Geometry-driven shift in the Tomonaga-Luttinger exponent of deformed cylinders

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We demonstrate the effects of geometric perturbation on the Tomonaga-Luttinger liquid (TLL) states in a long, thin, and hollow cylinder whose radius varies periodically. The variation in the surface curvature inherent to the system gives rise to a significant increase in the power-law exponent of the single-particle density of states. The increase in the TLL exponent is caused by a curvature-induced potential that attracts low-energy electrons to a region that has large curvature.

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Studying the quantum mechanics of a particle confined to curved surfaces has been a problem for more than 50 years. The difficulty arises from operator-ordering ambiguities,1 which permit multiple consistent quantizations for a curved system. The conventional method used to resolve the ambiguities is the confining-potential approach.2,3 In this approach, the motion of a particle on a curved surface (or, more generally, a curved space) is regarded as being confined by a strong force acting normal to the surface. Because of the confinement, quantum excitation energies in the normal direction are raised far beyond those in the tangential direction.

Hence, we can safely ignore the particle motion normal to the surface, which leads to an effective Hamiltonian for propagation along the curved surface with no ambiguity. It is well known that the effective Hamiltonian involves an effective scalar potential whose magnitude depends on the local surface curvature.2–5 As a result, quantum particles confined to a thin curved layer behave differently from those on a flat plane, even in the absence of any external field (except for the confining force). Such curvature effects have gained renewed attention in the last decade, mainly, because of the technological progress that has enabled the fabrication of low-dimensional nanostructures with complex geometry.5,6–17,11,12–14 From the theoretical perspective, many intriguing phenomena pertinent to electronic states,15–22 electron diffusion,23 and electron transport24–27 have been suggested. In particular, the correlation between surface curvature and spin-orbit interaction28,29 as well as with the external magnetic field30–32 has been recently considered as a fascinating subject.

Most of the previous works focused on noninteracting electron systems, though few have focused on interacting electrons33 and their collective excitations. However, in a low-dimensional system, Coulomb interactions may drastically change the quantum nature of the system. Particularly noteworthy are one-dimensional systems, where the Fermi-liquid theory breaks down so that the system is in a Tomonaga-Luttinger liquid (TLL) state.34 In a TLL state, many physical quantities exhibit a power-law dependence stemming from the absence of single-particle excitations near the Fermi energy; this situation naturally raises the question as to how geometric perturbation affects the TLL behaviors of quasi-one-dimensional curved systems.

We first considered noninteracting spinless electrons confined to a general two-dimensional curved surface S embedded in a three-dimensional Euclidean space. A point p on S is represented by \( p = [x(u^1, u^2), y(u^1, u^2), z(u^1, u^2)] \), where \((u^1, u^2)\) is a curvilinear coordinate spanning the surface and \((x, y, z)\) are the Cartesian coordinates in the embedding space. Using the notation \( p_i = \partial p / \partial u^i \) \((i = 1, 2)\), we introduced the following quantities \( g_{ij} = p_i p_j \), \( h_{ij} = p_i n \), and \( n = p_i \times p_j / ||p_i \times p_j|| \), where \( n \) is the unit vector normal to the surface. Using the confining-potential approach,2,3 we obtained the Schrödinger equation for noninteracting electron systems on curved surfaces as follows:

\[ \hat{H} \Psi = E \Psi \]

![FIG. 1. (Color online) Schematic illustration of a quantum hollow cylinder with periodic radius modulation.](image-url)
FIG. 2. (Color online) Profiles of the curvature-induced effective potential $U_n(\xi)$ for one period [0, λ]. Geometric parameters $r_0 = 4.0$ and $\lambda = 8.0$ in units of $a$ are fixed. Integers $n$ represent the angular momentum of eigenstates in the circumferential direction of a hollow tube.

$$-\frac{\hbar^2}{2m} \left[ \frac{1}{\sqrt{g}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i} \sqrt{g} g^{ij} \frac{\partial}{\partial u_j} + (\mathcal{H}^2 - K) \right] \Psi = E \Psi,$$

where $g = \det(g_{ij})$, $g^{ij} = g^{-1}_{ij}$, and $m^*$ is the effective mass of electrons. The quantities $K = (h_1 h_2 - h_2 h_1)/g$ and $\mathcal{H} = (g_{11} h_{22} + g_{22} h_{11} - 2 g_{12} h_{12})/(2g)$ are the so-called Gaussian curvature and mean curvature, respectively, both of which are functions of $(u^1, u^2)$. The term proportional to $\mathcal{H}^2 - K$ in Eq. (1) is the effective scalar potential induced by the surface curvature.

We next focus on a hollow tube with a periodically varying radius represented by $p = [r(z) \cos \theta, r(z) \sin \theta, z]$ (see Fig. 1). The tube radius $r(z)$ is periodically modulated in the axial $z$ direction as $r(z) = r_0 - \frac{\lambda}{2} \cos(2\pi z)$, where the parameter $r_0$ and $\lambda$ are introduced to express the maximum and minimum of $r(z)$ as $r_0$ and $r_0 - \lambda$, respectively. Because of the rotational symmetry, the eigenfunctions of the system have the form of $\Psi(z, \theta) = e^{i2\pi z b} \phi_n(z)$. Thus, the problem reduces to the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m} \left[ D - \frac{n^2}{r^2} + (\mathcal{H}^2 - K) \right] \phi_n(z) = E \phi_n(z),$$

where $D = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}$, $f(z) = \sqrt{1 + r^2}$, $K = -r''/(rf)^2$, and $\mathcal{H} = (f^2 - rr'')/(2rf)^2$ with $r' = dr/dz$.

Equation (2) is simplified by using a new variable $\xi = \xi(z) = \int f(\eta) d\eta$, which corresponds to the line length along the curve on the surface with a fixed $\theta$. Straightforward calculation yields

$$\left[ -\frac{\hbar^2}{2m} \frac{d^2}{d\xi^2} + U_0(\xi) \right] \phi_n(\xi) = E \phi_n(\xi), \quad E = \frac{2m^* a^2 E}{\hbar^2}$$

with $U_0(\xi) = (n^2 - \frac{1}{2})a^2/r^2 - r^2 a^2/(4\pi)$, where $r, r''$, and $f$ are regarded as the functions of $\xi$ using the inverse relation $z = \xi^{-1}(\xi)$. In order to derive Eq. (3), we introduced the length scale $a$ and then multiplied both sides of Eq. (2) with $2m^* a^2 \hbar^2$ to make the units of $U_0$ and $E$ dimensionless. Notice that by the definition of $\xi(z)$, $U_0$ is periodic with a period $\Lambda = \xi(\lambda)$ depending on $r_0$ and $\lambda$ (as well as $\lambda$). Figure 2 shows the spatial profile of $U_n$ within one period; throughout the present work, we fixed $r_0 = 4.0$ and $\lambda = 8.0$ in units of $a$ by simulating the geometry of actual peanut-shaped $C_{60}$ polymers whose geometry is reproduced by imposing $a = 1 \text{Å}$. Mapping the discrete atomic structure of one-dimensional $C_{60}$ polymers to a continuum curved surface is based on the result of first-principle calculations which indicated that $\pi$ electrons on the polymers are almost free from its atomic configurations. We found in Fig. 2 that $U_n$ exists extrema at $\xi = 0$ (or $\Lambda$) and $\xi = \Lambda/2$, where $r$ takes the maximum ($r = r_0$) and the minimum ($r = r_0 - \lambda$) values, respectively.

To solve Eq. (3), we use the Fourier series expansions $U_n(\xi) = \sum G_G(g) e^{iG_x}$ and $\phi_n(\xi) = \sum c_n^{(G)} e^{iG_x}$, where $G = \pm 2\pi j/\lambda$ ($j = 0, 1, 2, \ldots$). Substituting the expansions into Eq. (3), we obtain the secular equation ($\frac{\hbar^2}{2m} k^2 - E - G$) + $\sum G_G g^{(G)} c_n^{(G)} c_{n'}^{(G)} = 0$ that holds for all possible $k$ and $n$'s. The summation has been truncated by $G_J = 20 \pi/\lambda$ because of the rapid decay of $U_G$ with $|G|$. We then numerically calculated eigenvalues for $0 \leq k \leq \pi/\Lambda$ and evaluated the low-energy-band structure for several different $\delta r$, as depicted in Fig. 3. In all the cases, there is some energy gap at the Brillouin-zone boundary $G_J = \pi/\Lambda$, where a wider energy gap occurs for a larger $\delta r$, as expected from the large amplitude of $|U_n(\xi)|$ with increasing $\delta r$ (see Fig. 2).

We now consider the Coulombic interactions between spinless electrons. The interactions make the electron-hole pairs share the ground state of the noninteracting electron system, and the most strongly affected states are those lying in the vicinity of $E_F$. As a consequence, the single-particle density of states $n(\omega)$ near $E_F$ exhibits a power-law singularity of the following form:

$$n(\omega) \propto |\hbar \omega - E_F|^\alpha, \quad \alpha = \frac{K + K^{-1}}{2} - 1.$$
coupling constants. \( V(q, m) \) is the Fourier transform of the screened interaction \( V(r) = -\varepsilon e^{-qr}/(4\pi r) \), where \( \varepsilon \) is the dielectric constant and \( \kappa \) is the screening length. The transformation is performed in terms of the curvilinear coordinates \((\xi, 0)\), and thus, the resulting \( V(q, m) \) becomes a function of both the momentum and angular momentum transfers \( q \) and \( m \), respectively. To make concise arguments, \( E_F \) was assumed to lie in the lowest-energy band \((n=0)\). This allows us to eliminate the index \( m \) from \( V(q, m) \), which leads to \( V(q) = -\varepsilon^2/4\pi^2 \log[(q^2 + \kappa^2)^2] \) for \( qr_0 \ll 1 \), in which \( r_0 \) serves as the short-length-scale cutoff.

The aim of the present study is to examine the \( \delta r \) dependence of both the exponent of power-law decay in Friedel oscillation and \( \alpha \) for \( k_F \) for various kinds of power-law exponents observed in TLL states. This is because various kinds of power-law exponents observed in TLL states are related to the quantity \( \kappa a \); some such exponents are the exponent of power-law decay in Friedel oscillation and that of temperature (or voltage)-dependent conductance. The present theoretical predictions need to be confirmed experimentally, thus opening a field of science that deals with quantum electron systems on curved surfaces.

In conclusion, we reveal that the power-law exponent \( \alpha \) of the TLL states in deformed hollow nanocylinders shows a monotonic increase with an increase in the degree of surface curvature. The increase in \( \alpha \) is attributed to the curvature-driven effective potential \( U_s(\xi) \) that acts on electrons moving along the curved surface. The present results suggest that there are shifts in the power-law exponents of TLL states of real low-dimensional materials such as the peanut-shaped \( C_{60} \) polymers and MoS\(_2\).

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