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Twist Quantization of String and B Field Background

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Abstract: In a previous paper, we investigated the Hopf algebra structure in string theory and gave a unified formulation of the quantization of the string and the space-time symmetry. In this paper, this formulation is applied to the case with a nonzero B-field background, and the twist of the Poincaré symmetry is studied. The Drinfeld twist accompanied by the B-field background gives an alternative quantization scheme, which requires a new normal ordering. In order to obtain a physical interpretation of this twisted Hopf algebra structure, we propose a method to decompose the twist into two successive twists and we give two different possibilities of decomposition. The first is a natural decomposition from the viewpoint of the twist quantization, leading to a new type of twisted Poincaré symmetry. The second decomposition reveals the relation of our formulation to the twisted Poincaré symmetry on the Moyal type noncommutative space.

Contents

1. Introduction

The connection between string theory and noncommutative geometry is an important subject and has been developed especially in the case of D-branes in the NSNS B-field background (for a review see Refs.[1, 2]). In a background of constant B field, the effective theory on the D-branes in a certain zero-slope limit is described very well by the gauge theory on the noncommutative space [3], where all the products are replaced by the Moyal-Weyl product. The noncommutativity of the space,

$$[x^\mu \overset{*}{,} x^\nu] = i\theta^{\mu\nu}, \tag{1.1}$$

originates from the operator product expansion of open string vertex operators [4], or equivalently, from the canonical quantization of open strings in a B-field background [5]. This connection drew the interests towards phenomena like UV/IR mixing, noncommutative solitons, open Wilson lines, etc., however, the explicit breaking of the Lorentz symmetry was a bottleneck.

It is known that in a wide class of noncommutative spaces, Hopf algebras are considered to be the underlying symmetries rather than just groups or Lie algebras [6, 7, 8]. When we consider noncommutative spaces as deformations of commutative spaces, such Hopf algebras are obtained by a Drinfeld twist of the Hopf algebra of the corresponding symmetry group [9] of the underlying undeformed spaces. Therefore, it is natural to expect that

there is also a Hopf algebra structure corresponding to the deformed symmetry on the Moyal-Weyl space. In fact, such twisted Hopf algebra structures were studied and the twist element corresponding to the Moyal-Weyl product was discovered in Refs.[10, 11] for the translation symmetry, and in [10] for the Poincaré group. More recently, the twisted Poincaré symmetry of the noncommutative field theory has been fully realized in Refs.[12, 13, 14]. There, the Moyal-Weyl product is considered as a twisted product equipped with the Drinfeld twist of the Poincaré-Lie algebra. In other words, both the noncommutativity and the modification of the symmetry are controlled by a single twist. This approach is then generalized to the twisted version of the diffeomorphism on the Moyal-Weyl space [14, 15] (see for example Ref.[16] and references therein on these developments).

Considering the string theory, one may raise the question whether there is a string theory counterpart of these twisted Hopf algebra structure, and if so, how it is realized in the string theory. In this respect, the authors of Ref.[17] studied a effective theory on D-branes in a zero slope limit, where the leading brane-induced gravitational effects are visible, and are compared with the noncommutative gravity action proposed in Ref.[14]. The authors of Ref.[17] were led to a negative answer, since effective gravity in a B-field background has more interaction terms than that in Ref.[14]. However, we have to keep in mind that there are always ambiguities when comparing two theories only on-shell, and thus it is difficult to capture the essential difference between these two results by a comparison of the actions. It is plausible that a direct comparison of the underlying symmetries, without recourse to the actions, will lead to a more concrete answer. The purpose of this paper is to give a suitable framework to answer the above questions.

In a previous paper Ref.[18], we have investigated the Hopf algebra structure in string worldsheet theory in the Minkowski background and gave a unified formulation of the quantization of the string and the spacetime symmetry, by reformulating the path-integral quantization of the string as a Drinfeld twist at the worldsheet level. The coboundary relation showed that this was equivalent to operators with normal ordering. By the twist, the space-time diffeomorphism was deformed into a twisted Hopf algebra, while the Poincaré symmetry was kept unchanged. This result suggests a characterization of the symmetry as follows: unbroken symmetries are twist invariant Hopf subalgebras, while broken symmetries are realized as twisted ones.

Although in Ref.[18] only the case of a Minkowski spacetime as a background was considered explicitly, the process is applicable to a wider class of backgrounds. In this paper, we apply our formulation to the case of the presence of a nonzero B-field, as the most simple example of such a background, and show that there is also a natural structure of a twisted Hopf algebra, which depends on the B-field. As a quantization, this twisted Hopf algebra structure corresponds to a new normal ordering in the operator formulation, which was studied in Refs.[19, 20, 21, 22]. As a symmetry, it represents a twisted symmetry including space-time diffeomorphisms.

To understand the geometrical picture and the structure of the twisted symmetry in a B-field background, we formulate a method to decompose the twist into a combined operation of two successive twists. As we will see, we can interpret the first twist as a quantization and the second twist as a deformation of the first twisted Hopf algebra.

We give two types of decomposition. The first type is natural from the viewpoint of the twist quantization and useful for comparing the twisted Hopf algebra structure in $B \neq 0$ with the corresponding structure in the case $B = 0$, treated in Ref.[18]. The result indicates the existence of a geometrical spacetime description different from the commutative spacetime with B-field, where the effect of the B-field is hidden in a twist element. This is similar to the noncommutative spacetime description used in open string theory, however the formulation using the twist is valid for closed strings as well. The second type of decomposition is useful when we want to compare the Hopf algebra structure in the open string sector with the twisted Poincaré symmetry[12, 13, 14] in the Moyal-Weyl noncommutative space.

The paper is organized as follows: In section 2 we give a brief review of the results of our previous paper, Ref.[18] and explain some notations, which are necessary to proceed the analysis here. In section 3, applying the formulation proposed in Ref.[18] to the case of non-zero B-field background, we present the twist quantization of the string worldsheet theory and the corresponding twisted symmetry for the case $B \neq 0$. In section 4, we first give a method to decompose the twist discussed in section 3 into two successive twists, and discuss its interpretation. We apply this splitting method of Hopf algebra twist in two different ways. Section 5 is devoted to discussion and conclusion.

2. Hopf algebra in string theory

In this section, we briefly review the content of Ref.[18] with a slightly generalized form so that the discussions in the following section becomes more transparent.

2.1 Classical Hopf and module algebras

Our starting point is the classical Hopf algebra $\mathcal{H}$ of functional derivatives and its module algebra $\mathcal{A}$ of string functionals. The functionals depend on a target space manifold $\mathbb{R}^d$ but are independent of the background such as the metric and B field (see Ref.[18] for more details).

The classical string variable $X^\mu(z)\,(\mu = 0, \cdots d-1)$ is a set of functions defining an embedding $X : \Sigma \to \mathbb{R}^d$ of a worldsheet into a target space. Let $\mathcal{A}$ be the space of the complex valued embedding functionals $X^\mu(z)$ together with their worldsheet derivatives $\partial_a X^\mu(z)$ of the form

$$I[X] = \int d^2z\ \rho(z) F[X(z)]\ . \tag{2.1}$$

where $F[X(z)] \in C^\infty(\Sigma)$ is a component of a pull-back tensor field in the target space and $\rho(z)$ is some weight function (distribution). In particular, we call

$$F[X](z_i) = \int d^2z\ \delta^{(2)}(z - z_i) F[X(z)]\ , \tag{2.2}$$

a local functional at z_i, with an additional label $z_i \in \Sigma$. We also write it simply as $F[X(z_i)]$ when it is not confusing. These functionals (2.1) and (2.2) correspond to an integrated vertex operator and to a local vertex operator after quantization, respectively.

We define a multiplication of two functionals $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ as $I_1 I_2[X] = I_1[X] I_2[X]$, where the r.h.s. is the multiplication in $\mathbb{C}$, in particular, $FG[X](z_1, z_2) = F[X(z_1)]G[X(z_2)]$ for two local functionals. By including all multi-local functionals with countable labels, $\mathcal{A}$ forms an algebra over $\mathbb{C}$. Note that its product is commutative and associative.

Next, let $\mathfrak{X}$ be the space of all functional vector fields of the form

$$\xi = \int d^2 w \, \xi^\mu(w) \frac{\delta}{\delta X^\mu(w)}, \tag{2.3}$$

and denote its action on $I[X] \in \mathcal{A}$ as $\xi \triangleright I[X]$. Here $\xi^\mu(w)$ is a weight function (distribution) on the worldsheet including the following two classes.

i) $\xi^\mu(w)$ is a pull-back of a target space function $\xi^\mu(w) = \xi^\mu[X(w)]$. It is related to the variation of the functional under diffeomorphism $X^\mu(z) \to X^\mu(z) + \xi^\mu[X(z)]$ as $\delta_\xi F[X] = -\xi \triangleright F[X]$.

ii) $\xi^\mu(w)$ is a function of w but independent of $X(w)$ and its derivatives. It corresponds to a change of the embedding, $X^\mu(z) \to X^\mu(z) + \xi^\mu(z)$, and is used to derive the equation of motion.

By successive transformations, $\xi \triangleright (\eta \triangleright F)$, the vector fields ξ and η form a Lie algebra with the Lie bracket

$$[\xi, \eta] = \int d^2 w \left(\xi^\mu \frac{\delta \eta^\nu}{\delta X^\mu} - \eta^\mu \frac{\delta \xi^\nu}{\delta X^\mu} \right)(w) \frac{\delta}{\delta X^\nu(w)} \quad . \tag{2.4}$$

Now we can define the universal enveloping algebra $\mathcal{H} = U(\mathfrak{X})$ of $\mathfrak{X}$ over $\mathbb{C}$, which has a natural cocommutative Hopf algebra structure $(U(\mathfrak{X}); \mu, \iota, \Delta, \epsilon, S)$. The defining maps given on elements $\xi, \eta \in \mathfrak{X}$ are

$$\begin{aligned}
&\mu(\xi \otimes \eta) = \xi \cdot \eta \ , \quad \iota(k) = k \cdot 1 \ , \\
&\Delta(1) = 1 \otimes 1 \ , \quad \Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi \ , \\
&\epsilon(1) = 1 \ , \quad \epsilon(\xi) = 0 \ , \\
&S(1) = 1, \quad S(\xi) = -\xi \ ,
\end{aligned} \tag{2.5}$$

where $k \in \mathbb{C}$. As usual, these maps are uniquely extended to any such element of $U(\mathfrak{X})$ by the algebra (anti-) homomorphism. In particular, the Poincaré-Lie algebra $\mathcal{P}$ generated by

$$\begin{aligned}
P^\mu &= -i \int d^2 z \, \eta^{\mu\lambda} \frac{\delta}{\delta X^\lambda(z)}, \\
L^{\mu\nu} &= -i \int d^2 z X^{[\mu}(z) \eta^{\nu]\lambda} \frac{\delta}{\delta X^\lambda(z)},
\end{aligned} \tag{2.6}$$

where P^μ are the generators of the translation and $L^{\mu\nu}$ are the Lorentz generators, forms a Hopf subalgebra of $\mathcal{H} = U(\mathfrak{X})$, denoted as $U(\mathcal{P})$. We denote the abelian Lie subalgebra consisting of ξ (2.3) in class ii) as $\mathfrak{C}$. Then $U(\mathfrak{C})$ is also a Hopf subalgebra of $\mathcal{H}$.

The algebra $\mathcal{A}$ of functionals is now considered as a $\mathcal{H}$-module algebra. The action of an element $h \in \mathcal{H}$ on $F \in \mathcal{A}$ is denoted by $h \triangleright F$, as above. The action on the product of two elements $F, G \in \mathcal{A}$ is defined by

$$h \triangleright m(F \otimes G) = m \, \Delta(h) \triangleright (F \otimes G) \ , \tag{2.7}$$

which represents the covariance of the module algebra $\mathcal{A}$ under diffeomorphisms or worldsheet variations.

2.2 Quantization as a Hopf algebra twist

In Ref.[18] we gave a simple quantization procedure in terms of a Hopf algebra twist, in which the vacuum expectation value (VEV) coincides with the conventional path integral average. A general theory of Hopf algebra twist is presented in Ref.[6] and for details relating to our approach see also the Appendix in Ref.[18].

Suppose that there is a twist element (counital 2-cocycle), $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$, which is invertible, counital $(\mathrm{id} \otimes \epsilon)\mathcal{F} = 1$ and satisfies the 2-cocycle condition

$$(\mathcal{F} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)\mathcal{F} \ . \tag{2.8}$$

In this paper it is sufficient to assume that a twist element is in the abelian Hopf subalgebra $\mathcal{F} \in U(\mathfrak{C}) \otimes U(\mathfrak{C})$ of the form

$$\mathcal{F} = \exp\left(-\int d^2z \int d^2w \, G^{\mu\nu}(z,w) \frac{\delta}{\delta X^\mu(z)} \otimes \frac{\delta}{\delta X^\nu(w)} \right) , \tag{2.9}$$

which is specified by a Green function $G^{\mu\nu}(z,w)$ on the worldsheet. It is easy to show that this $\mathcal{F}$ in (2.9) satisfies all conditions for twist elements[18].

Given a twist element $\mathcal{F}$, the twisted Hopf algebra $\mathcal{H}_\mathcal{F}$ can be defined using the algebra and counit of the untwisted $\mathcal{H}$, but with coproduct and antipode twisted

$$\Delta_\mathcal{F}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \quad S_\mathcal{F}(h) = US(h)U^{-1} \tag{2.10}$$

for all $h \in \mathcal{H}$, where $U = \mu(\mathrm{id} \otimes S)\mathcal{F}$. We regard this procedure of twisting as a quantization with respect to the twist element $\mathcal{F}$.

Correspondingly, the consistency of the action, i.e. covariance with respect to the Hopf algebra action, requires that a $\mathcal{H}$-module algebra $\mathcal{A}$ is twisted to the $\mathcal{H}_\mathcal{F}$-module algebra $\mathcal{A}_\mathcal{F}$. As a vector space, $\mathcal{A}_\mathcal{F}$ is identical to $\mathcal{A}$, but is accompanied by the twisted product

$$m_\mathcal{F}(F \otimes G) = m \circ \mathcal{F}^{-1} \triangleright (F \otimes G) \ . \tag{2.11}$$

This twisted product is associative owing to the cocycle condition (2.8). We also denote it as $F *_\mathcal{F} G$ in a more familiar notation, using the star product, as a result of a quantization. See Ref.[18] for remarks about the similarities and differences with deformation quantization. Note that the twisted product remains commutative. Since each element in $\mathcal{H}_\mathcal{F}$ as well as in $\mathcal{A}_\mathcal{F}$ is the same as the corresponding classical element, the variation of the local functional has the same representation, $h \triangleright F[X]$, as the classical transformation. However, the Hopf algebra action is not the same as the classical one when $I[X]$ is a product of several local functionals, since

$$\begin{aligned} h \triangleright m_\mathcal{F}(F \otimes G) &= m \circ \Delta(h)\mathcal{F}^{-1} \triangleright (F \otimes G) \\ &= m_\mathcal{F}\Delta_\mathcal{F}(h) \triangleright (F \otimes G) \ . \end{aligned} \tag{2.12}$$

In this way the Hopf algebra and the module algebra are twisted in a consistent manner. The resulting twisted action is considered as a transformation in the quantized theories.

We define the VEV for the twisted module algebra $\mathcal{A}_{\mathcal{F}}$ as the map $\tau : \mathcal{A}_{\mathcal{F}} \to \mathbb{C}$. For any element $I[X] \in \mathcal{A}_{\mathcal{F}}$ it is defined as the evaluation at $X = 0$:

$$\tau\,(\,I[X]\,) := I[X]|_{X=0}. \tag{2.13}$$

If $I[X]$ is a product of two elements $F, G \in \mathcal{A}_{\mathcal{F}}$, their correlation function is

$$\sigma(z,w) = \tau(F[X(z)] *_{\mathcal{F}} G[X(w)]) = \tau \circ m \circ \mathcal{F}^{-1} \triangleright (F \otimes G), \tag{2.14}$$

Owing to the associativity, the correlation function of n local functionals is similarly given as

$$\sigma(z_1, ..., z_n) = \tau(F_1[X(z_1)] *_{\mathcal{F}} F_2[X(z_2)] \cdots *_{\mathcal{F}} F_n[X(z_n)]). \tag{2.15}$$

The action of $h \in \mathcal{H}_{\mathcal{F}}$ inside the VEV $\tau(I[X])$ is

$$\tau\,(h \triangleright I[X]) \tag{2.16}$$

and this action appears in the various relations related with the symmetry transformation.

A different choice of the twist element $\mathcal{F}$ gives a different quantization scheme. This completes the description of the procedure.

One advantage of this procedure is that both, the module algebra $\mathcal{A}$ (observables) and the Hopf algebra $\mathcal{H}$ (symmetry) are simultaneously quantized by a single twist element $\mathcal{F}$. This gives us a significantly simple understanding of the symmetry structure after the quantization, as discussed in Ref.[18] in great detail: There, we argued that the symmetry of the theory in the conventional sense is characterized as a twist invariant Hopf subalgebra of $\mathcal{H}_{\mathcal{F}}$, which consists of elements $h \in \mathcal{H}$ such that $\Delta_{\mathcal{F}}(h) = \Delta(h)$ and $S_{\mathcal{F}}(h) = S(h)$. The other elements of $\mathcal{H}_{\mathcal{F}}$, i.e., generic diffeomorphisms, should be twisted at the quantum level. Since each choice of twist element chooses a particular background (as we will see below when choosing an explicit $\mathcal{F}$), $\mathcal{H}_{\mathcal{F}}$ and $\mathcal{A}_{\mathcal{F}}$ are background dependent as well, but only through $\mathcal{F}$ in the coproduct $\Delta_{\mathcal{F}}$ or in the twisted product $m_{\mathcal{F}}$.

The action of this twisted diffeomorphism can be regarded as a remnant of the classical diffeomorphism, where the change of the background under diffeomorphisms is incorporated into the twisted diffeomorphism in such a way that the twist element (quantization scheme) is kept invariant.

2.3 Normal ordering and path integrals

Here we argue that the twist quantization described in the previous subsection is identical with the path integral quantization, by identifying the corresponding VEVs. As proved in Ref.[18], the twist element $\mathcal{F}$ in (2.9) can be written as

$$\begin{aligned} \mathcal{F} &= \partial\mathcal{N}^{-1} = (\mathcal{N}^{-1} \otimes \mathcal{N}^{-1})\Delta(\mathcal{N})\ , \\ \mathcal{F}^{-1} &= \partial\mathcal{N} = \Delta(\mathcal{N}^{-1})(\mathcal{N} \otimes \mathcal{N})\ , \end{aligned} \tag{2.17}$$

where $\mathcal{N} \in \mathcal{H}$ is defined by using the same Green's function as in $\mathcal{F}$:

$$\mathcal{N} = \exp\left\{ -\frac{1}{2} \int d^2z \int d^2w \, G^{\mu\nu}(z,w) \frac{\delta}{\delta X^\mu(z)} \frac{\delta}{\delta X^\nu(w)} \right\} \ . \tag{2.18}$$

This shows that the twist element $\mathcal{F}$ is a coboundary and thus it is trivial in the Hopf algebra cohomology. Consequently, there is an isomorphism between the Hopf algebras $\hat{\mathcal{H}}$ and $\mathcal{H}_\mathcal{F}$ and the module algebras $\hat{\mathcal{A}}$ and $\mathcal{A}_\mathcal{F}$, respectively, which can be summarized as:

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\text{twist}} & \mathcal{H}_\mathcal{F} & \xrightarrow{\sim} & \hat{\mathcal{H}} \\ \triangledown & & \triangledown & & \triangledown \\ \mathcal{A} & \xrightarrow{\text{twist}} & \mathcal{A}_\mathcal{F} & \xrightarrow{\sim} & \hat{\mathcal{A}} \end{array} \tag{2.19}$$

In the diagram, the left row is the classical pair $(\mathcal{H}, \mathcal{A})$, and the middle and the right rows are the quantum counterparts. Here the map $\mathcal{H}_\mathcal{F} \xrightarrow{\sim} \hat{\mathcal{H}}$ is given by an inner automorphism $h \mapsto \mathcal{N} h \mathcal{N}^{-1} \equiv \tilde{h}$, and the map $\mathcal{A}_\mathcal{F} \xrightarrow{\sim} \hat{\mathcal{A}}$ is given by $F \mapsto \mathcal{N} \triangleright F \equiv :F:$ (see below). We call $\hat{\mathcal{H}}$ ($\hat{\mathcal{A}}$) the normal ordered Hopf algebra (module algebra), respectively.

The normal ordered module algebra $\hat{\mathcal{A}}$ is the one which appears inside the VEV in the path integral. It consists of elements in normal ordered form

$$\mathcal{N} \triangleright I[X] \equiv :I[X]: \tag{2.20}$$

for any functional $I[X] \in \mathcal{A}_\mathcal{F}$. These elements correspond to vertex operators in the path integral. Note that the action of $\mathcal{N}$ (2.18) corresponds to subtractions of divergences at coincident points caused by self-contractions in the path integral. As an algebra, $\hat{\mathcal{A}}$ has the same multiplication $m : \hat{\mathcal{A}} \otimes \hat{\mathcal{A}} \to \hat{\mathcal{A}}$ as the classical functional $\mathcal{A}$. This is seen by the map of the product in $\mathcal{A}_\mathcal{F}$ as

$$\mathcal{N} \triangleright m \circ \mathcal{F}^{-1} \triangleright (F \otimes G) = m \circ (\mathcal{N} \otimes \mathcal{N}) \triangleright (F \otimes G) \ , \tag{2.21}$$

which is a direct consequence of the coboundary relation (2.17). An equivalent but more familiar expression, $:(F *_\mathcal{F} G):=:F::G:$, is simply the time ordered product of two vertex operators.

The normal ordered Hopf algebra $\hat{\mathcal{H}}$ is defined with the same algebraic operations as the classical Hopf algebra $\mathcal{H}$, but with elements dressed with a normal ordering as

$$\mathcal{N} h \mathcal{N}^{-1} \equiv \tilde{h} \tag{2.22}$$

for any $h \in \mathcal{H}_\mathcal{F}$. The isomorphism map relates the twisted Hopf algebra action $\mathcal{H}_\mathcal{F}$ on $\mathcal{A}_\mathcal{F}$ to the corresponding action of $\hat{\mathcal{H}}$ on $\hat{\mathcal{A}}$. For example,

$$h \triangleright F \ \xrightarrow{\sim} \ \mathcal{N} \triangleright (h \triangleright F) = \ \tilde{h} \triangleright :F: \tag{2.23}$$

$$h \triangleright (F *_\mathcal{F} G) \xrightarrow{\sim} \tilde{h} \triangleright (:F::G:). \tag{2.24}$$

As argued in Ref.[18], some elements in $\hat{\mathcal{A}}$ contain the formally divergent series which we have to deal with in the functional language (this is the reason why we distinguish $\mathcal{A}$

and $\hat{\mathcal{A}}$), and thus eqs. (2.23) and (2.24) have only a meaning inside the VEV. The VEV (2.13) for $\mathcal{A}_{\mathcal{F}}$ implies the definition of the VEV for $\hat{\mathcal{A}}$ to be a map, $\tau \circ \mathcal{N}^{-1} : \hat{\mathcal{A}} \to \mathbb{C}$, and it turns out that it coincides with the VEV $\langle \cdots \rangle$ in the path integral. For instance, the correlation of two local functionals is

$$\tau \circ \mathcal{N}^{-1} \triangleright (:F[X]:(z) :G[X]:(w)) = \langle :F[X]:(z) :G[X]:(w) \rangle \tag{2.25}$$

Here, the action $\mathcal{N}^{-1}$ gives the Wick contraction with respect to the Green function. The equality (2.25) can be easily verified using the standard path integral argument (see for example Ref.[23]). The VEV $\langle \cdots \rangle$ in the path integral means

$$\langle \mathcal{O} \rangle := \frac{\int \mathcal{D}X \mathcal{O} e^{-S}}{\int \mathcal{D}X e^{-S}} \ . \tag{2.26}$$

Here the action functional $S[X]$ is related to the choice of the Green function. It is quadratic $S[X] = \frac{1}{2} \int d^2z X^\mu D_{\mu\nu} X^\nu$ with a fixed second order derivative $D_{\mu\nu}$ such that the Green function in (2.18) is a solution of the equation $D_{\mu\rho} G^{\rho\nu}(z,w) = \delta^\nu_\mu \delta^{(2)}(z-w)$. Note that in (2.26), each local insertion is understood as being regulated by the (conformal) normal ordering $\mathcal{N}$. This must correspond to a (operator) normal ordering in the operator formulation [1].

The two descriptions in terms of a twisted pair $(\mathcal{H}_{\mathcal{F}}, \mathcal{A}_{\mathcal{F}})$ and of a normal ordered pair $(\hat{\mathcal{H}}, \hat{\mathcal{A}})$ (thus path integral) are equivalent. However, note that the background dependences in the two formulations are different. In the case of the normal ordered pair $(\hat{\mathcal{H}}, \hat{\mathcal{A}})$, both, an element $:F: \in \hat{\mathcal{A}}$ and the VEV $\tau \circ \mathcal{N}^{-1}$, contain $\mathcal{N}$, which depends on the background. This corresponds, in the operator formulation, to the property that a mode expansion of the string variable $X^\mu(z)$ as well as the oscillator vacuum are background dependent. Therefore, the description of the quantization that makes $\hat{\mathcal{A}}$ well-defined is only applicable to that particular background and we need a different mode expansion for a different background.

3. Twisted Hopf algebra in B field background

In this section, we study the Hopf algebra structure for the case of a non-zero B-field background, following the above quantization procedure based on the Hopf algebra twist. The relation between the quantization and twist as explained in section 2 leads to another quantization scheme for the case with B-field background, where, we need, in particular, a deformed normal ordering.

3.1 String worldsheet theory in B field background

Consider the bosonic closed strings as well as open strings and take a space-filling D-brane for simplicity. We start with the σ model of the bosonic string with flat d-dimensional

[1] It is also possible to define a path integral with regularization prescriptions other than the conformal normal ordering used here. In that case, the corresponding operator ordering is also changed. We do not exclude this possibility but claim that there is a natural choice of the normal ordering from the viewpoint of classical functionals, in the sense that the functionals themselves are not modified under the quantization.

Minkowski space as the target. The action in the conformal gauge is

$$S_0[X] = \frac{1}{2\pi\alpha'}\int_\Sigma d^2z\eta_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu, \tag{3.1}$$

where the world sheet Σ can be any Riemann surface with boundaries, and typically, we take it as the complex plane (upper half plane) for a closed string (open string). $z^a = (z, \bar{z})$ are the complex coordinates on the world sheet. The flat metric in the target space $\mathbb{R}^d$ is represented by $\eta^{\mu\nu}$.

The constant B field is turned on by adding a term

$$S_B = \frac{1}{2\pi\alpha'}\int_\Sigma d^2zB_{\mu\nu}\partial X^\mu\bar{\partial}X^\nu \tag{3.2}$$

to the free action (3.1). This does not affect the equation of motion for $X^\mu(z)$, but it changes the boundary condition from Neumann boundary condition to the mixed one. Therefore, we consider the worldsheet with boundaries in this section. To be more explicit, the action of the theory is given by

$$\begin{aligned} S_1 &= S_0 + S_B \\ &= \frac{1}{2\pi\alpha'}\int_\Sigma d^2z(\eta_{\mu\nu} + B_{\mu\nu})\partial X^\mu\bar{\partial}X^\nu \\ &= \frac{-1}{2\pi\alpha'}\int_\Sigma d^2z\eta_{\mu\nu}X^\mu\partial\bar{\partial}X^\nu + \frac{1}{2\pi\alpha'}\int_{\partial\Sigma} dtX^\mu(\eta_{\mu\nu}\partial_n X^\nu + B_{\mu\nu}\partial_t X^\nu), \end{aligned} \tag{3.3}$$

where integration by parts is used. $\partial\Sigma$ is in general a sum of boundary components of the worldsheet Σ, and t is its boundary coordinate. ∂_n is the normal derivative to $\partial\Sigma$, and ∂_t is the tangential derivative along t, respectively.

With the constant B-field background, the propagator can be calculated by solving the equation of motion on the worldsheet with the mixed boundary condition and we denote it by the subscript 1 according to S_1:

$$\langle X^\mu(z)X^\nu(w)\rangle_1 = G_1^{\mu\nu}(z, w). \tag{3.4}$$

This propagator satisfies

$$-\frac{1}{\pi\alpha'}\partial\bar{\partial}G_1^{\mu\nu}(z, w) = \eta^{\mu\nu}\delta^{(2)}(z - w), \quad (\eta_{\mu\nu}\partial_n + B_{\mu\nu}\partial_t)G_1^{\mu\nu}(t, w) = 0. \tag{3.5}$$

In the simplest case, Σ is the upper half plane of the complex plane where $G_1^{\mu\nu}$ is given by Refs.[24, 25, 26]

$$\begin{aligned} G_1^{\mu\nu}(z, w) = -\alpha'\Big[&\eta^{\mu\nu}\ln|z - w| - \eta^{\mu\nu}\ln|z - \bar{w}| \\ &+ G^{\mu\nu}\ln|z - \bar{w}|^2 + \Theta^{\mu\nu}\ln\frac{z - \bar{w}}{\bar{z} - w}\Big], \end{aligned} \tag{3.6}$$

where the open string metric $G^{\mu\nu}$ and the antisymmetric tensor $\Theta^{\mu\nu}$ are defined as

$$G^{\mu\nu} = \left(\frac{1}{\eta + B}\eta\frac{1}{\eta - B}\right)^{\mu\nu} \quad \Theta^{\mu\nu} = \frac{\theta^{\mu\nu}}{2\pi\alpha'} = -\left(\frac{1}{\eta + B}B\frac{1}{\eta - B}\right)^{\mu\nu}. \tag{3.7}$$

If we set $B = 0$, (3.4) reduces to the propagator $G_0^{\mu\nu}(z, w)$ in Ref.[18] which satisfies the Neumann boundary condition $\partial_n G_0^{\mu\nu}(t, w) = 0$.

3.2 Twist quantization with B field

Let $\mathcal{H} = U(\mathfrak{X})$ be the Hopf algebra of functional variations and let $\mathcal{A}$ be its module algebra given by the classical functionals of the embedding X. In the B field background, the total action of the theory is $S_1 = S_0 + S_B$, such that the propagator is $G_1^{\mu\nu}(z,w)$. Then, in our approach, the quantization of the system is defined by the twist element

$$\mathcal{F}_1 := \exp\left\{-\int d^2z \int d^2w\, G_1^{\mu\nu}(z,w) \frac{\delta}{\delta X^\mu(z)} \otimes \frac{\delta}{\delta X^\nu(w)}\right\} \quad . \tag{3.8}$$

The whole structure concerning the quantization is exactly the same as in the previous section. Since $\mathcal{F}_1$ satisfies the unital 2-cocycle condition, we obtain the twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_1}$ and its module algebra $\mathcal{A}_{\mathcal{F}_1}$ as a quantization of $\mathcal{H}$ and $\mathcal{A}$. As discussed in section 2.3, there are (formal) isomorphisms $\mathcal{H}_{\mathcal{F}_1} \simeq \mathcal{H}$ and $\mathcal{A}_{\mathcal{F}_1} \simeq \mathcal{A}$ relating the twisted Hopf (module) algebras and the normal ordered Hopf (module) algebras. In order to distinguish them from the classical ones, we denote them by $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{A}}_1$, respectively. They consist of elements of the form $\tilde{h} = \mathcal{N}_1 h \mathcal{N}_1^{-1} \in \hat{\mathcal{H}}_1$ and $\mathcal{N}_1 \triangleright F \in \hat{\mathcal{A}}_1$, respectively, as in the general construction in section 2.3. Here, the normal ordering element is

$$\mathcal{N}_1 = \exp\left\{-\frac{1}{2}\int d^2z d^2w\; G_1^{\mu\nu}(z,w) \frac{\delta}{\delta X^\mu(z)} \frac{\delta}{\delta X^\nu(w)}\right\}. \tag{3.9}$$

We denote this normal ordering as $\overset{\circ}{\circ} \cdots \overset{\circ}{\circ}$ and the star product as $F *_{\mathcal{F}_1} G = m\mathcal{F}_1^{-1} \triangleright (F \otimes G)$, then their relation analogous to (2.21) is written as

$$\overset{\circ}{\circ} F *_{\mathcal{F}_1} G \overset{\circ}{\circ} = \overset{\circ}{\circ} F \overset{\circ}{\circ} \overset{\circ}{\circ} G \overset{\circ}{\circ} \tag{3.10}$$

The VEV for a normal ordered functional $\overset{\circ}{\circ} I[X] \overset{\circ}{\circ} \in \hat{\mathcal{A}}_1$ is

$$\langle \overset{\circ}{\circ} I[X] \overset{\circ}{\circ} \rangle_1 = \tau\left(\mathcal{N}_1^{-1} \triangleright \overset{\circ}{\circ} I[X] \overset{\circ}{\circ}\right) = \tau\left(I[X]\right), \tag{3.11}$$

which coincides with the path integral average with respect to the action S_1

$$\langle \mathcal{O} \rangle_1 = \frac{\int \mathcal{D}X \mathcal{O} e^{-S_1}}{\int \mathcal{D}X e^{-S_1}}. \tag{3.12}$$

Note that here, for the case with the background $(\eta_{\mu\nu}, B_{\mu\nu})$, we denote the twist element as $\mathcal{F}_1$, and the related structures such as $\mathcal{H}_{\mathcal{F}_1}$, $\mathcal{A}_{\mathcal{F}_1}$, $\mathcal{N}_1$ and $*_{\mathcal{F}_1}$ with a suffix 1. On the other hand, the twist quantization in the background $(\eta_{\mu\nu}, B_{\mu\nu} = 0)$ as treated in Ref.[18] is determined by the twist element $\mathcal{F}_0$ with the propagator $G_0^{\mu\nu}(z,w)$. Following this notation we denote the resulting structures such by $\mathcal{H}_{\mathcal{F}_0}$, $\mathcal{A}_{\mathcal{F}_0}$, $\mathcal{N}_0$ and $*_{\mathcal{F}_0}$ with a suffix 0. The normal ordering in this case is denoted as $\mathcal{N}_0 \triangleright F =: F :$ and the normal ordered Hopf and module algebras are denoted by $\hat{\mathcal{H}}$ and $\hat{\mathcal{A}}$, respectively.[2].

Comparing with the $B = 0$ case, the change of the background to a non-zero value affects directly the normal ordered module algebra $\hat{\mathcal{A}}_1$, which is in contrast to the twisted

[2] In principle, all the formulas relating the quantization and normal ordering in Ref.[18] is valid for both cases with suffix 0 or 1.

counterpart $\mathcal{A}_{\mathcal{F}_1}$, because an element of $\hat{\mathcal{A}}_1$ has a different power series expansion, i.e., $\mathcal{N}_1 \triangleright F$, rather than $\mathcal{N}_0 \triangleright F$ for the case without B-field background in Ref.[18]. This modification of the normal ordering corresponds to the modification of the operator description, where a new mode expansion and a new Fock vacuum are required by the change of background. In fact, the normal ordering defined by (3.9) coincides with that proposed in Refs.[19, 20, 21, 22]. There, this normal ordering was introduced such that a normal ordered operator like $\overset{\circ}{\circ}\hat{X}^\mu(z)\hat{X}^\nu(w)\overset{\circ}{\circ}$ satisfies not only the equation of motion, but also the mixed boundary condition as an operator relation. The authors also observed the noncommutativity of position operators by an appropriate mode expansion. Our results here give a foundation of this proposal.

Twisted symmetry with B field. The discussion of the symmetry in the case with B-field background goes analogously to the case without B field studied in Ref.[18]. Using the above definition of the VEV, the formula of the twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_1}$ action on a functional inside the VEV can be written as

$$\langle\, \tilde{h} \ \triangleright \overset{\circ}{\circ} F[X] \overset{\circ}{\circ} \rangle_1 \ = \ \tau\left(h \triangleright F[X]\right) \ . \tag{3.13}$$

where $\tilde{h} = \mathcal{N}_1 h \mathcal{N}_1^{-1} \in \hat{\mathcal{H}}_1$. We can see that the key relation

$$\xi \triangleright I[X] = -m \circ (\xi \otimes 1) F_1 \triangleright (S_1 \otimes I[X]) \,, \tag{3.14}$$

for deriving the Ward-like identities holds as in the case without B field (See Ref.[18]). Here, F_1 is defined by $\mathcal{F}_1 = e^{F_1}$. The proof of (3.14) is as follows: The action of F_1 on $S_1 \otimes I[X]$ is written using (3.3) as

$$\begin{aligned} F_1 \triangleright (S_1 \otimes I[X]) = & -\frac{1}{\pi\alpha'} \int d^2z \int d^2w \, G_1^{\mu\nu}(z,w) \left(\partial\bar{\partial} X_\mu(z) \otimes \frac{\delta I[X]}{\delta X^\nu(w)} \right) \\ & + \frac{1}{2\pi\alpha'} \int_{\partial\Sigma} dt \int d^2w \, G_1^{\mu\nu}(t,w) \left((\eta_{\mu\nu}\partial_n X^\nu + B_{\mu\nu}\partial_t X^\nu)(t) \otimes \frac{\delta I[X]}{\delta X^\nu(w)} \right) . \end{aligned} \tag{3.15}$$

Integrating by parts, and using the defining relations of the Green function (3.5), we confirm (3.14). In particular, we obtain the Ward-like identity analogous to eq.(43) in Ref.[18] as

$$0 = \langle\, \tilde{\xi} \ \triangleright \overset{\circ}{\circ} I[X] \overset{\circ}{\circ} \rangle_1 - \langle \overset{\circ}{\circ}(\xi \triangleright S_1) \overset{\circ}{\circ} \ \overset{\circ}{\circ} I[X] \overset{\circ}{\circ} \rangle_1 - m \circ \left[\xi \otimes 1, \mathcal{F}_1^{-1}\right] \triangleright (S_1 \otimes I[X]) \ . \tag{3.16}$$

Since a infinitesimal Lorentz transformation in a generic B-field background does not preserve the action: $\xi \triangleright S_1 \neq 0$, nor does the commutator vanishes: $[\xi \otimes 1, \mathcal{F}_1^{-1}] \neq 0$, we have only broken-type Ward identities. This feature is equivalent to the statement that the Poincaré symmetry as well as the full space-time diffeomorphism should be twisted at the quantum level.[3] It will become clearer in the next section, how the original Poincaré symmetry is deformed by the B field.

[3] Of course, when some of component of $B_{\mu\nu}$ happens to be zero, an unbroken symmetry remains.

4. Decompositions of the twist and physical interpretations

We have shown that application of the twist quantization to the case with nontrivial B-field background gives a twisted Hopf algebra including the diffeomorphism of the target space. Thereby, the quantization and the deformation by the B-field are treated in a unified way, which was one of our aims. On the other hand, this unified approach makes the geometrical implication of the deformation by the B-field background less transparent.

For the open string case, it is well known that the effective theory with the B-field background can be formulated as a field theory on the noncommutative space. The effect of the B-field is reflected only on the noncommutativity and the corresponding geometrical picture is noncommutative space. On the other hand, for the closed string, such a geometrical interpretation of the B-field is missing. It is an interesting question whether we can have an appropriate geometrical structure for the closed string case.

In the following, we develop a method to decompose the twist into two successive twists, and discuss its interpretation. We propose two different decompositions of the twist and discuss the feature of the B-field effect as a deformation of the functional algebra and as a twist of the symmetry, respectively.

The first is a natural decomposition from the viewpoint of the twist quantization. There we can see the relation between the standard quantization and the deformation caused by a B-field background. With this decomposition, we obtain a new twisted Poincaré symmetry, which is different from the one considered in Refs.[12, 13, 14]. The twisted Poincaré symmetry obtained here should be considered as a symmetry structure underlying the string theory.

However, the first decomposition becomes singular when we take the Seiberg-Witten limit[3] to obtain the noncommutative field theory picture of the effective theory. The second decomposition is meaningful for the open string case. By restricting the functional space to the one corresponding to the open string vertex operators, we can derive the relation to the twisted Poincaré symmetry in the field theory on the noncommutative space considered in Refs.[12, 13, 14].

4.1 B field as a deformation

Using the twist quantization, the effect of the B field background is unified with the quantization twist into a single twist element $\mathcal{F}_1$. Therefore, to see the effect of the B field background separately, we have to extract it from $\mathcal{F}_1$. It is also more natural to compare directly the twisted Hopf algebras $\mathcal{H}_{\mathcal{F}_1}$ and $\mathcal{H}_{\mathcal{F}_0}$, without referring to the classical Hopf algebra $\mathcal{H}$.

Successive twists. For this purpose, we consider a decomposition of the propagator (3.4) into the following two parts:

$$\langle X^\mu(z)X^\nu(w)\rangle_1 = G_1^{\mu\nu}(z,w) = G_0^{\mu\nu}(z,w) + G_B^{\mu\nu}(z,w) \ , \tag{4.1}$$

where $G_0^{\mu\nu}$ is the propagator in $B=0$. Then, the propagators are given, on the upper half plane, for example, by

$$\begin{aligned} G_0^{\mu\nu}(z,w) &:= -\alpha'\eta^{\mu\nu}(\ln|z-w| + \ln|z-\bar{w}|) \\ G_B^{\mu\nu}(z,w) &:= -\alpha'\Big[(G-\eta)^{\mu\nu}\ln|z-\bar{w}|^2 + \Theta^{\mu\nu}\ln\frac{z-\bar{w}}{\bar{z}-w}\Big] \ . \end{aligned} \tag{4.2}$$

Accordingly, the twist element $\mathcal{F}_1$ (3.8) is also divided into

$$\mathcal{F}_1 = \mathcal{F}_B\mathcal{F}_0 \ , \tag{4.3}$$

where $\mathcal{F}_B$ is the pure effect of the B field defined by

$$\mathcal{F}_B := \exp\left\{-\int d^2z\int d^2w\, G_B^{\mu\nu}(z,w)\frac{\delta}{\delta X^\mu(z)}\otimes\frac{\delta}{\delta X^\nu(w)}\right\} \ . \tag{4.4}$$

The decomposition (4.3) defines two successive twists

$$\mathcal{H} \xrightarrow{\text{twist by } \mathcal{F}_0} \mathcal{H}_{\mathcal{F}_0} \xrightarrow{\text{twist by } \mathcal{F}_B} (\mathcal{H}_{\mathcal{F}_0})_{\mathcal{F}_B} = \mathcal{H}_{\mathcal{F}_1} \ . \tag{4.5}$$

We give a remark for the last equality. In general, for a given a twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_0}$ with a coproduct $\Delta_{\mathcal{F}_0}$, a further twisting by $\mathcal{F}_B$ is possible if $\mathcal{F}_B$ is a 2-cocycle in $\mathcal{H}_{\mathcal{F}_0}$

$$(\mathcal{F}_B\otimes\mathrm{id})(\Delta_{\mathcal{F}_0}\otimes\mathrm{id})\mathcal{F}_B = (\mathrm{id}\otimes\mathcal{F}_B)(\mathrm{id}\otimes\Delta_{\mathcal{F}_0})\mathcal{F}_B \ . \tag{4.6}$$

Then, it is equivalent to the 1-step twist $\mathcal{H}_{\mathcal{F}_1}$, where $\mathcal{F}_1 = \mathcal{F}_B\mathcal{F}_0$. This property holds because the l.h.s. of the 2-cocycle condition (2.8) for $\mathcal{F}_1$ can be written using (2.8) for $\mathcal{F}_0$ and (4.6) as

$$(\mathcal{F}_1\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\mathcal{F}_1 = [(\mathcal{F}_B\otimes\mathrm{id})(\Delta_{\mathcal{F}_0}\otimes\mathrm{id})\mathcal{F}_B]\ [(\mathcal{F}_0\otimes\mathrm{id})(\Delta\otimes\mathrm{id})\mathcal{F}_0]\,, \tag{4.7}$$

and similarly for the r.h.s.. Conversely, if two twist elements $\mathcal{F}_0$ and $\mathcal{F}_1$ in $\mathcal{H}$ have a relation $\mathcal{F}_1 = \mathcal{F}_B\mathcal{F}_0$, then $\mathcal{F}_B$ is a twist element in $\mathcal{H}_{\mathcal{F}_0}$ satisfying (4.6). Note that in our case, the decomposition of the Green function is directly related to the decomposition of the twist element, because these are all abelian twists and (4.6) holds. Therefore, the effect of the B-field is completely characterized as a (second) twist $\mathcal{F}_B$ of the Hopf algebra $\mathcal{H}_{\mathcal{F}_0}$ and the module algebra $\mathcal{A}_{\mathcal{F}_0}$.

Using the isomorphism (2.19), $\mathcal{H}_{\mathcal{F}_0}$ can be replaced with the normal ordered $\hat{\mathcal{H}}$ with the standard normal ordering $\mathcal{N}_0$. Of course, there is also an isomorphism $\mathcal{H}_{\mathcal{F}_1} \simeq \hat{\mathcal{H}}_1$ as described in section 3.2. In this case, the second twist by $\mathcal{F}_B$ in (4.5) can also be regarded as a twist relating the two normal ordered Hopf algebras $\hat{\mathcal{H}} \to \hat{\mathcal{H}}_{\mathcal{F}_B} = \hat{\mathcal{H}}_1$. Correspondingly, we have a map of normal ordered module algebras $\hat{\mathcal{A}} \to \hat{\mathcal{A}}_{\mathcal{F}_B} = \hat{\mathcal{A}}_1$ for the second twist. This enables us to study the effect of the B-field in terms of vertex operators as we will see in the following.

B field as a deformation of OPE. First, we compare the two normal ordered functionals $:F:\in\hat{\mathcal{A}}$ and ${}^{\circ}_{\circ}F{}^{\circ}_{\circ}\in\hat{\mathcal{A}}_1$. They correspond to operators in two different quantization schemes. According to (4.3), the normal orderings are related as $\mathcal{N}_1=\mathcal{N}_B\mathcal{N}_0$, which are defined in the same manner as (2.17). Then, we have a relation ${}^{\circ}_{\circ}F{}^{\circ}_{\circ}=\mathcal{N}_B\triangleright:F:$ between them. The relation is such that the VEVs, each with respect to the corresponding quantization scheme, give the same result

$$\langle {}^{\circ}_{\circ}F{}^{\circ}_{\circ}\rangle_1=\tau\left(\mathcal{N}_1^{-1}\triangleright {}^{\circ}_{\circ}F{}^{\circ}_{\circ}\right)=\tau\left(\mathcal{N}_0^{-1}\triangleright:F:\right)=\langle:F:\rangle_0. \tag{4.8}$$

At first sight, it may be curious that the two different theories give the same VEV. However, this is a requirement, since the definition of the VEV corresponds to the *normalized* path integral average (3.12). The difference of two quantization schemes lies in the Wick contraction of several local functionals.

Next, the twisted product of two local functional $F,G\in\mathcal{A}_{\mathcal{F}_1}$ is rewritten by using (4.3) in terms of the elements in $\hat{\mathcal{A}}_{\mathcal{F}_B}$ as

$$\begin{aligned}F*_{\mathcal{F}_1}G&=m\circ\mathcal{F}_1^{-1}\triangleright(F\otimes G)\\&=m\circ\mathcal{F}_B^{-1}\Delta(\mathcal{N}_0^{-1})(\mathcal{N}_0\otimes\mathcal{N}_0)\triangleright(F\otimes G)\\&=\mathcal{N}_0^{-1}\triangleright m\circ\mathcal{F}_B^{-1}(\mathcal{N}_0\triangleright F\otimes\mathcal{N}_0\triangleright G)\\&=\mathcal{N}_0^{-1}\triangleright(:F:*_{\mathcal{F}_B}:G:),\end{aligned} \tag{4.9}$$

where the coboundary relation (2.17) is used for $\mathcal{F}_0^{-1}$. Note that this simply means the equivalence of the two star products $:F*_{\mathcal{F}_1}G:=:F:*_{\mathcal{F}_B}:G:$. By applying τ on both sides, this leads to the equation of the VEV

$$\begin{aligned}\langle {}^{\circ}_{\circ}F{}^{\circ}_{\circ}{}^{\circ}_{\circ}G{}^{\circ}_{\circ}\rangle_1&=\tau(F*_{\mathcal{F}_1}G)\\&=\tau\circ\mathcal{N}_0^{-1}\triangleright(:F:*_{\mathcal{F}_B}:G:)\\&=\langle:F:*_{\mathcal{F}_B}:G:\rangle_0.\end{aligned} \tag{4.10}$$

On the l.h.s., the definition of the VEV as well as the normal ordering are with respect to the new quantization $\hat{\mathcal{A}}_1$ including the B field. The r.h.s. is written in terms of the standard quantization scheme $\hat{\mathcal{A}}$, except that the OPE is twisted by $\mathcal{F}_B$ (thus $\hat{\mathcal{A}}_{\mathcal{F}_B}$).

Now we relate the twisted Hopf algebra action to the normal ordered Hopf algebra action. As above, the action of $h\in\mathcal{H}_{\mathcal{F}_1}$ on any functional $I\in\mathcal{A}_{\mathcal{F}_1}$ is related to the action of $\tilde{h}=\mathcal{N}_0h\mathcal{N}_0^{-1}\in\hat{\mathcal{H}}$ on $:I:\in\hat{\mathcal{A}}$ as $h\triangleright I=\mathcal{N}_0^{-1}\tilde{h}\triangleright:I:$, or equivalently, ${}^{\circ}_{\circ}h\triangleright I{}^{\circ}_{\circ}=\mathcal{N}_B\tilde{h}\triangleright:I:$. Using (4.9), we get for the product of two local functional $I=F*_{\mathcal{F}_1}G$,

$$\begin{aligned}\langle {}^{\circ}_{\circ}h\triangleright(F*_{\mathcal{F}_1}G){}^{\circ}_{\circ}\rangle_1&=\tau\circ\mathcal{N}_0^{-1}\tilde{h}\triangleright(:F:*_{\mathcal{F}_B}:G:)\\&=\langle\tilde{h}\triangleright(:F:*_{\mathcal{F}_B}:G:)\rangle_0,\end{aligned} \tag{4.11}$$

where the action of $\tilde{h}$ in the last line is written as

$$\tilde{h}\triangleright(:F:*_{\mathcal{F}_B}:G:)=m\circ\mathcal{F}_B^{-1}\Delta_{\mathcal{F}_B}(\tilde{h})\triangleright(:F:\otimes:G:). \tag{4.12}$$

This formula is a direct consequence of the second twist $\hat{\mathcal{H}}\to\hat{\mathcal{H}}_{\mathcal{F}_B}$, and shows that the transformation on the operator algebra deformed by the B field should also be twisted.

In summary, at the level of vertex operators, the quantization scheme with B field (twist by $\mathcal{F}_1$) can also be considered in the standard quantization scheme without B field (by $\mathcal{F}_0$), but with the operator product and the Hopf algebra action being twisted by the twist element $\mathcal{F}_B$.

Twisted Poincaré symmetry. Here we discuss about the fate of the Poincaré symmetry $U(\mathcal{P})$. As already observed in section 3.2, the universal envelope of the Poincaré-Lie algebra $U(\mathcal{P})$ has to be twisted in a generic B-field background. Note that this is true even for closed string vertex operators in the presence of the boundary $\partial\Sigma \neq 0$.[4] By extracting the effect of the B-field as the second twist $\mathcal{F}_B$, the structure of this twisted Poincaré symmetry becomes transparent as follows.

For $B = 0$ case, $U(\mathcal{P})$ is a twist invariant Hopf subalgebra of $\mathcal{H}$ under the twisting by $\mathcal{F}_0$, as argued in Ref.[18]. Equivalently, $U(\mathcal{P})$ is a Hopf subalgebra of $\hat{\mathcal{H}}$ with $\tilde{P}_\mu = P_\mu$ and $\tilde{L}_{\mu\nu} = L_{\mu\nu}$ as elements in $\hat{\mathcal{H}}$. Thus, $U(\mathcal{P})$ remains a symmetry at the quantum level, and each (normal ordered) vertex operator in $\hat{\mathcal{A}}$ is in a representation of a Poincaré-Lie algebra. We emphasize that this fact guarantees the spacetime meaning of a vertex operator. For example, a graviton vertex operator $V =: \partial X^\mu \bar{\partial} X^\nu e^{ikX} :$ transforms as spin 2 representation under $U(\mathcal{P})$ corresponding to the spacetime graviton field $h_{\mu\nu}(x)$.

However, in a B-field background, the second twist $\hat{\mathcal{H}} \to \hat{\mathcal{H}}_{\mathcal{F}_B}$ modifies the coproduct and antipode of the Lorentz generator in $U(\mathcal{P})$ as

$$\Delta_{\mathcal{F}_B}(L_{\mu\nu}) = L_{\mu\nu} \otimes 1 + 1 \otimes L_{\mu\nu} - 2 \int d^2z d^2w \, \eta_{[\mu\alpha} G_B^{\alpha\beta}(z,w) \frac{\delta}{\delta X^\beta(z)} \otimes \frac{\delta}{\delta X^{\nu]}(z)} \tag{4.13}$$

$$S_{\mathcal{F}_B}(L_{\mu\nu}) = S(L^{\mu\nu}) + 2 \int d^2u d^2z \, G_B^{\rho[\mu}(u,z) \frac{\delta}{\delta X^\rho(u)} \frac{\delta}{\delta X_{\nu]}(z)}. \tag{4.14}$$

Apparently, this $L_{\mu\nu}$ is not primitive and this means that the $U(\mathcal{P})$ is twisted. We give several remarks about this twisted Poincaré symmetry read from (4.13), (4.14) and (4.4):

1. The twisting of $U(\mathcal{P})$ is only due to the second twist $\mathcal{F}_B$. This guarantees that a single local vertex operator $: F :$ in $\hat{\mathcal{A}}_{\mathcal{F}_B}$ is still Poincaré covariant after the twist (in the same representation under $U(\mathcal{P}) \subset \hat{\mathcal{H}}$), and thus the spacetime meaning of a vertex operator (e.g., a graviton as spin 2 representation) is unchanged.

2. The twisting is not closed within $U(\mathcal{P})$, but the additional terms in (4.13) are in $U(\mathfrak{C}) \otimes U(\mathfrak{C})$, and similarly for (4.14). This shows that the twisting does not mix the Lorentz generator with other diffeomorphisms even for products of vertex operators. Therefore, from the spacetime point of view, a Poincaré transformation on products of fields is still a global transformation even after the twist.

3. The twisted product $*_{\mathcal{F}_B}$ of two vertex operators reduces to the product of corresponding spacetime fields, but the latter depends on the representations (spins) of $U(\mathcal{P})$, because $\mathcal{F}_B$ in (4.4) depends on the Green function G_B. For instance, the twisted product of two gravitons is different from that of two tachyons due to this

[4] Of course if $\partial\Sigma = 0$, then G_B can be set to 0.

fact. Similarly, the Lorentz transformation law for the product of two spacetime fields depends on their spins.

It is still an open question how this twisted product and Lorentz transformations are seen in the effective field theory. At least, if we restrict our attention to open string vertex operators only, there is another natural decomposition of the twist $\mathcal{F}_1$, which is closely related to the twisted Poincaré algebra on the Moyal-Weyl space as we will see in the next subsection.

Note also that there is another (and more conventional) treatment of a B-field background as a perturbation, where the boundary action S_B is considered as an interaction vertex operator and e^{-S_B} is treated as perturbative insertion. This corresponds to considering the B-field as a matter field in the effective field theory. In that case, the quantization is the standard procedure, but the Poincaré symmetry is explicitly broken by the background flux B. It would be interesting to establish the equivalence to our treatment. (The situation is similar to the equivalence between commutative and noncommutative description of a D-brane worldvolume in Ref.[3].) The decomposition of the twist element here is suitable for that purpose, because the quantization scheme remains the standard one by $\mathcal{F}_0$. Thus, it would be possible to show that infinitely many perturbative insertions of S_B reproduces the second twist $\mathcal{F}_B$. But we do not discuss this issue in this paper.

To summarize, from the decomposition of the twist element $\mathcal{F}_1 = \mathcal{F}_B\mathcal{F}_0$, we obtain a description of a spacetime with a B-field background, such that the effect of background B-field is hidden in the twist element $\mathcal{F}_B$, while the other matter fields acquires a deformation caused by this twist element. It seems that there is no other natural decomposition in the case of closed string vertex operators. This description of spacetime is different from the description formulated on a commutative spacetime with a matter field $B_{\mu\nu}$ or from the description using a noncommutative space as in the effective theory of open string. Although it is not clear yet how this new description translates into the effective field theory language, our method clearly demonstrates how a background field (other than the metric) can be incorporated into a formulation of a stringy spacetime.

4.2 Relation to field theory and twisted Poincaré symmetry

In this subsection, we study the relation between the Hopf algebra $\mathcal{H}_{\mathcal{F}_1}$ of string worldsheet theory and the corresponding Hopf algebra structure in the effective field theory on D-branes, where the twisted Poincaré symmetry is realized in the field theory on the Moyal-type noncommutative space [12, 13, 14]. As mentioned in section 1, one of the motivations of this work is to understand the Hopf algebra symmetry and its twist in the full string theory. Schematically, we want to establish the diagram

$$\begin{array}{ccc} \mathcal{H}_{\mathcal{F}_1} & \xrightarrow{\text{field theory}} & U_{\mathcal{F}_M}(\mathcal{P}') \\ \triangledown & & \triangledown \\ \mathcal{A}_{\mathcal{F}_1} & \xrightarrow{\text{field theory}} & A_{\mathcal{F}_M}. \end{array} \tag{4.15}$$

$\mathcal{H}_{\mathcal{F}_1}$ and $\mathcal{A}_{\mathcal{F}_1}$ are the Hopf algebra and module algebra in the string worldsheet theory, respectively. On the r.h.s., $U_{\mathcal{F}_M}(\mathcal{P}')$ and $A_{\mathcal{F}_M}$ are the corresponding Hopf algebra and

module algebra structures in the effective field theory on the D-brane worldvolume, respectively. $U_{\mathcal{F}_M}(\mathcal{P}')$ is the twisted Poincar'e symmetry with the Moyal twist $\mathcal{F}_M$ and $A_{\mathcal{F}_M}$ denotes the twisted module algebra with the Moyal product. The structure of the objects on the r.h.s. is studied in Refs.[12, 13, 14]. The structure of the objects on the l.h.s. is investigated in the first part of this paper. The lower arrow in (4.15), i.e., the relation between OPE of vertex operators and the Moyal product is clarified in [3]. In the following, we will complete diagram by showing the missing relation, i.e., the upper arrow in (4.15) between twisted Hopf algebra and the twisted Poincaré symmetry on the Moyal-Weyl space.

Let the worldsheet Σ be the upper half plane, so that we can see the correspondence with the tree-level effective field theory. Recall that the effective field theory on D-branes in the B-field background is obtained from the worldsheet theory by an appropriate moduli integral of the correlation functions of boundary vertex operators (see for example Ref.[3]). There are two new ingredients when we move from correlation functions to the effective theory: i) fixing the cyclic ordering of the insertions, and ii) zero slope limit. Note that each cyclic ordering corresponds to a different region in the moduli space, and thus gives a different interaction term, e.g. $\int \Phi_1 \cdot (\Phi_2 * \Phi_3) \neq \int \Phi_1 \cdot (\Phi_3 * \Phi_2)$ for fields Φ_i on D-branes. This is the source for the non-equivalence between planar and non-planar diagrams in the case of the noncommuative field theory. It turns out that the ordering of insertions is the essential part to establish the above correspondence, however the zero slope limit becomes relevant when we consider an explicit form of the effective action. Here we concentrate on the structure independent of a zero slope limit.

For our purpose here, it is appropriate to consider the twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_1}$ with successive twists[5] different from those in section 4.1. Therefore, we consider here the second decomposition, where the first twist is a quantization with respect to the open string metric, when restricted to boundary vertex operators, and the second is its deformation. This second twist turns out to contain the same information like the Moyal-twist on the D-brane and the universal R-matrix.

The second decomposition. In section 4.1, the successive twist is characterized by the decomposition of the propagator (3.4). However, the decomposition of propagator is in fact arbitrary and we can choose an appropriate decomposition depending on which physical property of the twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_1}$ we want to see. Here it is useful to decompose the propagator according to the symmetry of the tensor indices $\mu\nu$. We denote the "symmetric part" G_S and the "anti-symmetric part" G_A of the propagator[6], respectively. For the upper half plane, the decomposition of the propagator is

$$G_1^{\mu\nu}(z,w) = G_S^{\mu\nu}(z,w) + G_A^{\mu\nu}(z,w), \tag{4.16}$$

[5]Our argument here is restricted only to boundary vertex operators for open strings, unless stated explicitly. We will give a remark about closed string vertex operators at the end of this subsection.

[6]Of course, the propagator $G_1^{\mu\nu}(z,w)$ is symmetric under the simultaneous exchanges $\mu \leftrightarrow \nu$ and $z \leftrightarrow w$. This corresponds to the fact that the twisted module algebra with the product $*_{\mathcal{F}_1}$ is commutative, and that the product in the path integral should be the time-ordered product.

where

$$
\begin{aligned}
G_S^{\mu\nu}(z,w) &:= -\alpha' \left[\eta^{\mu\nu}\ln|z-w| - \eta^{\mu\nu}\ln|z-\bar{w}| + G^{\mu\nu}\ln|z-\bar{w}|^2\right], \\
G_A^{\mu\nu}(z,w) &:= -\alpha'\Theta^{\mu\nu}\ln\frac{z-\bar{w}}{\bar{z}-w}\ .
\end{aligned}
\tag{4.17}
$$

If z and w are at the boundary, they reduce to the form

$$
G_S^{\mu\nu}(s,t) = -\alpha' G^{\mu\nu}\ln(s-t)^2 G_A^{\mu\nu}(s,t) = \frac{i}{2}\theta^{\mu\nu}\epsilon(s-t), \tag{4.18}
$$

where $\epsilon(t)$ is the sign function.

The following procedure is rather analogous to the one in section 4.1, that is, we divide the twist into $\mathcal{F}_1 = \mathcal{F}_A\mathcal{F}_S$, and we regard the first twist by the twist element

$$
\mathcal{F}_S = \exp\left(-\int d^2z\int d^2w\, G_S^{\mu\nu}(z,w)\frac{\delta}{\delta X^\mu(z)}\otimes\frac{\delta}{\delta X^\nu(w)}\right), \tag{4.19}
$$

as a quantization, and the second twist by the twist element

$$
\mathcal{F}_A = \exp\left(-\int d^2z\int d^2w\, G_A^{\mu\nu}(z,w)\frac{\delta}{\delta X^\mu(z)}\otimes\frac{\delta}{\delta X^\nu(w)}\right), \tag{4.20}
$$

as a deformation of the operator product, respectively.

First twist as quantization. The twisting by $\mathcal{F}_S$ converts $\mathcal{H}$ into a twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_S}$, which is isomorphic to $\hat{\mathcal{H}}_S$ by elements of the form $\tilde{h} = \mathcal{N}_S h \mathcal{N}_S^{-1}$. Correspondingly, we have a twisted module algebra $\mathcal{A}_{\mathcal{F}_S}$ with the product $*_{\mathcal{F}_S}$, and the normal ordered module algebra $\hat{\mathcal{A}}_S$. We denote elements of $\hat{\mathcal{A}}_S$ as $\bullet F \bullet = \mathcal{N}_S \triangleright F$. This defines a quantization scheme, but it turns out that it is natural only for boundary elements of $\mathcal{H}$ and $\mathcal{A}$.[7]

First, the Lorentz generator $L_{\mu\nu}$ in the Poincaré-Lie algebra acquire the twist and becomes non-primitive. This is the same situation considered in section 4.1. However, in this case, there are other boundary elements in the classical Hopf algebra $\mathcal{H}$,

$$
L'_{\mu\nu} = \int_{\partial\Sigma} dt\, G_{[\mu\rho}X^\rho(t)\frac{\delta}{\delta X^{\nu]}(t)} \tag{4.21}
$$

These $L'_{\mu\nu}$ and the translation generators P_μ (as boundary elements) constitute another Poincaré-Lie algebra $\mathcal{P}'$ when acting on boundary local functionals, where the commutation relations are written with respect to the open string metric $G_{\mu\nu}$. It is easy to show that the Hopf subalgebra $U(\mathcal{P}')$ of $\mathcal{H}$ is invariant under the twist $\mathcal{F}_S$. In particular, $L'_{\mu\nu}$ is primitive. In terms of the normal ordered Hopf algebra, this means $\tilde{P}_\mu = P_\mu$ and $\tilde{L}'_{\mu\nu} = L'_{\mu\nu}$ as boundary elements in $\hat{\mathcal{H}}_S$. Therefore, $U(\mathcal{P}')$ is considered to be a quantum Poincaré symmetry in this quantization scheme when restricted on the boundary.

Consequently, boundary elements of the module algebra $\hat{\mathcal{A}}_S$ are classified by the representation of $U(\mathcal{P}')$ with a fixed momentum k_μ. A local functional $F[X] = e^{ik_\mu X^\mu(t)}$ defines

[7] They are defined by the functionals and functional derivatives of the embedding $X^\mu(t)$ of the boundary, or equivalently, defined by inserting a delta function of the form $\int_{\partial\Sigma} dt\delta^{(2)}(z-t)$ into $\mathcal{H}$ and $\mathcal{A}$.

a tachyon vertex operator $\mathcal{N}_S \triangleright F = {}^{\bullet}_{\bullet} e^{ik\cdot X(t)} {}^{\bullet}_{\bullet}$ with momentum k_μ. In general, a local boundary vertex operator $V_k(t)$ with momentum k_μ consists of the worldsheet derivatives of X and $e^{ik\cdot X}$, having the form

$$V_k(t) = {}^{\bullet}_{\bullet} P[\partial X(t)] e^{ik\cdot X(t)} {}^{\bullet}_{\bullet} \quad . \tag{4.22}$$

Here, $P[\partial X]$ is a polynomial in $\partial_t^l X^\mu(t)$ and $\partial_n^l X^\mu(t)$ $(l = 1, 2, \cdots)$. The functional form of $P[\partial X]$ determines the representation of $U(\mathcal{P}')$. Since the action of $h \in U(\mathcal{P}')$ is given by $h \triangleright V_k(t) = {}^{\bullet}_{\bullet} h \triangleright P[\partial X(t)] e^{ik\cdot X(t)} {}^{\bullet}_{\bullet}$, this representation is the same as that of a classical functional. For example, ${}^{\bullet}_{\bullet} \partial_t X^\mu e^{ik\cdot X(t)} {}^{\bullet}_{\bullet}$ transforms as a 1-form and the lowering and the raising of tensor indices are taken with respect to $G_{\mu\nu}$. Note that above boundary vertex operators $V_k(t)$ are equivalent to the vertex operators used in Ref.[3]. Their anomalous dimension and the on-shell condition are determined with respect to the open string metric $G_{\mu\nu}$.

Second twist. Twisting by $\mathcal{F}_A$, we obtain the twisted Hopf algebra $(\hat{\mathcal{H}}_S)_{\mathcal{F}_A}$, which is equivalent to the full twisted Hopf algebra $\mathcal{H}_{\mathcal{F}_1}$ as in the case discussed in section 4.1. Correspondingly, we obtain the twisted module algebra $\mathcal{A}_{\mathcal{F}_1} \simeq (\hat{\mathcal{A}}_S)_{\mathcal{F}_A}$. The twisted product $*_{\mathcal{F}_1}$ in $\mathcal{A}_{\mathcal{F}_1}$ is seen as a deformation of the product in $\hat{\mathcal{A}}_S$:

$$\langle {}^{\circ}_{\circ} F {}^{\circ}_{\circ} {}^{\circ}_{\circ} G {}^{\circ}_{\circ} \rangle_1 = \tau(F *_{\mathcal{F}_1} G) = \langle {}^{\bullet}_{\bullet} F {}^{\bullet}_{\bullet} *_{\mathcal{F}_A} {}^{\bullet}_{\bullet} G {}^{\bullet}_{\bullet} \rangle_S. \tag{4.23}$$

This is valid for arbitrary functionals $F, G \in \mathcal{A}$, including bulk local functionals corresponding to the closed string vertex operators.

If we consider only the boundary vertex operators, there is a great simplification in the structure of the second twist. This is because the boundary-boundary propagator $G_A^{\mu\nu}(s,t)$ in (4.18) is topological, i.e., it depends only on the sign of $s - t$. The twist element acting on the boundary vertex operators has the form

$$\mathcal{F}_A = \exp\left\{ -\frac{i}{2}\theta^{\mu\nu} \int ds \int dt\, \epsilon(s-t) \frac{\delta}{\delta X^\mu(s)} \otimes \frac{\delta}{\delta X^\nu(t)} \right\} \quad . \tag{4.24}$$

First, worldsheet derivatives of X do not feel this deformation: e.g., $\partial_a X^\mu(s) *_{\mathcal{F}_A} X^\nu(t) = \partial_a X^\mu(s) X^\nu(t)$ holds for $s \neq t$. Therefore, for any correlation function of boundary vertex operators of the form (4.22), the product $*_{\mathcal{F}_A}$ is only sensitive to the tachyon part, and we have the relation

$$\langle V_{k_1}(t_1) *_{\mathcal{F}_A} \cdots *_{\mathcal{F}_A} V_{k_n}(t_n) \rangle_S = e^{-\frac{i}{2}\sum_{i>j} k_{i\mu}\theta^{\mu\nu}k_{j\nu}\epsilon(t_i - t_j)} \langle V_{k_1}(t_1) \cdots V_{k_n}(t_n) \rangle_S \tag{4.25}$$

Next, by fixing the order of insertion points $t_1 > t_2 > \cdots > t_n$, the extra phase factor in the r.h.s. above becomes independent of the precise locations, and gives the factor of the Moyal product. As explained in Ref.[3], this enables us to take the following prescription: *replace ordinary multiplication in the effective field theory written in the open string metric by the Moyal product.* We emphasize here that this works independently of the kind of the field (representation of $U(\mathcal{P}')$) on D-branes, contrary to the case in section 4.1. On the contrary, if $*_{\mathcal{F}_A}$ would be sensitive to the derivatives of X, then the definition of the

product for scalar fields and vector fields, such as $\phi * \phi$ and $\phi * A_\mu$, would be different. In the Hopf algebra language, this means that with a fixed ordering the twist element (4.24) acts as a Moyal-twist

$$\mathcal{F}_M = e^{\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} \quad , \tag{4.26}$$

given in the literature Refs.[10, 11, 12, 13, 14]. Here, the integrals of functional derivatives in (4.24) are replaced by its zero mode part, the translation generators $P_\mu \in U(\mathcal{P}')$.

The Poincaré symmetry is also simplified. The coproduct of the Lorentz generators $L'_{\mu\nu} \in U(\mathcal{P}')$ acting on boundary vertex operators is twisted by $\mathcal{F}_A$ as

$$\Delta_{\mathcal{F}_A}(L'_{\mu\nu}) = \Delta(L'_{\mu\nu}) + \frac{1}{2}\theta^{\alpha\beta} \int ds \int dt\, \epsilon(s-t) G_{\alpha[\mu} \frac{\delta}{\delta X^{\nu]}(s)} \otimes \frac{\delta}{\delta X^\beta(t)} \tag{4.27}$$

but it reduces by fixing the ordering $s > t$ to

$$\Delta_{\mathcal{F}_M}(L'_{\mu\nu}) = \Delta(L'_{\mu\nu}) + \frac{1}{2}\theta^{\alpha\beta} \left\{ G_{\alpha[\mu} P_{\nu]} \otimes P_\beta + P_\alpha \otimes G_{\beta[\mu} P_{\nu]} \right\}, \tag{4.28}$$

This gives the twisted Poincaré-Hopf algebra $U_{\mathcal{F}_M}(\mathcal{P}')$ structure found in Refs.[12, 13, 14]. Note that since the twist element itself now belongs to $U(\mathcal{P}') \otimes U(\mathcal{P}')$, the twisting is closed in $U(\mathcal{P}')$, contrary to the case in section 4.1. To summarize, the twisted Poincaré symmetry on the Moyal-Weyl noncommutative space is derived from string worldsheet theory in a B-field background. We give two further remarks.

Relation between $\mathcal{F}_A$ and $\mathcal{F}_M$. Before fixing the ordering of insertions, we have

$$\begin{aligned}
&\mathcal{F}_A^{-1} \triangleright (V_{k_1}(t_1) \otimes V_{k_2}(t_2)) \\
&= \exp\left\{ -\frac{i}{2}\theta^{\mu\nu} \int ds_1 \int ds_2\, \epsilon(s_1 - s_2) k_{1\mu} k_{2\nu} \delta(s_1 - t_1) \delta(s_2 - t_2) \right\} (V_{k_1}(t_1) \otimes V_{k_2}(t_2)) \\
&= e^{-\frac{i}{2}\epsilon(t_1 - t_2)\theta^{\mu\nu} k_{1\mu} k_{2\nu}} \left(V_{k_1}(t_1) \otimes V_{k_2}(t_2)\right) \\
&= \left[\theta(t_1 - t_2) e^{-\frac{i}{2}\theta^{\mu\nu} k_{1\mu} k_{2\nu}} + \theta(t_2 - t_1) e^{\frac{i}{2}\theta^{\mu\nu} k_{1\mu} k_{2\nu}} \right] (V_{k_1}(t_1) \otimes V_{k_2}(t_2)) \\
&= \left[\theta(t_1 - t_2) \mathcal{F}_M^{-1} + \theta(t_2 - t_1) \mathcal{F}_M \right] \triangleright (V_{k_1}(t_1) \otimes V_{k_2}(t_2)),
\end{aligned} \tag{4.29}$$

using the fact that $\mathcal{F}_A$ acts only on the tachyon part, and $\epsilon(t) = \theta(t) - \theta(-t)$. Therefore, on the tensor product $V_{k_1}(t_1) \otimes V_{k_2}(t_2)$ of boundary vertex operators, we get a relation

$$\begin{aligned}
\mathcal{F}_A^{-1} &= \theta(t_1 - t_2) \mathcal{F}_M^{-1} + \theta(t_2 - t_1) \mathcal{F}_M \\
&= \mathcal{F}_M^{-1} \left[\theta(t_1 - t_2) + \theta(t_2 - t_1) \mathcal{R}_M^{-1} \right].
\end{aligned} \tag{4.30}$$

Here $\mathcal{R}_M := \mathcal{F}_{M21} \mathcal{F}_M^{-1}$ is the universal R-matrix for the Moyal deformation, and its inverse is $\mathcal{R}_M^{-1} = \mathcal{F}_M^2 = e^{i\theta^{\mu\nu} P_\mu \otimes P_\nu}$. In general, the universal R-matrix is given for an *almost* cocommutative Hopf algebra, in the sense that the coproduct is cocommutative up to conjugation by $\mathcal{R}$.

$$\Delta^{op}(h) = \mathcal{R} \Delta(h) \mathcal{R}^{-1}. \tag{4.31}$$

Correspondingly, the twisted product $f * g$ in the module algebra $\mathcal{A}_{\mathcal{F}}$ is related to its opposite product $g * f$ as

$$\begin{aligned} g * f &= m \circ \mathcal{F}^{-1} \triangleright (g \otimes f) \\ &= m \circ \mathcal{F}_{21}^{-1} \triangleright (f \otimes g) \\ &= m \circ \mathcal{F}^{-1} \mathcal{R}^{-1} \triangleright (f \otimes g) \ . \end{aligned} \tag{4.32}$$

Hence, $*$ is almost commutative up to the insertion of $\mathcal{R}^{-1}$.

In our case, the twist $\mathcal{F}_A$ in the string theory is strictly commutative, and therefore the R-matrix is trivial, $\mathcal{R}_A = 1 \otimes 1$. It corresponds to the time ordered product in the operator formulation. However, as shown in (4.30) the product of functionals is rewritten by the operator product with fixed ordering, where each operator product defined by $\mathcal{F}_M$ is noncommutative but almost commutative. In this way the noncommutative product in the effective field theory is derived from the twisted but commutative product in the worldsheet theory.

Remark on closed strings In the above discussion, the insertion points of the vertex operators are restricted to the boundary of the world sheet. Owing to this restriction, the second twist $\mathcal{F}_A$ reduces to the Moyal-twist, and we obtain the twisted Poincaré-Hopf algebra $U_{\mathcal{F}_M}(\mathcal{P}')$. On the other hand, if we consider the correlation function of closed string, or couplings of D-branes to the closed string, the insertion points are not always at the boundary and it is inevitable to use the bulk-bulk or the bulk-boundary propagator. Therefore, $U_{\mathcal{F}_M}(\mathcal{P}')$ is not sufficient to describe this situation, rather we should work within a full diffeomorphism $\mathcal{H}$ and twisting by $\mathcal{F}_1$. This is one reason why the simple generalization of the twisted Poincaré symmetry to a diffeomorphism [14, 15] based on the Moyal-twist $\mathcal{F}_A$ is not sufficient to describe a symmetry in a brane induced gravity, as pointed out in Ref.[17]. It is an interesting question how the twisted diffeomorphism in the effective field theory has to be modified in order to recover the string theory result.

5. Conclusion and discussion

Considering the Hopf algebra structure of the symmetry in string theory, the quantization of the string can be considered as the Drinfeld twist. For the case of the trivial background, i.e., in the Minkowski target space with vanishing B field, we have shown in Ref.[18] that the path integral quantization of the string worldsheet theory is reformulated as the Drinfeld twist of the Hopf algebra $\mathcal{H}$ and the module algebra $\mathcal{A}$ of functionals on the worldsheet, with twist element $\mathcal{F}_0$. The quantized algebra of the vertex operators $\hat{\mathcal{A}}$ appeared via the coboundary relation (2.17) which showed that the twist characterizes also the normal ordering $\mathcal{N}_0$, a property which is rather implicit in the path integral.

In the present paper, we applied the same formalism to the case with non-vanishing B-field background. The quantization is characterized by the twist element $\mathcal{F}_1$. In the twist quantization, we can also obtain the natural choice of the normal ordering. We have discussed the mechanism to obtain the information on the normal ordering and have shown that the coboundary relation for the twist element $\mathcal{F}_1$ yields that the natural normal

ordering in this approach is given by $\mathcal{N}_1$. The normal ordering conditions obtained in this way is most natural from the point of view of the Hopf algebra as the symmetry of the theory.

One of the advantages of the twist quantization is that there is a direct relation between quantization and symmetry via the Hopf algebra. For example, the Poincaré symmetry $U(\mathcal{P})$ is also twisted by $\mathcal{F}_1$ in the twist quantization. In order to compare the twist quantization with known formulations, we have decomposed the twist into two successive twists. We have discussed two types of decomposition.

In the first decomposition, $\mathcal{F}_1 = \mathcal{F}_B\mathcal{F}_0$, the first twist by $\mathcal{F}_0$ is the standard quantization of string with $B = 0$ and the second twist by $\mathcal{F}_B$ is a twist of the operator product of vertex operators by the B-field. The resulting structure is the twisted Poincaré symmetry, which indicates the new description of spacetime, with the B-field as a twist rather than a background field. The description here can be considered as an example, for incorporating a closed string state into a more general kind of geometry (stringy geometry). This decomposition is also natural if one wants to compare with the perturbative calculation of the effect of the B-field background.

The second decomposition, $\mathcal{F}_1 = \mathcal{F}_A\mathcal{F}_S$, is meaningful only for open strings, where the first twist $\mathcal{F}_S$ is interpreted as a quantization with respect to the open string metric, and the second twist $\mathcal{F}_A$ reduces to the Moyal twist $\mathcal{F}_M$ equipped with the universal R-matrix. This explains clearly the relation between the twisted Poincaré symmetries in the string worldsheet theory and in the low energy effective theory on D-branes. From the viewpoint of noncommutative field theory, this second decomposition gives an alternative way to Refs.[14, 15], for generalizing the noncommutativity of the Moyal plane, where in our case the (open string) metric $G_{\mu\nu}$ is also incorporated in the twist $\mathcal{F}_1$. Moreover, the concept of the twisted diffeomorphism in Refs.[14, 15] is realized not at the level of the noncommutative field theory, but at the level of the worldsheet theory. As stated in the end of section 4.2, the true twisted diffeomorphism on the Moyal plane should be different from that in Refs.[14, 15] as a low energy effective theory on D-branes.

The decomposition is rather arbitrary and possible as far as the second twist satisfies the cocycle condition under the first twist, the condition corresponding to (4.6). Therefore, the method proposed in this paper would be applicable to more non-trivial backgrounds, such as non-constant metric or B-field. The corresponding perturbative calculations in these cases have been done in Refs. [27, 28, 29, 30]. It is an interesting question whether their results can be read off as an analogue of the first decomposition, $\mathcal{F}_1 = \mathcal{F}_0\mathcal{F}_B$ of some (possibly non-abelian) twist element. This would help to get a better understanding of the structure of the stringy geometry.

As already remarked in Ref.[17], the string field theory is a formulation which can, in principle, represent the space-time symmetry as a kind of gauge symmetry. The constant B-field background in such a formulation has been considered in Refs.[31, 32]. There, a product of string fields (string vertex) acquires a modification by a phase factor, which is similar to the twist element $\mathcal{F}_A$ here. Thus, the Hopf algebra structure and the method presented here would also be helpful to understand the structure of the gauge symmetry in the string field theory.

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