+, \nabla^+_A\} = 0, \\ &\{\nabla^+, \nabla^+\} = \mathcal{Z} - i\mathcal{F}, \quad \{\nabla^+_\mu, \nabla^+_\nu\} = \delta_{\mu\nu}(\mathcal{Z} - i\mathcal{W}), \quad \{\nabla^+_A, \nabla^+_B\} = \delta^+_{A,B}(\mathcal{Z} - i\mathcal{F}), \\ &\{\nabla_{\underline{\mu}}, \nabla_{\underline{\nu}}\} = -i\mathcal{F}_{\underline{\mu},\underline{\nu}}, \end{aligned} \tag{4.27}$$

where $\mathcal{F}$ and $\mathcal{W}$ are bosonic curvature superfields and appearing in the twisted vector multiplet and $\mathcal{F}_{\underline{\mu},\underline{\nu}}$ are supercurvatures with the gauge fields. The curvature superfields $\mathcal{F}$, $\mathcal{W}$ and $\mathcal{F}_{\underline{\mu},\underline{\nu}}$ commute with the central charge $\mathcal{Z}$.

In the covariant case the constraints of the superfield corresponding to the constraints (3.9) are

$$\begin{aligned} &\delta^+_{A,\mu\nu}\nabla^{+\nu}\mathcal{V}^+ + \nabla^+_\mu\mathcal{V}^+_A = 0, \\ &\nabla^+_A\mathcal{V}^+ + \nabla^+\mathcal{V}^+_A = 0, \\ &\nabla^+_A\mathcal{V}^+_B + \nabla^+_B\mathcal{V}^+_A - 2\delta^+_{A,B}\nabla^+\mathcal{V}^+ = 0, \\ &\nabla^+_A\mathcal{V}^+_B - \nabla^+_B\mathcal{V}^+_A - \frac{1}{4}\Gamma^+_{ABC}(\nabla^{+C}\mathcal{V}^+ - \nabla^+\mathcal{V}^{+C}) = 0. \end{aligned} \tag{4.28}$$

We define component fields of superfields which is consistent with Abelian case.

$$\begin{aligned} &\mathcal{F}| = \phi, \qquad \nabla^+_\mu\mathcal{F}| = C_\mu, \qquad \frac{1}{4}\delta^+_{A,\mu\nu}\nabla^{+\mu}\nabla^{+\nu}\mathcal{F}| = -\phi^+_A, \\ &\mathcal{W}| = \overline{\phi}, \qquad \nabla^+_A\mathcal{W}| = \chi^+_A, \qquad \nabla^+\mathcal{W}| = \chi, \\ &\mathcal{V}^+| = v, \qquad \mathcal{V}^+_A| = v^+_A, \qquad \mathcal{Z}\mathcal{V}^+| = K, \qquad \mathcal{Z}\mathcal{V}^+_A| = K^+_A, \end{aligned}$$

$$
\begin{aligned}
\nabla^+_A\mathcal{V}^+| = \lambda^+_A, \qquad \nabla^+\mathcal{V}^+| &= \lambda, \qquad\qquad \nabla^+_\mu\mathcal{V}^+| = \psi_\mu, \\
\nabla_{\underline{\mu}}| = \mathcal{D}_\mu, \qquad \Gamma_{\underline{\mu}}| &= A_\mu,
\end{aligned} \tag{4.29}
$$

where $\mathcal{D}_\mu$ and $A_\mu$ stand for the usual covariant derivatives and the gauge fields, respectively. Taking into account the Wess-Zumino gauge, we eliminate the lowest components of the fermionic superconnections and some fields appearing in higher components of $\theta's$. For example a nontrivial TSUSY transformation can be derived in the following manner,

$$
\begin{aligned}
s^+\lambda &\equiv \mathcal{Q}^+\nabla^+\mathcal{V}^+| = \nabla^+\nabla^+\mathcal{V}^+| \\
&= \frac{1}{2}\mathcal{Z}\mathcal{V}^+| - \frac{i}{2}\{\mathcal{F},\mathcal{V}^+\}| = \frac{1}{2}K - \frac{i}{2}[\phi, v].
\end{aligned} \tag{4.30}
$$

We can also find the other twisted supertransformations for all the component fields. We show the $N=2$ twisted SUSY transformations of the hypermultiplet in Appendix A.

These transformations satisfy the following commutation relations,

$$
\begin{aligned}
&\{s^+, s^+\}\varphi = Z\varphi - i[\phi,\varphi], && \{s^+, s^+_\mu\}\varphi = -iD_\mu\varphi, && [s^+, Z]\varphi = 0, \\
&\{s^+_\mu, s^+_\nu\}\varphi = \delta_{\mu\nu}Z\varphi - i\delta_{\mu\nu}[\overline{\phi},\varphi], && \{s^+_\mu, s^+_A\}\varphi = i\delta^+_{\mu\nu,A}D^\nu\varphi, && [s^+_\mu, Z]\varphi = 0, \\
&\{s^+_A, s^+_B\}\varphi = \delta^+_{A,B}Z\varphi - i\delta^+_{A,B}[\phi,\varphi], && \{s^+, s^+_A\}\varphi = 0, && [s^+_A, Z]\varphi = 0,
\end{aligned} \tag{4.31}
$$

where $\varphi = v, v^+_A, \lambda, \lambda^+_A, \psi_\mu, K, K^+_A$ and $D_\mu\varphi = \partial_\mu\varphi - i[\omega_\mu,\varphi]$. These algebras are closed at the off-shell level up to the gauge transformation.

A genaral method of the constructing (twisted) supersymmetric actions with a central charge is unknown unfortunately. Adopting the Sohnius' method [70], we can construct a covariantized action of the real form of the hypermultiplet. We derive the action by taking the lowest components of the $\theta$'s. We then obtain the covariantized action as follows:

$$
\begin{aligned}
\mathcal{S}_H &= \frac{1}{2}\int d^4x\mathcal{L}_H\Big| \\
&= \frac{1}{2}\int d^4x \mathrm{Tr} \\
&\times \Bigg\{-\frac{1}{12}\Big(\frac{1}{16}\Gamma^+_{AB\mu\nu}\nabla^{+\mu}\nabla^{+\nu}(\mathcal{V}^{+A}\mathcal{Z}\mathcal{V}^{+B}) + \nabla^{+\mu}\nabla^{+\nu}(\mathcal{V}^+\mathcal{Z}\mathcal{V}^+_{\mu\nu} - \mathcal{V}^+_{\mu\nu}\mathcal{Z}\mathcal{V}^+)\Big) \\
&\quad +\frac{1}{12}\Big(\frac{1}{64}\Gamma^+_{ABC}\nabla^{+A}\nabla^{+B}(\mathcal{V}^{+C}\mathcal{Z}\mathcal{V}^+) - \frac{1}{64}\Gamma^+_{ABC}\nabla^{+A}\nabla^{+B}(\mathcal{V}^+\mathcal{Z}\mathcal{V}^{+C}) \\
&\quad -\frac{1}{64}\Gamma^+_{ABC}\nabla^+\nabla^{+A}(\mathcal{V}^{+B}\mathcal{Z}\mathcal{V}^{+C}) + \frac{1}{64}\Gamma^+_{ABC}\nabla^{+A}\nabla^+(\mathcal{V}^{+B}\mathcal{Z}\mathcal{V}^{+C})\Big) \\
&\quad +\frac{1}{12}\Big(\frac{1}{4}\nabla^{+A}\nabla^+(\mathcal{V}^+\mathcal{Z}\mathcal{V}^+_A) + \frac{1}{4}\nabla^+\nabla^{+A}(\mathcal{V}^+_A\mathcal{Z}\mathcal{V}^+) + \frac{1}{16}\nabla^{+A}\nabla^{+B}(\mathcal{V}^+_B\mathcal{Z}\mathcal{V}^+_A) \\
&\quad -\frac{1}{4}\nabla^{+A}\nabla^+(\mathcal{V}^+_A\mathcal{Z}\mathcal{V}^+) - \frac{1}{4}\nabla^+\nabla^{+A}(\mathcal{V}^+\mathcal{Z}\mathcal{V}^+_A) - \frac{1}{16}\nabla^{+A}\nabla^{+B}(\mathcal{V}^+_A\mathcal{Z}\mathcal{V}^+_B)\Big)\Bigg\}\Bigg|,
\end{aligned} \tag{4.32}
$$

where we impose the R-symmetry (3.6) on the action to restrict forms of the action. The invariance of the TSUSY is guaranteed as follows:

$$\begin{aligned}\mathcal{Q}^{+}\mathcal{L}_{H}\Big| &= \nabla^{+}\mathcal{L}_{H}\Big| \\ &= i\partial^{\mu}\Big\{\mathrm{Tr}(\mathcal{V}^{+}\mathcal{Z}\nabla_{\mu}^{+}\mathcal{V}^{+}-\mathcal{Z}\mathcal{V}^{+}\nabla_{\mu}^{+}\mathcal{V}^{+} \\ &\qquad -\mathcal{V}_{\mu\nu}^{+}\mathcal{Z}\nabla^{+\nu}\mathcal{V}^{+}+\mathcal{Z}\mathcal{V}_{\mu\nu}^{+}\nabla^{+\nu}\mathcal{V}^{+})\Big\}\Big|. \quad (4.33)\end{aligned}$$

This implies that the lowest component of $\mathcal{L}_H$ transforms as a total divergence under the $s^+$ transformation. The rest of the TSUSY transformations have the same character. The action is invariant for all the TSUSY transformations, but the action cannot be expressed by $\{s^+, s^+_\mu, s^+_A\}$ exact form.

We construct an off-shell Donaldson-Witten theory coupled to the twisted real form of the hypermultiplet. The off-shell Donaldson-Witten theory are given by using the twisted superspace formalism [19, 21]:

$$\begin{aligned}S_{DW} &= \frac{1}{2}\int d^4x d^4\theta\ \mathrm{Tr}\mathcal{F}^2 \\ &= \frac{1}{2}\int d^4x \mathrm{Tr}\Big(-\phi D^{\mu}D_{\mu}\overline{\phi}-iC^{\mu}(D_{\mu}\chi-D^{\nu}\chi_{\mu\nu}^{+})+(F_{\mu\nu}^{-})^2 \\ &\qquad +\frac{i}{2}\phi\{\chi,\chi\}+\frac{i}{8}\phi\{\chi^{+A},\chi_A^{+}\}+\frac{i}{2}\overline{\phi}\{C^{\mu},C_{\mu}\} \\ &\qquad +\frac{1}{4}[\phi,\overline{\phi}]^2-\frac{1}{4}(\phi_{\mu\nu}^{+})^2\Big). \quad (4.34)\end{aligned}$$

We redefine the component fields as

$$\lambda\to\frac{1}{2}\lambda,\qquad \lambda_A^{+}\to\frac{1}{2}\lambda_A^{+},\qquad \psi_{\mu}\to-\frac{1}{2}\psi_{\mu},\qquad \overline{\phi}\to-\overline{\phi}, \quad (4.35)$$

in order to adjust coefficients in the action. We then derive the off-shell Donaldson-Witten theory coupled to the twisted real form of the hypermultiplet as follows:

$$\begin{aligned}\mathcal{S} &= -(\mathcal{S}_H+\mathcal{S}_{DW}) \\ &= -\frac{1}{2}\int d^4x \mathrm{Tr}\Big(-\mathcal{D}^{\mu}v\mathcal{D}_{\mu}v-\frac{1}{4}\mathcal{D}^{\mu}v^{A}\mathcal{D}_{\mu}v_A+\phi D^{\mu}D_{\mu}\overline{\phi}+(F_{\mu\nu}^{-})^2 \\ &\qquad -i\psi^{\mu}(\mathcal{D}_{\mu}\lambda-\mathcal{D}^{\nu}\lambda_{\mu\nu})+\frac{i}{2}\phi\{\psi^{\mu},\psi_{\mu}\}-\frac{i}{2}\overline{\phi}\{\lambda,\lambda\}-\frac{i}{8}\overline{\phi}\{\lambda^{+A},\lambda_A^{+}\} \\ &\qquad -iC^{\mu}(D_{\mu}\chi-D^{\nu}\chi_{\mu\nu}^{+})-\frac{i}{2}\overline{\phi}\{C^{\mu},C_{\mu}\}+\frac{i}{2}\phi\{\chi,\chi\}+\frac{i}{8}\phi\{\chi^{+A},\chi_A^{+}\} \\ &\qquad -\frac{i}{4}v\{\chi^{+A},\lambda_A^{+}\}+\frac{i}{64}\Gamma^{+ABC}v_A^{+}\{\chi_B^{+},\lambda_C^{+}\}-iv_{\mu\nu}^{+}\{C^{\mu},\psi^{\nu}\} \\ &\qquad +iv\{C^{\mu},\psi_{\mu}\}-\frac{i}{4}v^{+A}\{\chi_A^{+},\lambda\}+\frac{i}{4}v^{+A}\{\chi,\lambda_A^{+}\}-iv\{\chi,\lambda\}\end{aligned}$$

$$+\frac{1}{2}v[\phi,[\overline{\phi},v]]+\frac{1}{2}v[\overline{\phi},[\phi,v]]+\frac{1}{8}v^{+A}[\phi,[\overline{\phi},v^+_A]]$$
$$+\frac{1}{8}v^{+A}[\overline{\phi},[\phi,v^+_A]]+\frac{1}{4}[\phi,\overline{\phi}]^2+\frac{i}{2}\phi^{+A}[v^+_A,v]$$
$$-\frac{i}{64}\Gamma^{+ABC}\phi^+_A[v^+_B,v^+_C]+KK+\frac{1}{4}K^{+A}K^+_A-\frac{1}{4}\phi^{+A}\phi^+_A\Big). \qquad (4.36)$$

This action possesses the off-shell $N$=2 twisted supersymmetry. We then investigate a symmetry when we eliminate the auxiliary fields $K, K^+_A$ and $\phi^+_A$. The equations of motion of the auxiliary fields are

$$\phi^+_A=-\frac{i}{32}\Gamma^+_{ABC}[v^{+B},v^{+C}]+i[v^+_A,v], \qquad K=K^+_A=0. \qquad (4.37)$$

We find that the $Z$ transformations for the components $\{v, v^+_A, \lambda, \lambda^+_A, \psi_\mu\}$ of the twisted hypermultiplet are equivalent to the on-shell conditions and therefore these transformations disappear at the on-shell level. The on-shell action is given by the following form:

$$\begin{aligned}
\mathcal{S}_{\text{on-shell}} &= -\frac{1}{2}\int d^4x\mathrm{Tr}\\
&\times \Big(-\mathcal{D}^\mu v\mathcal{D}_\mu v-\frac{1}{4}\mathcal{D}^\mu v^A\mathcal{D}_\mu v_A+\phi D^\mu D_\mu\overline{\phi}+(F^-_{\mu\nu})^2\\
&\quad -i\psi^\mu(\mathcal{D}_\mu\lambda-\mathcal{D}^\nu\lambda_{\mu\nu})+\frac{i}{2}\phi\{\psi^\mu,\psi_\mu\}-\frac{i}{2}\overline{\phi}\{\lambda,\lambda\}-\frac{i}{8}\overline{\phi}\{\lambda^{+A},\lambda^+_A\}\\
&\quad -iC^\mu(D_\mu\chi-D^\nu\chi^+_{\mu\nu})-\frac{i}{2}\overline{\phi}\{C^\mu,C_\mu\}+\frac{i}{2}\phi\{\chi,\chi\}+\frac{i}{8}\phi\{\chi^{+A},\chi^+_A\}\\
&\quad -\frac{i}{4}v\{\chi^{+A},\lambda^+_A\}+\frac{i}{64}\Gamma^{+ABC}v^+_A\{\chi^+_B,\lambda^+_C\}-iv^+_{\mu\nu}\{C^\mu,\psi^\nu\}\\
&\quad +iv\{C^\mu,\psi_\mu\}-\frac{i}{4}v^{+A}\{\chi^+_A,\lambda\}+\frac{i}{4}v^{+A}\{\chi,\lambda^+_A\}-iv\{\chi,\lambda\}\\
&\quad +\frac{1}{2}v[\phi,[\overline{\phi},v]]+\frac{1}{2}v[\overline{\phi},[\phi,v]]+\frac{1}{8}v^{+A}[\phi,[\overline{\phi},v^+_A]]+\frac{1}{8}v^{+A}[\overline{\phi},[\phi,v^+_A]]\\
&\quad -\frac{1}{32}[v^+_A,v^+_B][v^{+A},v^{+B}]-\frac{1}{4}[v^+_A,v][v^{+A},v]\Big). \qquad (4.38)
\end{aligned}$$

We then find that the action (4.38) possesses the following symmetries:

$$\begin{array}{llll}
\phi\to-\overline{\phi}, & \overline{\phi}\to-\phi, & v\to v, & v^+_A\to-v^+_A,\\
\chi\leftrightarrow\lambda, & \chi^+_A\leftrightarrow\lambda^+_A, & C_\mu\leftrightarrow\lambda_\mu, & A_\mu\to A_\mu.
\end{array} \qquad (4.39)$$

Applying these discrete symmetries to the twisted SUSY transformations, we obtain new fermionic symmetries which is shown in Appendix A. We express new scalar, vector and anti-self-dual tensor supercharges as $\{\overline{s}^+, \overline{s}^+_\mu, \overline{s}^+_A\}$. We then find that the commutation relations of the supercharge form a new $N$=4 twisted SUSY algebra and this action possesses the $N=4$ twisted supersymmetry. The explicite form of the $N$=4 twisted SUSY algebras are given in Appendix A. These algebras closed

| | $\nabla^+$ | $\nabla^+_B$ | $\overline{\nabla}^+_\nu$ | $\overline{\nabla}^+$ | $\overline{\nabla}^+_B$ | $\nabla^+_\nu$ | $\nabla_{\underline{\nu}}$ |
|---|---|---|---|---|---|---|---|
| $\nabla^+$ | $-i\mathcal{F}$ | $0$ | $0$ | $-i\mathcal{V}^+$ | $-i\mathcal{V}^+_B$ | $-i\nabla_{\underline{\nu}}$ | $-i\mathcal{F}_{\underline{\nu}}$ |
| $\nabla^+_A$ | | $-i\delta^+_{A,B}\mathcal{F}$ | $0$ | $i\mathcal{V}^+_A$ | $-i\delta^+_{A,B}\mathcal{V}^+ - \frac{i}{4}\Gamma^+_{ABC}\mathcal{V}^{+C}$ | $i\delta^+_{A,\nu\rho}\nabla^{\underline{\rho}}$ | $-i\mathcal{F}_{A,\underline{\nu}}$ |
| $\overline{\nabla}^+_\mu$ | | | $-i\delta_{\mu\nu}\mathcal{F}$ | $-i\nabla_{\underline{\mu}}$ | $i\delta^+_{B,\mu\nu}\nabla^{\underline{\nu}}$ | $i\delta_{\mu\nu}\mathcal{V}^+ - i\mathcal{V}^+_{\mu\nu}$ | $-i\overline{F}_{\mu\underline{\nu}}$ |
| $\overline{\nabla}^+$ | | | | $-i\mathcal{W}$ | $0$ | $0$ | $-i\overline{\mathcal{F}}_{\underline{\nu}}$ |
| $\overline{\nabla}_A$ | | | | | $-i\delta^+_{A,B}\mathcal{W}$ | $0$ | $-i\overline{F}_{A,\underline{\nu}}$ |
| $\nabla^+_\mu$ | | | | | | $-i\delta_{\mu\nu}\mathcal{W}$ | $-i\mathcal{F}_{\mu\underline{\nu}}$ |
| $\nabla_{\underline{\mu}}$ | | | | | | | $-i\mathcal{F}_{\underline{\mu}\underline{\nu}}$ |

Table 1: $N$=4 twisted supercurvatures

at the on-shell level up to the gauge transformations. In this case the ghost numbers of twisted scalar supercharges $s^+$ and $\overline{s}^+$ are $+1$ and $-1$, respectively. $s^+$ is BRST charge and $\overline{s}^+$ is anti-BRST charge. We conclude that this $N$=4 twisted supersymmetric theory is equivalent to the Vafa-Witten theory.

# 5 $N = 4$ twisted superconnection formalism

The Marcus type $N$=4 twisted supersymmetric theory was derived from $N$=4 twisted superconnection formalism [73]. We then reconstruct the $N = 4$ twisted super Yang-Mills theory (4.38) which is derived from the $N$=2 superspace formalism by using $N$=4 twisted superconnection formalism, similarly. We represent sixteen supercovariant derivatives as $\{\nabla^+, \nabla^+_\mu, \nabla^+_A, \overline{\nabla}^+, \overline{\nabla}^+_\mu, \overline{\nabla}^+_A\}$. We impose special constraints on general curvature superfields based on the twisted superalgebras in Appendix A, and construct the $N = 4$ twisted super Yang-Mills theory directly. We define the (anti)commutation relations of these supercovariant derivatives in Table 1, where $\mathcal{F}$, $\mathcal{W}$, $\mathcal{V}^+$ and $\mathcal{V}^+_A$ are bosonic supercurvatures, $\mathcal{F}_{\underline{\mu}}$, $\mathcal{F}_{A,\underline{\nu}}$, $\overline{\mathcal{F}}_{\mu,\underline{\nu}}$, $\overline{\mathcal{F}}_{\underline{\nu}}$, $\overline{\mathcal{F}}_{A,\underline{\nu}}$ and $\mathcal{F}_{\mu,\underline{\nu}}$ are fermionic supercurvatures and $\mathcal{F}_{\underline{\mu},\underline{\nu}}$ is a bosonic supercurvature which includes the ordinary curvature $F_{\mu\nu}$. We can gauge away some superfluous fields in the superconnections by taking Wess-Zumino gauge. From Jacobi identities of the supercovariant derivatives, we derive the following nontrivial relations:

$$
\begin{aligned}
\nabla^+\mathcal{F} &= \nabla^+_A\mathcal{F} = \overline{\nabla}^+_\mu\mathcal{F} = 0, \\
\overline{\nabla}^+\mathcal{W} &= \overline{\nabla}^+_A\mathcal{W} = \nabla^+_\mu\mathcal{W} = 0,
\end{aligned} \tag{5.40}
$$

$$
\begin{aligned}
&\mathcal{F}_{\underline{\mu}} = -\frac{i}{2}\nabla^+_\mu\mathcal{F}, \quad \mathcal{F}_{A,\underline{\mu}} = -\frac{i}{2}\delta^+_{A,\mu\nu}\nabla^{+\nu}\mathcal{F}, \quad \mathcal{F}_{\mu\underline{\nu}} = -\frac{i}{2}\delta_{\mu\nu}\overline{\nabla}^+\mathcal{F} + \frac{i}{2}\overline{\nabla}^+_{\mu\nu}\mathcal{F}, \\
&\overline{\mathcal{F}}_{\underline{\mu}} = -\frac{i}{2}\overline{\nabla}^+_\mu\mathcal{W}, \quad \overline{\mathcal{F}}_{A,\underline{\mu}} = -\frac{i}{2}\delta^+_{A,\mu\nu}\overline{\nabla}^{+\nu}\mathcal{W}, \quad \overline{\mathcal{F}}_{\mu\underline{\nu}} = -\frac{i}{2}\delta_{\mu\nu}\nabla^+\mathcal{W} + \frac{i}{2}\nabla^+_{\mu\nu}\mathcal{W},
\end{aligned} \tag{5.41}
$$

$$
\mathcal{F}_{\underline{\mu}\underline{\nu}} = -\frac{1}{2}\nabla^+\nabla^+_{\mu\nu}\mathcal{W} + \frac{1}{4}[\nabla^+_\mu, \nabla^+_\nu]\mathcal{F}
$$

$$= -\frac{1}{2}\overline{\nabla}^{+}\overline{\nabla}^{+}_{\mu\nu}\mathcal{F} + \frac{1}{4}[\overline{\nabla}^{+}_{\mu}, \overline{\nabla}^{+}_{\nu}]\mathcal{W}, \tag{5.42}$$

$$\begin{array}{lll}
\nabla^{+}\mathcal{V}^{+} = -\frac{1}{2}\overline{\nabla}^{+}\mathcal{F}, & \overline{\nabla}^{+}_{\mu}\mathcal{V}^{+} = \frac{1}{2}\nabla^{+}_{\mu}\mathcal{F}, & \nabla^{+}_{A}\mathcal{V}^{+} = -\frac{1}{2}\overline{\nabla}^{+}_{A}\mathcal{F}, \\
\nabla^{+}\mathcal{V}^{+}_{A} = -\frac{1}{2}\overline{\nabla}^{+}_{A}\mathcal{F}, & \nabla^{+}_{\mu}\mathcal{V}^{+}_{A} = -\frac{1}{2}\delta^{+}_{A,\mu\nu}\nabla^{+\nu}\mathcal{F}, & \\
\overline{\nabla}^{+}\mathcal{V}^{+} = -\frac{1}{2}\nabla^{+}\mathcal{W}, & \nabla^{+}_{\mu}\mathcal{V}^{+} = \frac{1}{2}\overline{\nabla}^{+}_{\mu}\mathcal{W}, & \overline{\nabla}^{+}_{A}\mathcal{V}^{+} = -\frac{1}{2}\nabla^{+}_{A}\mathcal{W}, \\
\overline{\nabla}^{+}\mathcal{V}^{+}_{A} = \frac{1}{2}\nabla^{+}_{A}\mathcal{W}, & \overline{\nabla}^{+}_{\mu}\mathcal{V}^{+}_{A} = \frac{1}{2}\delta^{+}_{A,\mu\nu}\overline{\nabla}^{+\nu}\mathcal{W}, &
\end{array}$$

$$\begin{aligned}
\nabla^{+}_{A}\mathcal{V}^{+}_{B} &= \frac{1}{2}\delta^{+}_{A,B}\overline{\nabla}^{+}\mathcal{F} + \frac{1}{8}\Gamma^{+}_{ABC}\overline{\nabla}^{+C}\mathcal{F}, \\
\overline{\nabla}^{+}_{A}\mathcal{V}^{+}_{B} &= -\frac{1}{2}\delta^{+}_{A,B}\nabla^{+}\mathcal{W} - \frac{1}{8}\Gamma^{+}_{ABC}\nabla^{+C}\mathcal{W},
\end{aligned}$$

It should be noted that the curvature superfields $\mathcal{F}$ and $\mathcal{W}$ are (anti)chiral superfield of non-Abelian type from eq. (5.40). $\mathcal{F}_{\underline{\mu}}$, $\mathcal{F}_{A,\underline{\mu}}$, $\mathcal{F}_{\mu,\underline{\nu}}$, $\overline{\mathcal{F}}_{\underline{\mu}}$, $\overline{\mathcal{F}}_{A,\underline{\mu}}$ and $\overline{\mathcal{F}}_{\mu,\underline{\nu}}$ are expressed by the superfields $\mathcal{F}$ and $\mathcal{W}$. We define the component fields of these superfields $\mathcal{F}$, $\mathcal{W}$, $\mathcal{V}^{+}$ and $\mathcal{V}^{+}_{A}$ as the following forms:

$$\begin{array}{lllll}
\mathcal{F}| = \phi, & \overline{\nabla}\mathcal{F}| = \lambda, & \nabla_{\mu}\mathcal{F}| = C_{\mu}, & \overline{\nabla}_{A}\mathcal{F}| = \lambda^{+}_{A}, & \mathcal{V}| = -v, \\
\mathcal{W}| = -\overline{\phi}, & \nabla\mathcal{W}| = \chi, & \overline{\nabla}_{\mu}\mathcal{W}| = \psi_{\mu}, & \nabla_{A}\mathcal{W}| = \chi^{+}_{A}, & \mathcal{V}^{+}_{A}| = v^{+}_{A}.
\end{array} \tag{5.43}$$

We can derive $N = 4$ twisted supertransformations by using the above equations. The $N = 4$ twisted supertransformations strictly correspond with the twisted ones for Donaldson-Witten theory coupled to the real form of the hypermultiplet at on-shell level in Appendix A.

# 6 Euclidean $N = 4$ super Yang-Mills Action

We derive an ordinary $N = 4$ super Yang-Mills theory by untwisting the twisted $N = 4$ super Yang-Mills theory of the Vafa-Witten type in this section. We construct Dirac-Kähler fermions from the tensor fermions $\{\lambda, \lambda^{+}_{A}, \psi_{\mu}, \chi, \chi^{+}_{A}, C_{\mu}\}$ appearing in the twisted $N = 4$ super Yang-Mills theory. We, however, cannot construct one Dirac-Kähler fermion form these fermion[3]. The reason is that the Dirac-Kähler fermion needs to a self-dual tensor field and a anti-self-dual tensor field but this action does not contain self-dual one. Thus we define two Dirac-Kähler fermions with the chiral projection $(P_{+})_{IJ}$ where $I, J = 1, \cdots, 4$:

$$\Psi^{iN=2}_{\alpha} \equiv \psi_{\alpha J}(P_{+})_{JI} = \frac{1}{2}\{(\lambda + \psi^{\mu}\gamma_{\mu} + \frac{1}{4}\lambda^{+\mu\nu}\gamma_{\mu\nu})P_{+}\}_{\alpha I},$$

[3]It should be noted that in the case of Marcus type an ordinary Dirac-Kähler fermion without projections is constructed by fermions $\{\tilde{\psi}_{\mu}, \tilde{\eta}, \chi^{-}_{A}, \chi, \chi^{+}_{A}, C_{\mu}\}$ in four dimensions [25].

$$\chi_\alpha^{iN=2} \equiv \chi_{\alpha J}(P_+)_{JI} = \frac{1}{2}\{(\chi + C^\mu\gamma_\mu + \frac{1}{4}\chi^{+\mu\nu}\gamma_{\mu\nu})P_+\}_{\alpha I}. \tag{6.44}$$

$I = 3$ and $I = 4$ components are vanished in these fermions because of the chiral projection $P_+$. It means that these fermions are the $N$=2 fermions.

The action (4.38) can be represented by the Dirac-Kähler fermions (6.44).

$$\begin{aligned}
\mathcal{S} &= -\frac{1}{2}\int d^4x \mathrm{Tr} \\
&\times \Big( -\mathcal{D}^\mu v \mathcal{D}_\mu v - \frac{1}{4}\mathcal{D}^\mu v^A \mathcal{D}_\mu v_A + \phi D^\mu D_\mu \overline{\phi} + (F^-_{\mu\nu})^2 \\
&-i\overline{\Psi}^{iN=2}\gamma^\mu \mathcal{D}_\mu \Psi^{iN=2} - i\overline{\chi}^{iN=2}\gamma^\mu\mathcal{D}_\mu\chi^{iN=2} \\
&-2i\overline{\phi}\ \overline{\Psi}^{iN=2}P_+\Psi^{iN=2} + 2i\phi\ \overline{\Psi}^{iN=2}P_-\Psi^{iN=2} \\
&+2i\phi\ \overline{\chi}^{iN=2}P_+\chi^{iN=2} - 2i\overline{\phi}\ \overline{\chi}^{iN=2}P_-\chi^{iN=2} \\
&-2iv\ \overline{\Psi}^{iN=2}P_+\chi^{iN=2} + 2iv\ \overline{\Psi}^{iN=2}P_-\chi^{iN=2} \\
&-2iv\ \overline{\chi}^{iN=2}P_+\Psi^{iN=2} + 2iv\ \overline{\chi}^{iN=2}P_-\Psi^{iN=2} \\
&-2i(-\frac{1}{4}\hat{\sigma}_{\mu\nu}v^{+\mu\nu})^{ij}\overline{\Psi}^{iN=2}P_+\chi^{jN=2} + 2i(-\frac{1}{4}\hat{\sigma}_{\mu\nu}v^{+\mu\nu})^{ij}\overline{\Psi}^{iN=2}P_-\chi^{jN=2} \\
&+2i(-\frac{1}{4}\hat{\sigma}_{\mu\nu}v^{+\mu\nu})^{ij}\overline{\chi}^{iN=2}P_+\Psi^{jN=2} - 2i(-\frac{1}{4}\hat{\sigma}_{\mu\nu}v^{+\mu\nu})^{ij}\overline{\chi}^{iN=2}P_-\Psi^{jN=2} \\
&+\frac{1}{2}v[\phi,[\overline{\phi},v]] + \frac{1}{2}v[\overline{\phi},[\phi,v]] + \frac{1}{8}v^{+A}[\phi,[\overline{\phi},v^+_A]] + \frac{1}{8}v^{+A}[\overline{\phi},[\phi,v^+_A]] \\
&+\frac{1}{4}[\phi,\overline{\phi}]^2 - \frac{1}{32}[v^+_A,v^+_B][v^{+A},v^{+B}] - \frac{1}{4}[v^+_A,v][v^{+A},v]\Big),
\end{aligned} \tag{6.45}$$

where $\hat{\sigma}^{\mu\nu} = \frac{1}{2}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)$, $\sigma^\mu = (\sigma^1,\sigma^2,\sigma^3,\sigma^4)$,$\overline{\sigma}^\mu = (-\sigma^1,-\sigma^2,-\sigma^3,\sigma^4)$, $\sigma^4 = i\mathbf{1}_{2\times2}$. We furthermore construct a fermion $\Psi^I_\alpha$ with the internal SU(4) R-symmetry from the $\Psi^{iN=2}_\alpha$ and $\chi^{iN=2}_\alpha$ with the internal SU(2) R-symmetry. We define the fermion $\Psi^I_\alpha$ with the SU(4) R-symmetry as follows:

$$\begin{aligned}
\Psi^I_\alpha &\equiv \Psi^{iN=2}_\alpha, \qquad \{I = 1,2\}, \\
\Psi^I_\alpha &\equiv \chi^{iN=2}_\alpha, \qquad \{I = 3,4\}.
\end{aligned} \tag{6.46}$$

We construct fields $\phi^{IJ}$ and $\tilde{\phi}^{IJ}$ of a $\mathbf{6}$ representation with SU(4) symmetry from the bosonic fields: $\phi$, $\overline{\phi}$, $v$ and $v^+_A$ as follows:

$$\begin{aligned}
\phi^{IJ} &= \left(\begin{array}{cc|cc}
\overline{\phi} & 0 & v+iv_{12} & -v_{13}-iv_{14} \\
0 & \overline{\phi} & v_{13}-iv_{14} & v-iv_{12} \\
\hline
v-iv_{12} & v_{13}+iv_{14} & -\phi & 0 \\
-v_{13}+iv_{14} & v+iv_{12} & 0 & -\phi
\end{array}\right), \\
\tilde{\phi}^{IJ} &= \left(\begin{array}{cc|cc}
-\phi & 0 & -v-iv_{12} & v_{13}+iv_{14} \\
0 & -\phi & -v_{13}+iv_{14} & -v+iv_{12} \\
\hline
-v+iv_{12} & -v_{13}-iv_{14} & \overline{\phi} & 0 \\
v_{13}-iv_{14} & -v-iv_{12} & 0 & \overline{\phi}
\end{array}\right).
\end{aligned} \tag{6.47}$$

The $\phi^{IJ}$ and $\tilde{\phi}^{IJ}$ are satisfied with the following relations:

$$\phi^\dagger_{IJ} = -\phi_{IJ}, \qquad \tilde{\phi}^\dagger_{IJ} = -\tilde{\phi}_{IJ}, \tag{6.48}$$

$$(C\tilde{\phi})^*_{IJ} = -\frac{1}{2}\epsilon_{IJKL}(C\phi)^{KL}, \tag{6.49}$$

where $\phi$, $\overline{\phi}$, $v$ and $v^+_A$ are the anti-hermitian. We can interpret the $\Psi^I_\alpha$ as spinors in the $N = 4$ SYM theory since the Lorentz symmetry in twisted theory separate the Lorentz symmetry into the internal R-symmetry. We reparametrize component fields as

$$\Psi^I \to i\Psi^I, \qquad \phi^{IJ} \to -i\phi^{IJ}, \qquad \tilde{\phi}^{IJ} \to -i\tilde{\phi}^{IJ}, \tag{6.50}$$

and we derive a $N = 4$ super Yang-Mills action as follows:

$$\begin{aligned} \mathcal{S} \;=\; &-\int d^4x \mathrm{Tr}\Big(\frac{1}{8}\tilde{\phi}^{IJ}\mathcal{D}^\mu\mathcal{D}_\mu\phi^{JI} + \frac{i}{2}\overline{\Psi}^I\gamma^\mu\mathcal{D}_\mu\Psi^I + \frac{1}{2}(F^-_{\mu\nu})^2 \\ &+\overline{\Psi}^I P_+\Psi^J\phi^{IJ} + \overline{\Psi}^I P_-\Psi^J\tilde{\phi}^{IJ} - \frac{1}{64}[\tilde{\phi}^{IJ},\tilde{\phi}^{LM}][\phi^{JI},\phi^{ML}]\Big). \end{aligned} \tag{6.51}$$

# 7 Conclusions and Discussions

We have constructed the Vafa-Witten theory by using $N = 2$ or $N = 4$ twisted superspace formalism based on the Dirac-Kähler mechanism. We summarize the Dirac-Kähler mechanism in this paragraph. The Dirac-Kähler mechanism gives the way to identify the $N = 4$ extended SUSY suffix $\{i\}$ as the Lorentz spinor suffix $\{\alpha\}$, i.e. , diagonal subgroup $SO(4) \otimes SO(4)_I$ [21]. The $N$=4 fermions based on the Dirac-Kähler mechanism consist of two scalar, two vector, a self-dual tensor and an anti-self-dual tensor. These fields just correspond to the fields which appear in the Marcus's theory. Since Vafa-Witten theory possesses two anti-self-dual fields, we cannot construct one Dirac-Kähler fermion. We can, therefore, construct two Dirac-Kähler fermion on which is imposed the chiral projection $(P_+)_{ij}$ with respect to the R-symmetry.

We proposed the $N = 2$ twisted superspace formalism of the twisted real form of the hypermultiplet with the central charge based on the Dirac-Kähler twist. We then introduced the bosonic superfields with a scalar and an anti-self-dual tensor indices, $\mathcal{V}^+$ and $\mathcal{V}^+_A$ respectively, while we introduced the bosonic superfields $\mathcal{V}_\mu$ with a vector index in the previous paper. The theory given by using the superfield with the vector index is as a result Marcus's theory [25]. In the scalar and the tensor superfields case we have derived the off-shell action with the auxiliary fields $K^+$ and $K^+_A$ by imposing the constraints on the superfields from the R-symmetry. We have also construct the twisted gauge covariant action with the Wess-Zumino gauge and have derived the off-shell Donaldson-Witten theory coupled to the covariantized real form of the hypermultiplet. Integrating out these auxiliary fields $K^+$, $K^+_A$ and $\phi^+_A$, we then found that the Donaldson-Witten theory coupled to the real form of the hypermultiplet possessed the discrete symmetries. From these discrete symmetries we can derive other fermionic symmetries whose generators are $\{\overline{s}^{+}, \overline{s}^{+}_{\mu}, \overline{s}^{+}_{A}\}$. Thus the symmetries of the theory is enhanced to the $N = 4$ twisted SUSY without a central charge. The ghost number of $\{\overline{s}^{+}, \overline{s}^{+}_{\mu}, \overline{s}^{+}_{A}\}$ are $\{-1, +1, -1\}$, while the ghost number of $\{s^{+}, s^{+}_{\mu}, s^{+}_{A}\}$ are $\{+1, -1, +1\}$, where two scalar supercharges $s^{+}$ and $\overline{s}^{+}$ are identified with the BRST charge and the anti-BRST charge, respectively. We claim that this twist is identified with Vafa-Witten twist. We explain this more precisely. We can immediately represent the Dirac-Kähler fermions by using these twisted supercharges as follows:

$$
\begin{aligned}
Q^{1}_{\alpha i} &\equiv \left\{ \left( s^{+} + \gamma^{\mu} s^{+}_{\mu} + \frac{1}{4}\gamma^{\mu\nu} s^{+}_{\mu\nu} \right) P_{+} \right\}_{\alpha i}, \\
Q^{2}_{\alpha i} &\equiv \left\{ \left( \overline{s}^{+} + \gamma^{\mu} \overline{s}^{+}_{\mu} + \frac{1}{4}\gamma^{\mu\nu} \overline{s}^{+}_{\mu\nu} \right) P_{+} \right\}_{\alpha i},
\end{aligned}
\tag{7.52}
$$

where we define $\{Q^{1}_{\alpha i}, Q^{2}_{\alpha i}\}$ as $Q^{\mathcal{A}}_{\alpha i}$. We can then represent $\{i, \mathcal{A}\} \in \{1, 2\}$ as the suffixes of the $SU(2)_{i}$ and $SU(2)_{\mathcal{A}}$ group, respectively. In this theory, the internal R-symmetry group is $SU(2)_{i} \otimes SU(2)_{\mathcal{A}} \simeq SO(4)_{I}$. We have shown that the untwisted theory possesses $N$=4 SUSY with $SO(4)_{I}$ R-symmetry. As the manner of the twist are well known in the papers [30, 31], the Vafa-Witten's twist is given by taking the diagonal sum of $SU(2)_{i}$ of the R-symmetry group $SU(2)_{i} \otimes SU(2)_{\mathcal{A}}$ and $SU(2)_{L}$ of the rotation group $SU(2)_{L} \otimes SU(2)_{R}$. In two component spinor notation, $Q^{\mathcal{A}}_{\alpha i}$ is divided into $\mathsf{Q}^{\mathcal{A}i}_{\alpha}$ and $\overline{\mathsf{Q}}^{\mathcal{A}}_{\dot{\alpha} i}$ where $\overline{\mathsf{Q}}^{\mathcal{A}}_{\dot{\alpha} i}$ is independent of $\mathsf{Q}^{\mathcal{A}i}_{\alpha}$ in Euclidean spacetime. We then identify internal $SU(2)$ index $i$ with $SU(2)_{L}$ index $\beta$ : $\mathsf{Q}^{\mathcal{A}i}_{\alpha} \to \mathsf{Q}^{\mathcal{A}\beta}_{\alpha}, \quad \overline{\mathsf{Q}}^{\mathcal{A}}_{\dot{\alpha} i} \to \overline{\mathsf{Q}}^{\mathcal{A}}_{\dot{\alpha}\beta}$. The supercharge $\mathsf{Q}^{\mathcal{A}\beta}_{\alpha}$ is expressed by the two scalar charges and the two anti-self-dual charges. On the other hand the supercharge $\overline{\mathsf{Q}}^{\mathcal{A}}_{\dot{\alpha}\beta}$ is expressed by two vector charges. We then found that the Dirac-Kähler twist of $Q^{\mathcal{A}}_{\alpha i}$ with chiral projection is equivalent to the Vafa-Witten's twist after rewriting the tensor representations with the spinor representations. We found that this action corresponded to the Vafa-Witten theory because of possessing the same ghost number of charges and the same algebras. Untwisting the theory, we have immediately derived the $N = 4$ super Yang-Mills theory by using the Dirac-Kähler mechanism.

One of the main aim is to establish the off-shell theory in the $N = 4$ twisted superspace. We have tried to construct the $N = 4$ twisted superconnection formalism of the twisted vector multiplet without a central charge. We then introduced the bosonic curvature superfields $\{\mathcal{F}, \mathcal{W}\}$ and $\{\mathcal{V}^{+}, \mathcal{V}^{+}_{A}\}$ corresponding to the Donaldson-Witten theory and the twisted hypermultiplet, respectively. We have imposed the special constraints on the anti-commutation relations of the fermionic supercovariant derivatives $\{\nabla^{+}, \nabla^{+}_{\mu}, \nabla^{+}_{A}, \overline{\nabla}^{+}, \overline{\nabla}^{+}_{\mu}, \overline{\nabla}^{+}_{A}\}$ and constructed the $N = 4$ twisted super Yang-Mills theory. Unfortunately, we have naturally derived the equations of motion from the Jacobi identities of the supercovariant derivatives and derived the same twisted supertransformations as the Donaldson-Witten theory coupled to the twisted real form of the hypermultiplet at the on-shell level.

In $N$=2 supersymmetric theory the action (3.21) has the internal automorphism group $SU(2)_i \otimes SU(2)_{PG}$. Thus we can construct three different twisted models: (i) diagonal subgroup of $SU(2)_i \otimes SU(2)_R$ and $SU(2)_{PG} \otimes SU(2)_R$, (ii) diagonal subgroup of $SU(2)_i \otimes SU(2)_R$ and $SU(2)_{PG} \otimes SU(2)_L$, (iii) diagonal subgroup of $SU(2)_i \otimes SU(2)_R$. In this paper we have proposed the type (i) twisted real form of the hypermultiplet model. The action (3.12) is similar to a vector-tensor multiplet [74–77]. A difference is that the vector-tensor multiplet is a gauge multiplet with central charge. One can derive a twisted version of the vector-tensor multiplet by using the Dirac-Kähler twist.

Some topological twists were classified by the way of taking diagonal sum for two global symmetries, i.e., the Lorentz symmetry and the internal R-symmetry in Euclidean flat spacetime [1, 26, 30, 31]. We are particularly interested in the Dirac-Kähler twist which is one of the representations of the topological twists because the Dirac-Kähler fermions correspond to the Kogut-Susskind or the staggered fermions on the lattice. In the recent years the twisted supersymmetric theory is applied to the lattice theory. We have constructed the Vafa-Witten theory by using the Dirac-Kähler mechanism in this paper. We may therefore construct this theory on the lattice.

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# Appendix A

The $N$=2 twisted SUSY transformations of the real form of the hypermultiplet coupling to the vector multiplet are given in the following lists.

| | $s^+$ | $s^+_\mu$ |
|---|---|---|
| $v$ | $\lambda$ | $\psi^\mu$ |
| $v^+_B$ | $-\lambda^+_B$ | $-\delta^+_{B\mu\nu}\psi^\nu$ |
| $\lambda$ | $\frac{1}{2}K - \frac{i}{2}[\phi, v]$ | $-\frac{i}{2}\mathcal{D}_\mu v + \frac{i}{2}\mathcal{D}^\nu v^+_{\mu\nu}$ |
| $\lambda^+_B$ | $-\frac{1}{2}K^+_B + \frac{i}{2}[\phi, v^+_A]$ | $\frac{i}{2}\delta^+_{B,\mu\nu}\mathcal{D}^\nu v + \frac{i}{2}\mathcal{D}_\mu v^+_B - \frac{i}{8}\Gamma^+_{BC\mu\nu}\mathcal{D}^\nu v^{+C}$ |
| $\psi_\nu$ | $-\frac{i}{2}\mathcal{D}_\nu v - \frac{i}{2}\mathcal{D}^\rho v^+_{\nu\rho}$ | $\frac{1}{2}\delta_{\mu\nu}K - \frac{1}{2}K^+_{\mu\nu} - \frac{i}{2}\delta_{\mu\nu}[\overline{\phi}, v] + \frac{i}{2}[\overline{\phi}, v^+_{\mu\nu}]$ |
| $K$ | $-i\mathcal{D}^\mu\psi_\mu + \frac{i}{2}[\chi, v]$ | $-i\mathcal{D}_\mu\lambda + i\mathcal{D}^\nu\lambda^+_{\mu\nu} + \frac{i}{2}[C_\mu, v]$ |
| | $+\frac{i}{8}[\chi^+_A, v^{+A}] + i[\overline{\phi}, \lambda]$ | $+\frac{i}{2}[C^\nu, v^+_{\mu\nu}] + i[\phi, \psi_\mu]$ |
| $K^+_B$ | $i\delta^+_{B,\mu\nu}\mathcal{D}^\mu\psi^\nu - \frac{i}{2}[\chi^+_B, v] + \frac{i}{2}[\chi, v^+_B]$ | $i\delta^+_{B,\mu\nu}\mathcal{D}^\nu\lambda + i\mathcal{D}_\mu\lambda^+_B + \frac{i}{4}\Gamma^+_{BC\mu\nu}\mathcal{D}^\nu\lambda^{+C}$ |
| | $-\frac{i}{32}\Gamma^+_{BCD}[\chi^{+C}, v^{+D}] - i[\overline{\phi}, \lambda^+_B]$ | $-\frac{i}{2}\delta^+_{B,\mu\nu}[C^\nu, v] + \frac{i}{2}[C_\mu, v^+_B]$ |
| | | $+\frac{i}{8}\Gamma^+_{BC\mu\nu}[C^\nu, v^{+C}] - i\delta^+_{B,\mu\nu}[\phi, \psi^\nu]$ |

| | $s^+_A$ |
|---|---|
| $v$ | $\lambda^+_A$ |
| $v^+_B$ | $\delta^+_{A,B}\lambda + \frac{1}{4}\Gamma^+_{ABC}\lambda^{+C}$ |
| $\lambda$ | $\frac{1}{2}K^+_A - \frac{i}{2}[\phi, v^+_A]$ |
| $\lambda^+_B$ | $\frac{1}{2}\delta^+_{A,B}K - \frac{1}{8}\Gamma^+_{ABC}K^{+C} - \frac{i}{2}\delta^+_{A,B}[\phi, v] + \frac{i}{8}\Gamma^+_{ABC}[\phi, v^{+C}]$ |
| $\psi_\nu$ | $+\frac{i}{2}\delta^+_{A,\nu\rho}\mathcal{D}^\rho v - \frac{i}{2}\mathcal{D}_\nu v^+_A + \frac{i}{8}\Gamma^+_{AB\nu\rho}\mathcal{D}^\rho v^{+B}$ |
| $K$ | $-i\delta^+_{A,\mu\nu}\mathcal{D}^\mu\psi^\nu - \frac{i}{2}[\chi, v^+_A] + \frac{i}{2}[\chi^+_A, v] + \frac{i}{32}\Gamma^+_{ABC}[\chi^{+B}, v^{+C}] + i[\overline{\phi}, \lambda^+_A]$ |
| $K^+_B$ | $-i\delta^+_{A,B}\mathcal{D}^\mu\psi_\mu - i\Gamma^+_{AB\mu\nu}\mathcal{D}^\mu\psi^\nu + \frac{i}{2}[\chi^+_A, v^+_B] - \frac{i}{2}[\chi^+_B, v^+_A] + \frac{i}{2}\delta^+_{A,B}[\chi, v]$ |
| | $+\frac{i}{8}\delta^+_{A,B}[\chi^{+C}, v^+_C] + \frac{i}{8}\Gamma^+_{ABC}[\chi^{+C}, v] - \frac{i}{8}\Gamma^+_{ABC}[\chi, v^{+C}]$ |
| | $+i\delta^+_{A,B}[\overline{\phi}, \lambda] + \frac{i}{4}\Gamma^+_{ABC}[\overline{\phi}, \lambda^{+C}]$ |

| | $Z$ |
|---|---|
| $v$ | $K$ |
| $v^+_B$ | $K^+_B$ |
| $\lambda$ | $-i\mathcal{D}^\mu\psi_\mu + \frac{i}{2}[\chi, v] + \frac{i}{8}[\chi^{+A}, v^+_A] + i[\overline{\phi}, \lambda]$ |
| $\lambda^+_B$ | $-i\delta^+_{B,\mu\nu}\mathcal{D}^\mu\psi^\nu + \frac{i}{2}[\chi^+_B, v] - \frac{i}{2}[\chi, v^+_B] + \frac{i}{32}\Gamma^+_{BCD}[\chi^{+C}, v^{+D}] + i[\overline{\phi}, \lambda^+_B]$ |
| $\psi_\nu$ | $-i\mathcal{D}_\nu\lambda + i\mathcal{D}^\rho\lambda^+_{\nu\rho} + \frac{i}{2}[C_\nu, v] + \frac{i}{2}[C^\rho, v^+_{\nu\rho}] + i[\phi, \psi_\nu]$ |
| $K$ | $-\mathcal{D}^\mu\mathcal{D}_\mu v + i\{C^\mu, \psi_\mu\} + \frac{i}{4}\{\chi^{+A}, \lambda^+_A\} + i\{\chi, \lambda\} - \frac{i}{4}[\phi^{+A}, v^+_A]$ |
| | $+i[\phi, K] + i[\overline{\phi}, K] + \frac{1}{2}[\phi, [\overline{\phi}, v]] + \frac{1}{2}[\overline{\phi}, [\phi, v]]$ |
| $K^+_B$ | $-\mathcal{D}^\mu\mathcal{D}_\mu v^+_B - i\delta^+_{B,\mu\nu}\{C^\mu, \psi^\nu\} + i\{\chi^+_B, \lambda\} - i\{\chi, \lambda^+_B\}$ |
| | $-\frac{i}{16}\Gamma^+_{BCD}\{\chi^{+C}, \lambda^{+D}\} + i[\phi^+_B, v] + i[\phi, K^+_B] + i[\overline{\phi}, K^+_B]$ |
| | $+\frac{i}{16}\Gamma^+_{BCD}[\phi^{+C}, v^{+D}] + \frac{1}{2}[\phi, [\overline{\phi}, v^+_B]] + \frac{1}{2}[\overline{\phi}, [\phi, v^+_B]]$ |

We show the full list of the on-shell $N$=4 twisted SUSY transformations and the R-transformations.

| | gh$\sharp$ | $s^+$ | $s^+_\mu$ |
|---|---|---|---|
| $v$ | 0 | $\frac{1}{2}\lambda$ | $-\frac{1}{2}\psi_\mu$ |
| $v^+_B$ | 0 | $-\frac{1}{2}\lambda^+_B$ | $\frac{1}{2}\delta^+_{B\mu\nu}\psi^\nu$ |
| $\lambda$ | 1 | $-i[\phi, v]$ | $-i\mathcal{D}_\mu v + i\mathcal{D}^\nu v^+_{\mu\nu}$ |
| $\lambda^+_B$ | 1 | $i[\phi, v^+_B]$ | $i\delta^+_{B,\mu\nu}\mathcal{D}^\nu v + i\mathcal{D}_\mu v^+_B - \frac{i}{4}\Gamma^+_{BC\mu\nu}\mathcal{D}^\nu v^{+C}$ |
| $\psi_\nu$ | $-1$ | $i\mathcal{D}_\nu v + i\mathcal{D}^\rho v^+_{\nu\rho}$ | $-i\delta_{\mu\nu}[\overline{\phi}, v] + i[\overline{\phi}, v^+_{\mu\nu}]$ |
| $\phi$ | 2 | 0 | $C_\mu$ |
| $\overline{\phi}$ | $-2$ | $-\chi$ | 0 |
| $\chi$ | $-1$ | $\frac{i}{2}[\phi, \overline{\phi}]$ | $i\mathcal{D}_\mu\overline{\phi}$ |
| $\chi^+_B$ | $-1$ | $\frac{i}{32}\Gamma^+_{BCD}[v^{+C}, v^{+D}]$ $-i[v^+_B, v] - 2F^+_B$ | $-i\delta^+_{B,\mu\nu}\mathcal{D}^\nu\overline{\phi}$ |
| $C_\nu$ | 1 | $-i\mathcal{D}_\nu\phi$ | $\frac{i}{32}\Gamma^+_{AB\mu\nu}[v^{+A}, v^{+B}] - i[v^+_{\mu\nu}, v]$ $+2F^-_{\mu\nu} - \frac{i}{2}\delta_{\mu\nu}[\phi, \overline{\phi}]$ |
| $A_\nu$ | 0 | $-\frac{i}{2}C_\nu$ | $-\frac{i}{2}(\delta_{\mu\nu}\chi - \chi^+_{\mu\nu})$ |

| | $s^+_A$ | $R^+_A$ |
|---|---|---|
| $v$ | $\frac{1}{2}\lambda^+_A$ | $-\frac{i}{2}v^+_A$ |
| $v^+_B$ | $\frac{1}{2}\delta^+_{A,B}\lambda + \frac{1}{8}\Gamma^+_{ABC}\lambda^{+C}$ | $\frac{i}{2}\delta^+_{A,B}v - \frac{i}{8}\Gamma^+_{ABC}v^{+C}$ |
| $\lambda$ | $-i[\phi, v^+_A]$ | 0 |
| $\lambda^+_B$ | $-i\delta^+_{A,B}[\phi, v] + \frac{i}{4}\Gamma^+_{ABC}[\phi, v^{+C}]$ | 0 |
| $\psi_\nu$ | $-i\delta^+_{A,\nu\rho}\mathcal{D}^\rho v + i\mathcal{D}_\nu v^+_A - \frac{i}{4}\Gamma^+_{AB\nu\rho}\mathcal{D}^\rho v^{+B}$ | 0 |
| $\phi$ | 0 | 0 |
| $\overline{\phi}$ | $-\chi^+_A$ | 0 |
| $\chi$ | $-\frac{i}{32}\Gamma^+_{ABC}[v^{+B}, v^{+C}] + i[v^+_A, v] + 2F^+_A$ | $-\frac{i}{2}\chi^+_A$ |
| $\chi^+_B$ | $i[v^+_A, v^+_B] - \frac{i}{4}\Gamma^+_{ABC}[v^{+C}, v] + \frac{1}{2}\Gamma^+_{ABC}F^{+C}$ $+\frac{i}{2}\delta^+_{A,B}[\phi, \overline{\phi}]$ | $\frac{i}{2}\delta^+_{A,B}\chi - \frac{i}{8}\Gamma^+_{ABC}\chi^{+C}$ |
| $C_\nu$ | $i\delta^+_{A,\nu\rho}\mathcal{D}^\rho\phi$ | $-\frac{i}{2}\delta^+_{A,\nu\rho}C^\rho$ |
| $A_\nu$ | $-\frac{i}{2}\delta^+_{A,\nu\rho}C^\rho$ | 0 |

| | $\overline{s}^+$ | $\overline{s}^+_\mu$ |
|---|---|---|
| $v$ | $\frac{1}{2}\chi$ | $-\frac{1}{2}C_\mu$ |
| $v^+_B$ | $\frac{1}{2}\chi^+_B$ | $-\frac{1}{2}\delta^+_{B\mu\nu}C^\nu$ |
| $\lambda$ | $-\frac{i}{2}[\phi,\overline{\phi}]$ | $-i\mathcal{D}_\mu\phi$ |
| $\lambda^+_B$ | $\frac{i}{32}\Gamma^+_{BCD}[v^{+C},v^{+D}]+i[v^+_B,v]-2F^+_B$ | $i\delta^+_{B,\mu\nu}\mathcal{D}^\nu\phi$ |
| $\psi_\nu$ | $i\mathcal{D}_\nu\overline{\phi}$ | $\frac{i}{32}\Gamma^+_{AB\mu\nu}[v^{+A},v^{+B}]+i[v^+_{\mu\nu},v]+2F^-_{\mu\nu}$ $+\frac{i}{2}\delta_{\mu\nu}[\phi,\overline{\phi}]$ |
| $\phi$ | $\lambda$ | $0$ |
| $\overline{\phi}$ | $0$ | $-\psi_\mu$ |
| $\chi$ | $i[\overline{\phi},v]$ | $-i\mathcal{D}_\mu v-i\mathcal{D}^\nu v^+_{\mu\nu}$ |
| $\chi^+_B$ | $i[\overline{\phi},v^+_B]$ | $i\delta^+_{B,\mu\nu}\mathcal{D}^\nu v-i\mathcal{D}_\mu v^+_B+\frac{i}{4}\Gamma^+_{BC\mu\nu}\mathcal{D}^\nu v^{+C}$ |
| $C_\nu$ | $i\mathcal{D}_\nu v-i\mathcal{D}^\rho v^+_{\nu\rho}$ | $i\delta_{\mu\nu}[\phi,v]+i[\phi,v^+_{\mu\nu}]$ |
| $A_\nu$ | $-\frac{i}{2}\psi_\nu$ | $-\frac{i}{2}(\delta_{\mu\nu}\lambda-\lambda^+_{\mu\nu})$ |

| | $\overline{s}^+_A$ | $\overline{R}^+_A$ |
|---|---|---|
| $v$ | $\frac{1}{2}\chi^+_A$ | $\frac{i}{2}v^+_A$ |
| $v^+_B$ | $-\frac{1}{2}\delta^+_{A,B}\chi-\frac{1}{8}\Gamma^+_{ABC}\chi^{+C}$ | $-\frac{i}{2}\delta^+_{A,B}v-\frac{i}{8}\Gamma^+_{ABC}v^{+C}$ |
| $\lambda$ | $-\frac{i}{32}\Gamma^+_{ABC}[v^{+B},v^{+C}]-i[v^+_A,v]+2F^+_A$ | $-\frac{i}{2}\lambda^+_A$ |
| $\lambda^+_B$ | $i[v^+_A,v^+_B]+\frac{i}{4}\Gamma^+_{ABC}[v^{+C},v]+\frac{1}{2}\Gamma^+_{ABC}F^{+C}$ $-\frac{i}{2}\delta^+_{A,B}[\phi,\overline{\phi}]$ | $\frac{i}{2}\delta^+_{A,B}\lambda-\frac{i}{8}\Gamma^+_{ABC}\lambda^{+C}$ |
| $\psi_\nu$ | $-i\delta^+_{A,\nu\rho}\mathcal{D}^\rho\overline{\phi}$ | $-\frac{i}{2}\delta^+_{A,\mu\nu}\psi^\nu$ |
| $\phi$ | $\lambda^+_A$ | $0$ |
| $\overline{\phi}$ | $0$ | $0$ |
| $\chi$ | $-i[\overline{\phi},v^+_A]$ | $0$ |
| $\chi^+_B$ | $i\delta^+_{A,B}[\overline{\phi},v]+\frac{i}{4}\Gamma^+_{ABC}[\overline{\phi},v^{+C}]$ | $0$ |
| $C_\nu$ | $-i\delta^+_{A,\nu\rho}\mathcal{D}^\rho v-i\mathcal{D}_\nu v^+_A+\frac{i}{4}\Gamma^+_{AB\nu\rho}\mathcal{D}^\rho v^{+B}$ | $0$ |
| $A_\nu$ | $-\frac{i}{2}\delta^+_{A,\nu\rho}\psi^\rho$ | $0$ |

The $N$=4 twisted supercharge $\{s^+,s^+_\mu,s^+_A,\overline{s}^+,\overline{s}^+_\mu,\overline{s}^+_A\}$ satisfy the following commutation relations up to the gauge transformation at on-shell level.

$$\{s^+,s^+_\mu\}\varphi=-i(\partial_\mu\varphi+\delta_{\mathrm{g}(-\omega_\mu)}\varphi),\qquad \{s^+_\mu,s^+_A\}\varphi=i\delta^+_{A,\mu\nu}(\partial^\nu\varphi+\delta_{\mathrm{g}(-\omega^\nu)}\varphi)$$
$$\{\overline{s}^+,\overline{s}^+_\mu\}\varphi=-i(\partial_\mu\varphi+\delta_{\mathrm{g}(-\omega_\mu)}\varphi),\qquad \{\overline{s}^+_\mu,\overline{s}^+_A\}\varphi=i\delta^+_{A,\mu\nu}(\partial^\nu\varphi+\delta_{\mathrm{g}(-\omega^\nu)}\varphi)$$

$$\begin{aligned}
&\{s^+,s^+\}\varphi=\delta_{\mathrm{g}(-\phi)}\varphi, && \{s^+_\mu,s^+_\nu\}\varphi=\delta_{\mu\nu}\delta_{\mathrm{g}(\overline{\phi})}\varphi, && \{s^+_A,s^+_B\}\varphi=\delta^+_{A,B}\delta_{\mathrm{g}(-\phi)}\varphi,\\
&\{\overline{s}^+,\overline{s}^+\}\varphi=\delta_{\mathrm{g}(\overline{\phi})}\varphi, && \{\overline{s}^+_\mu,\overline{s}^+_\nu\}\varphi=\delta_{\mu\nu}\delta_{\mathrm{g}(-\phi)}\varphi, && \{\overline{s}^+_A,\overline{s}^+_B\}\varphi=\delta^+_{A,B}\delta_{\mathrm{g}(\overline{\phi})}\varphi,\\
&\{s^+,\overline{s}^+_\mu\}\varphi=0, && \{s^+,\overline{s}^+_A\}\varphi=\delta_{\mathrm{g}(-v^+_A)}\varphi, && \{s^+_A,\overline{s}^+_\mu\}\varphi=0\\
&\{\overline{s}^+,s^+_\mu\}\varphi=0, && \{\overline{s}^+,s^+_A\}\varphi=\delta_{\mathrm{g}(v^+_A)}\varphi, && \{\overline{s}^+_A,s^+_\mu\}\varphi=0
\end{aligned}$$

$$\{s_A^+, \overline{s}_B^+\}\varphi = \delta_{A,B}^+ \delta_{\mathrm{g}(v)}\varphi - \frac{1}{4}\Gamma_{ABC}^+ \delta_{\mathrm{g}(v^{+C})}\varphi, \qquad \{s^+, \overline{s}^+\}\varphi = \delta_{\mathrm{g}(v)}\varphi,$$
$$\{s_\mu^+, \overline{s}_\nu^+\}\varphi = -\delta_{\mu\nu}\delta_{\mathrm{g}(v)}\varphi + \delta_{\mathrm{g}(v_{\mu\nu}^+)}\varphi,$$
$$\{s^+, s_A^+\}\varphi = 0, \qquad \{\overline{s}^+, \overline{s}_A^+\}\varphi = 0,$$
where $\delta_{\mathrm{g}(\epsilon)}\varphi \equiv i[\epsilon, \varphi]$ and $\delta_{\mathrm{g}(\epsilon)}A_\mu = D_\mu \epsilon$ .

# Appendix B

In this appendix we define Euclidean four dimensional $\gamma$-matrices:
$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \gamma^{\mu\dagger} = \gamma^\mu, \tag{B.1}$$
where $\gamma_\mu$ satisfies the Clifford algebra. We use the following notations:
$$\gamma_{\mu\nu} \equiv \frac{1}{2}[\gamma_\mu, \gamma_\nu], \qquad \tilde{\gamma}_\mu \equiv \gamma_\mu\gamma_5. \tag{B.2}$$
$\gamma^\mu$ can be constructed from the 2×2 matrices $\sigma^\mu$ and $\bar{\sigma}^\mu$ in the following way:
$$\gamma^\mu = \begin{pmatrix} 0 & i\sigma^\mu \\ i\bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{B.3}$$
where $\sigma^\mu = (\sigma^1, \sigma^2, \sigma^3, \sigma^4)$ and $\bar{\sigma}^\mu = (-\sigma^1, -\sigma^2, -\sigma^3, \sigma^4)$. $\sigma^i$ are Pauli matrices for $i \in \{1, 2, 3\}$ and $\sigma^4 = i\mathbf{1}_{2\times 2}$. A matrix $B$ and a charge conjugation matrix $C$ are defined as follows:
$$\begin{aligned} \gamma_\mu &= \eta B^{-1}\gamma_\mu^* B, & B^*B &= \epsilon\mathbf{1}, \\ \gamma_\mu &= \eta C^{-1}\gamma_\mu^T C, & C^T &= \epsilon C, \end{aligned} \tag{B.4}$$
where $(\eta, \epsilon) = (\pm 1, -1)$.

In a four dimensional Euclidean space Majorana fermions do not exist because the factor $\epsilon$ should be equal to $-1$ [78]. A Majorana spinor satisfies the following condition,
$$\psi^* = B\psi, \tag{B.5}$$
which leads
$$\psi = B^*\psi = B^*B\psi. \tag{B.6}$$
Thus the existence of Majorana fermion requires $B^*B = 1$. This condition can not be taken in four dimensional Euclidean space. We can, however, take a SU(2)≃USp(2) Majorana fermion and a USp(4) Majorana fermion which satisfy the following condition, respectively,
$$\psi^{i*} = \epsilon^{ij}B\psi^j, \tag{B.7}$$
$$\psi^{l*} = C^{lm}B\psi^m, \tag{B.8}$$
where $i, j \in \{1, 2\}, \quad l, m \in \{1, 2, 3, 4\}$ and these fermions correspond to the fermions which appear in $N$=2 and $N$=4 supersymmetric theory, respectively. In this paper we choose $(\eta, \epsilon) = (1, -1)$ and $C = B = -\gamma_1\gamma_3$.

# Appendix C

In this appendix we explain the $N$=4 SUSY algebra with USp(4) Majorana condition in Euclidean spacetime. $N$=4 SUSY algebra is

$$\{Q_{\alpha I}, \overline{Q}_{J\beta}\} = 2\mathbf{1}_{IJ}(\gamma^\mu)_{\alpha\beta}P_\mu. \tag{B.9}$$

We can also represent the four components supercharge as the following form,

$$Q_{\alpha I} \ = \ \begin{pmatrix} \mathsf{Q}_\alpha{}^i & i\tilde{\mathsf{Q}}_{\alpha\dot{i}} \\ i\overline{\mathsf{Q}}^{\dot{\alpha}i} & \overline{\tilde{\mathsf{Q}}}^{\dot{\alpha}}{}_{\dot{i}} \end{pmatrix}, \tag{B.10}$$

$$\mathbf{1}_{IJ} = \begin{pmatrix} \delta_i{}^j & 0 \\ 0 & \delta_{\dot{i}}{}^{\dot{j}} \end{pmatrix}, \qquad (\gamma^\mu)_{\alpha\beta} = \begin{pmatrix} 0 & i(\sigma^\mu)_{\alpha\dot{\beta}} \\ i(\overline{\sigma}^\mu)^{\dot{\alpha}\beta} & 0 \end{pmatrix}. \tag{B.11}$$

The USp(4) Majorana condition (B.8) is the following form,

$$(Q_{\alpha I})^* \ = \ C_{\alpha\beta}Q_{\beta J}C^{-1}_{JI}, \tag{B.12}$$

$$\begin{pmatrix} \mathsf{Q}_\alpha{}^i & i\tilde{\mathsf{Q}}_{\alpha\dot{i}} \\ i\overline{\mathsf{Q}}^{\dot{\alpha}i} & \overline{\tilde{\mathsf{Q}}}^{\dot{\alpha}}{}_{\dot{i}} \end{pmatrix}^* \ = \ \begin{pmatrix} -\mathsf{Q}^\alpha{}_i & i\tilde{\mathsf{Q}}^{\alpha\dot{i}} \\ i\overline{\mathsf{Q}}_{\dot{\alpha}i} & -\overline{\tilde{\mathsf{Q}}}_{\dot{\alpha}}{}^{\dot{i}} \end{pmatrix}. \tag{B.13}$$

Using the USp(4) Majorana condition, we define a $\overline{Q}_{I\alpha}$ as:

$$\overline{Q}_{I\alpha} = (Q_{\alpha I})^\dagger = \begin{pmatrix} (\mathsf{Q}_\alpha{}^i)^* & (i\overline{\mathsf{Q}}^{\dot{\alpha}i})^* \\ (i\tilde{\mathsf{Q}}_{\dot{\alpha}\dot{i}})^* & (\overline{\tilde{\mathsf{Q}}}^{\dot{\alpha}}{}_{\dot{i}})^* \end{pmatrix} = \begin{pmatrix} -\mathsf{Q}^\alpha{}_i & i\overline{\mathsf{Q}}_{\dot{\alpha}i} \\ i\tilde{\mathsf{Q}}^{\alpha\dot{i}} & -\overline{\tilde{\mathsf{Q}}}_\alpha{}^{\dot{i}} \end{pmatrix}, \tag{B.14}$$

We can then describe the algebra (B.9) with respect to theses two-component supercharges,

$$\{\mathsf{Q}_\alpha{}^i, \overline{\mathsf{Q}}_{\dot{\alpha}j}\} = 2\delta^i{}_j(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu, \qquad \{\tilde{\mathsf{Q}}_\alpha{}^i, \overline{\tilde{\mathsf{Q}}}_{\dot{\alpha}j}\} = 2\delta^i{}_j(\sigma^\mu)_{\alpha\dot{\alpha}}P_\mu, \tag{B.15}$$

where supercharges with upper and lower indices are related through the $\epsilon$-tensor: $\mathsf{Q}^\alpha = \epsilon^{\alpha\beta}\mathsf{Q}_\beta,\ \ \mathsf{Q}_\alpha = -\epsilon_{\alpha\beta}\mathsf{Q}^\beta$.

In two component spinor notation, the Dirac-Kähler twist give the following relations,

$$\begin{aligned} \mathsf{Q}_\alpha{}^i \to \mathsf{Q}_\alpha{}^\beta &= s^+\delta_\alpha{}^\beta - \frac{1}{4}s^+_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta, \qquad & \overline{\mathsf{Q}}^{\dot{\alpha}i} \to \overline{\mathsf{Q}}^{\dot{\alpha}\beta} &= s^+_\mu(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}, \\ \overline{\tilde{\mathsf{Q}}}^{\dot{\alpha}}{}_{\dot{i}} \to \overline{\tilde{\mathsf{Q}}}^{\dot{\alpha}}{}_{\dot{\beta}} &= s^-\delta^{\dot{\alpha}}{}_{\dot{\beta}} - \frac{1}{4}s^-_{\mu\nu}(\sigma^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}}, \qquad & \tilde{\mathsf{Q}}_{\alpha\dot{i}} \to \tilde{\mathsf{Q}}_{\alpha\dot{\beta}} &= s^-_\mu(\sigma^\mu)_{\alpha\dot{\beta}}. \end{aligned}$$

On the other hand, Vafa-Witten twist is the following relations,

$$\mathsf{Q}^{\mathcal{A}i}_\alpha \to \mathsf{Q}^{\mathcal{A}\beta}_\alpha = s^{\mathcal{A}+}\delta_\alpha{}^\beta - \frac{1}{4}s^{\mathcal{A}+}_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta, \qquad \overline{\mathsf{Q}}^{\mathcal{A}\dot{\alpha}i} \to \overline{\mathsf{Q}}^{\mathcal{A}\dot{\alpha}\beta} = s^{\mathcal{A}+}_\mu(\overline{\sigma}^\mu)^{\dot{\alpha}\beta}, \tag{B.16}$$

where $\mathcal{A} \in \{1,2\}$ and $\{s^{1+}, s^{1+}_\mu, s^{1+}_{\mu\nu}, s^{2+}, s^{2+}_\mu, s^{2+}_{\mu\nu}\} = \{s^+, s^+_\mu, s^+_{\mu\nu}, \overline{s}^+, \overline{s}^+_\mu, \overline{s}^+_{\mu\nu}\}$.

# Appendix D

In this appendix we give the definition of $\delta^{\pm}_{A,B}$ and $\Gamma^{\pm}_{ABC}$. The suffix $A$ is the second rank tensor which denotes the suffix $\mu\nu$: $\mu,\nu \in \{1,\cdots,4\}$. The definition of $\delta^{\pm}_{A,B}$ is

$$\delta^{\pm}_{A,B} = \delta^{\pm}_{\mu\nu,\rho\sigma} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} \mp \epsilon_{\mu\nu\rho\sigma}, \tag{D.1}$$

where $\delta^{\pm A,B}\delta^{\mp}{}_{A,B} = 0$. (Anti-)self-dual tensors $\chi^{\pm A}$ satisfy

$$\chi^{\pm A} = \frac{1}{4}\delta^{\pm A,B}\chi_B, \tag{D.2}$$

where $\chi_B = \chi^+_B + \chi^-_B$.

Variants of the definition of $\Gamma_{ABC}$ which stand for the third anti-symmetric tensor for $ABC$ is

$$\begin{aligned}
\Gamma^{\pm\mu\alpha,\nu\beta,\rho\gamma} &= \delta^{\alpha\nu}\delta^{\beta\rho}\delta^{\gamma\mu} + \delta^{\mu\nu}\delta^{\beta\gamma}\delta^{\rho\alpha} + \delta^{\alpha\beta}\delta^{\nu\gamma}\delta^{\rho\mu} + \delta^{\mu\beta}\delta^{\nu\rho}\delta^{\gamma\alpha} \\
&\quad -(\delta^{\alpha\nu}\delta^{\beta\gamma}\delta^{\rho\mu} + \delta^{\mu\nu}\delta^{\beta\rho}\delta^{\gamma\alpha} + \delta^{\alpha\beta}\delta^{\nu\rho}\delta^{\gamma\mu} + \delta^{\mu\beta}\delta^{\nu\gamma}\delta^{\rho\alpha}) \\
&\quad \mp\epsilon^{\mu\alpha\beta\gamma}\delta^{\beta\gamma} \mp \epsilon^{\mu\alpha\nu\rho}\delta^{\beta\gamma} \pm \epsilon^{\mu\alpha\nu\gamma}\delta^{\beta\rho} \pm \epsilon^{\mu\alpha\beta\rho}\delta^{\nu\gamma} \\
&= \delta^{\pm\mu\alpha,\nu\rho}\delta^{\beta\gamma} + \delta^{\pm\mu\alpha,\beta\gamma}\delta^{\nu\rho} - \delta^{\pm\mu\alpha,\nu\gamma}\delta^{\beta\rho} - \delta^{\pm\mu\alpha,\beta\rho}\delta^{\nu\gamma}, &\text{(D.3)} \\
\Gamma^{\pm AB\rho\sigma} &= \frac{1}{2}(\delta^{\pm A,\nu\rho}\delta^{\pm B}{}_{,\nu}{}^{\sigma} - \delta^{\pm A,\nu\sigma}\delta^{\pm B}{}_{,\nu}{}^{\sigma}), &\text{(D.4)} \\
\Gamma^{\pm ABC} &= \frac{1}{4}\delta^{\pm A,\nu\rho}\delta^{\pm B}{}_{,\nu}{}^{\sigma}\delta^{\pm C}{}_{,\rho\sigma}. &\text{(D.5)}
\end{aligned}$$

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