On the Property of the Curl–Curl Matrix in Finite Element Analysis With Edge Elements

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***Abstract*—This paper discusses properties of the curl–curl matrix in the finite element formulation with edge elements. Moreover the observed deceleration in convergence of the CG and ICCG methods applied to magnetostatic problems through the tree–cotree gauging is explained on the basis of the eigenvalue separation property. From the eigenvalue separation property it follows that neither minimum nonzero eigenvalue of the curl–curl matrix nor maximum one increase through the tree–cotree gauging. Hence it is concluded that the condition number of the curl–curl matrix tends to grow by its definition. Moreover the maximum eigenvalue tends to keep constant whereas the minimum nonzero eigenvalue reduces. This property also makes the condition number worse.**

***Index Terms*—Edge elements, finite element method, graph theory, ICCG, magnetostatic fields, network theory.**

I. Introduction

MAGNETOSTATIC problems are effectively solved by the finite element method with edge elements. In those analyzes unknown vector potentials are assigned to element edges to guarantee their tangential continuity and allow necessary discontinuity in their normal components. The finite element discretization of the differential curl–curl operator gives the curl–curl matrix, which is known to be singular, and its nullity equals to the number of the edges in the spanning tree of a finite element mesh [1], [2]. The curl–curl matrix can be regularized by eliminating the unknown vector potentials assigned to the edges in the spanning tree [3]. This ingenuous gauging technique, sometimes called the tree–cotree gauging [4], has reasonably been constructed to bridge the gap between continuous and discontinuous systems. However this gauging unfortunately results in slow convergence of the CG and ICCG methods which are extensively used for matrix inversion.

On the other hand it has been observed in numerical experiments that the ICCG method applied to nongauged curl–curl matrices converges in spite of its singularity, provided that the system of equations is compatible. Moreover convergence of the ICCG has been shown to be much faster than that for matrices regularized by the above gauging technique [5], [6]. Consequently the solutions without gauging seem most effective and have extensively been used recently.

However it still remains unclear why the above gauging technique undermines convergence of the ICCG. In this paper properties of the curl–curl matrix are discussed in order to give an answer to this question.

This paper is organized as follows: The next section discusses rank of the curl–curl matrix with the help of the network theory, and surveys the tree–cotree gauging. The third section describes the condition number of the curl–curl matrix, and its influence on convergence of the CG and ICCG method. The influence of the tree–cotree gauging on the condition number of the curl–curl matrix is analyzed on the basis of the eigenvalue separation property in the forth section. The fifth section provides a numerical example, and the last section includes some concluding remarks.

II. The Curl–Curl Matrix

We consider here a magnetostatic field governed by

$$\operatorname{curl} \nu(\operatorname{curl} \boldsymbol{A}) = \boldsymbol{J}_0, \tag{1}$$

where

$\boldsymbol{A}$ is the vector potential,
ν the magnetic reluctivity, and
$\boldsymbol{J}_0$ the external current density.

Let us assume for simplicity that (1) is discretized by the finite element method with the tetrahedral edge elements of the lowest order. The finite element mesh considered here has n nodes, e edges and f faces. We then obtain a system of equations of the form

$$C\boldsymbol{a} = \boldsymbol{b}, \tag{2}$$

where C denotes $e \times e$ matrix, referred to as the curl–curl matrix, which is the discrete counterpart of the operator curl νcurl in (1). The column vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, both with e entities, consist of the projection a_i of $\boldsymbol{A}$ to element edge i and the source terms relevant to $\boldsymbol{J}_0$, respectively.

The matrix C is decomposed as $C = B^t M B$, where M is the $f \times f$ matrix, whose entities M_{ij} are given by $M_{ij} = \int_\Omega \nu \boldsymbol{w}_i \cdot \boldsymbol{w}_j \, dv$, where $\boldsymbol{w}_k$ denote the basis vectors for the Whitney second form (or facet element) [1]. Since the independent vectors $\boldsymbol{w}_k$ span f-dimensional space, M, whose determinant is the Gramian, must be regular. Otherwise there exist nontrivial constants α and β for different i, j after appropriate elementary transformations such that $\boldsymbol{w}_k \cdot (\alpha \boldsymbol{w}_i + \beta \boldsymbol{w}_j) = 0$ for $^\forall k$. However this means that $\alpha \boldsymbol{w}_i + \beta \boldsymbol{w}_j$ is not in the f-dimensional space spanned by $\boldsymbol{w}_k$. This is contradiction.

The matrix B included in C is the $f \times e$ matrix whose entities are given by $B_{ij} = 1(-1)$ when face (or loop) i includes edge

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j and the former direction is parallel (antiparallel) to the latter, and $B_{ij} = 0$ for others. The matrix B, which depends only on the topological property of the mesh, is called the loop or circuit matrix in the network theory. It is known that rank of B is equal to the number of edges in the cotree, that is $e-n+1$ (see standard textbooks of the network theory).

Rank C can be evaluated as follows: Noticing theorems in the linear algebra we have

$$\begin{aligned} rank\, C &\leq rank\,(MB) \\ &\leq rank\, B. \end{aligned} \tag{3}$$

On the other hand we can see that

$$\begin{aligned} rank\, C &\geq rank\,(B^t M^{-1} MB) \\ &= rank\,(B^t B) \\ &= rank\, B, \end{aligned} \tag{4}$$

from which it follows that $rank\, C = rank\, B = e - n + 1$.

The curl–curl matrix C can be regularized by eliminating the rows and columns corresponding to edges in the spanning tree of the finite element mesh. This is the tree–cotree gauging mentioned in the previous section. Physically this determines a unique value of the projection a_i for a given magnetic flux passing through the closed loop. As mentioned above this gauging deteriorates convergence of the CG and ICCG methods when applied to (2). This phenomenon will be analyzed in Section IV after convergence of the CG and ICCG methods are discussed in the next section.

III. Convergence of CG and ICCG

A. CG Applied to Singular Matrices

We consider here the CG method for solution of a linear system of order N

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{5}$$

where A is a symmetric matrix. The algorithm of CG method is as follows: choose the initial solution $\boldsymbol{x}_0$ and set $\boldsymbol{p}_0 = \boldsymbol{r}_0 = \boldsymbol{b} - A\boldsymbol{x}_0$. Then for $k = 0, 1, \ldots$ compute

$$\begin{aligned} \alpha_k &= (\boldsymbol{p}_k, \boldsymbol{r}_k)/(\boldsymbol{p}_k, A\boldsymbol{p}_k), \\ \boldsymbol{x}_{k+1} &= \boldsymbol{x}_k + \alpha_k \boldsymbol{p}_k, \\ \boldsymbol{r}_{k+1} &= \boldsymbol{r}_k - \alpha_k A\boldsymbol{p}_k, \\ \beta_k &= (\boldsymbol{r}_{k+1}, A\boldsymbol{p}_k)/(\boldsymbol{p}_k, A\boldsymbol{p}_k), \\ \boldsymbol{p}_{k+1} &= \boldsymbol{r}_{k+1} + \beta_k \boldsymbol{p}_k. \end{aligned} \tag{6}$$

As can be seen in (6), the approximated solution $\boldsymbol{x}_{k+1}$ is constructed by the linear combination of the search directions $\boldsymbol{p}_j$, which is in the space spanned by residual vectors $\boldsymbol{r}_j$, where $j = 1, 2, \ldots, k$. Hence $\boldsymbol{x}_{k+1}$ is searched in the space spanned by the residual vectors $\boldsymbol{r}_j$.

Moreover from (6) it follows that $\boldsymbol{r}_k$ is expressed in the form

$$\boldsymbol{r}_k = \boldsymbol{r}_0 + \sum_{m=1}^{k} a_{mk} A^m \boldsymbol{r}_0, \tag{7}$$

where a_{mk} are constants.

Now $\boldsymbol{r}_0$ is expanded by the eigenvectors $\boldsymbol{e}_i$ of A as $\boldsymbol{r}_0 = \sum_{i=1}^{N} c_i \boldsymbol{e}_i$. We assume that $nul\, A = \nu$, and its nonzero eigenvalues are $\lambda_{\nu+1}, \lambda_{\nu+2}, \ldots, \lambda_N$, where $rank\, A + nul\, A = N$. The residue $\boldsymbol{r}_k$ can then be written as

$$\boldsymbol{r}_k = \boldsymbol{r}_0 + \sum_{i=\nu+1}^{N} c_i \boldsymbol{e}_i \sum_{m=1}^{k} \lambda_i^m a_{mk}. \tag{8}$$

We can see from (8) that in the CG method the solution is searched in the space spanned by the nonzero eigenvectors $\boldsymbol{e}_{\nu+1}, \ldots, \boldsymbol{e}_N$ and $\boldsymbol{r}_0$. Therefore the solution processes are independent of the zero-eigenvalues, and one of the solutions to (5) is found even if A is singular provided that the solution exists.

B. Condition Number and Convergence

One can qualitatively expect that convergence of the CG method has substantial influence from the range of eigenvalues of A because it searches for the solution in the space spanned by its eigenvectors.

Quantitatively it is known that the CG method linearly converges in the form [7], [8]

$$\|\boldsymbol{x} - \boldsymbol{x}_k\|_A \leq 2\|\boldsymbol{x} - \boldsymbol{x}_0\|_A \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^k, \tag{9}$$

where κ denotes the condition number of A, defined by $\kappa = \lambda_{\max}/\lambda_{\min}$, $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of A, and $\|\boldsymbol{y}\|_A = \sqrt{\boldsymbol{y}^t A \boldsymbol{y}}$.

Assuming that $\kappa \gg 1$ in (9), we can obtain the condition that the relative error becomes smaller than ϵ as follows:

$$k > \frac{\sqrt{\kappa}}{2} \log\left(\frac{2}{\epsilon}\right). \tag{10}$$

Hence the number of iterations until convergence is proportional to $\sqrt{\kappa}$.

When the CG is applied to the singular system (2), the eigenvectors corresponding to the zero eigenvalues give no influence on the iterative processes as mentioned above. Therefore, in this case, it would be reasonable to define the condition number by $\kappa = \lambda_{\max}/\lambda_{\min}^0$, where $\lambda_{\min}^0$ is the minimum nonzero eigenvalue of C.

The preconditioned CG methods such as ICCG transform (5) to

$$\tilde{A}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}}, \tag{11}$$

where

$\tilde{A} = P^{-1} A P^{-1}$,
$\tilde{\boldsymbol{x}} = P\boldsymbol{x}$,
$\tilde{\boldsymbol{b}} = P^{-1}\boldsymbol{b}$, and
P is symmetric positive definite.

By this transformation $\tilde{A}$ is made as near to the unit matrix as possible in order that the resultant eigenvalues become nearly unity, and hence $\sqrt{\kappa}$ becomes smaller. A look at the estimate (10) reveals that this preconditioning improves the convergence.

It is difficult to make a quantitative estimation of convergence of the ICCG. One can expect however that it would be positively correlative with κ of the original matrix A since the essential

property of the original eigenvalues is inherited to the preconditioned system.

IV. Tree–Cotree Gauging and Condition Number

A. Eigenvalue Separation Property

The minimax principle in the linear algebra states that the ith eigenvalue λ_i of a symmetric matrix A of order N can be expressed by

$$\lambda_i = \max_{\boldsymbol{y}_j} \left\{ \min_{\boldsymbol{y}_j^t \boldsymbol{x}=0} R(\boldsymbol{x}) \right\}, \qquad \text{for } i = 1, 2, \ldots, N, \quad (12)$$

where $\boldsymbol{y}_j$, $j = 1, 2, \ldots, i-1$ are arbitrary vectors in N dimensional space, and $R(\boldsymbol{x})$ represents the Rayleigh quotient defined by

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^t A \boldsymbol{x}}{\boldsymbol{x}^t \boldsymbol{x}}. \quad (13)$$

When $i = 1$, (12) reduces to the Rayleigh principle.

Suppose that we eliminate the end row and column from A to obtain the reduced matrix A' of order $N-1$. Now the ith eigenvalue μ_i of A' can be expressed by

$$\mu_i = \max_{\boldsymbol{y}_j} \left\{ \min_{\boldsymbol{y}_j^t \boldsymbol{x}=0} R(\boldsymbol{x}) \right\}, \quad (14)$$

where $\boldsymbol{y}_j$ are again arbitrary for $j = 1, 2, \ldots, i-1$ but $\boldsymbol{y}_i$ is chosen as $\boldsymbol{y}_i = (0, 0, \ldots, 0, 1)^t$.

By comparing (12) and (14) we can see that

$$\lambda_i \le \mu_i \le \lambda_{i+1} \quad (15)$$

since the restriction for λ_{i+1} is severer than μ_i while μ_i has a severer restriction than λ_i.

We can easily generalize (15) as

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \mu_2 \le \lambda_3 \le \cdots \le \mu_{N-1} \le \lambda_N. \quad (16)$$

This is known as the eigenvalue separation property, see e.g., [9]. Note here that since an arbitrary row and column can be permutated to the end of a matrix through the elementary operation, the property (16) is valid for the elimination of a set of a row and column corresponding to arbitrary degree of freedom.

B. Condition Number of Singular Systems

Now let us consider what happens when the tree–cotree gauging is applied to our singular system (2). This gauging eliminates degree of freedom assigned to all the edges in the spanning tree. Each process of this gauging is nothing else but the elimination of a set of a row and column from C.

Suppose that we eliminate one degree of freedom in the first process of the tree–cotree gauging. It can be seen from (16) that the first $\nu - 1$ zero eigenvalues remain the same while the minimum nonzero eigenvalue $\mu^0_{\min}$ appears so that

$$0 \le \mu^0_{\min} \le \lambda^0_{\min}. \quad (17)$$

Similarly we have

$$\lambda_{N-1} \le \mu_{N-1} = \mu_{\max} \le \lambda_N = \lambda_{\max}. \quad (18)$$

The change in the condition number κ caused by this process is indefinite. [In contrast to this, elimination of a degree of freedom, i.e., an increase in the restriction, in a regular system always improves κ, as can be seen from (16).] Nevertheless one can expect that this process tends to make κ worse since changes in the denominator give stronger influence on κ than those in the numerator. For instance we assume that the condition number κ_r for the reduced system is written as

$$\kappa_r = \frac{\mu_{\max}}{\mu^0_{\min}} = \frac{\lambda_{\max} - \delta_N}{\lambda^0_{\min} - \delta_1}, \quad (19)$$

where δ_1 and δ_N are the reductions in the eigenvalues through elimination of a degree of freedom. If the condition number becomes smaller after the elimination process, that is $\kappa_r < \kappa$, then

$$\frac{\delta_1}{\delta_N} < \frac{\lambda^0_{\min}}{\lambda_{\max}} \quad (20)$$

must hold. This is, however, a very severe condition because usually $\lambda^0_{\min}/\lambda_{\max} \ll 1$.

Moreover the maximum eigenvalue, which corresponds to an eigenvector with a short characteristic wavelength, would be insensitive to introduction of the new restriction to (2). The minimum nonzero eigenvalue $\lambda^0_{\min}$, whose eigenvector has a long characteristic wavelength, increases its value after introduction of the new restriction. Hence this would correspond to the second minimum nonzero eigenvalue in the reduced system. On the other hand the minimum nonzero eigenvalue $\mu^0_{\min}$, which newly appears between zero and $\lambda^0_{\min}$, is possibly very different from $\lambda^0_{\min}$. This fact also helps the increase in κ by the gauging.

Consequently we conclude that the tree–cotree gauging, which is regarded as the successive elimination of degrees of freedom in the spanning tree, tends to deteriorate the condition number κ of the curl–curl matrix C, and result in deceleration of the CG and ICCG. In the next section this theoretical prediction will be tested through a simple numerical example.

V. Numerical Results

For a numerical test we analyze the magnetostatic field in the cube which has unit-length edges parallel to the axes of the Cartesian coordinates [3]. In the region a uniform source current $\boldsymbol{J}_0$ flows in parallel with one of the normal vectors of the cube surface, e.g., the direction of the z-axis. Assuming that the magnetic field lines are confined in the cube, the tangential components of vector potentials are set to zero on the surfaces.

The matrix equation (2) is solved with the CG and ICCG methods. The degree of freedom of the system is varied by gradually eliminating the unknown vector potentials assigned to edges in the spanning tree. The eigenvalues of the curl–curl matrix C are also computed to evaluate its condition number and nullity.

The cubic region is subdivided into 234 tetrahedral elements with 77 nodes and 364 edges. The resultant numbers of edges in the tree and co-tree after imposing the boundary conditions are

TABLE I
CONVERGENCE OF CG AND ICCG

N_{elim}	N_{zero}	N_{CG}	N_{ICCG}	λ^0_{min}	λ_{max}	$\sqrt{\kappa}$
0	21	21	15	1.43	53.6	6.13
1	20	27	17	6.13×10^{-1}	53.6	9.35
5	16	40	21	8.54×10^{-2}	53.4	25.0
10	11	55	26	3.69×10^{-2}	52.8	37.8
15	6	68	31	1.91×10^{-2}	52.8	52.6
21	0	81	36	1.23×10^{-2}	48.6	62.9

N_{elim}: Number of eliminated DOFs on the spanning tree
N_{zero}: Number of zero-eigenvalues (nullity)
N_{CG}: Number of CG iterations
N_{ICCG}: Number of ICCG iterations

21 and 181. The tolerance for convergence of the CG and ICCG methods is set to 1×10^{-6}.

Table I summarizes the result, where $\lambda^0_{\min}$ and $\lambda_{\max}$ represents minimum nonzero and maximum eigenvalues under each computational condition. We can see that $\lambda^0_{\min}$ monotonously decreases while $\lambda_{\max}$ keeps almost constant as $N_{e\ \mathrm{lim}}$ increases, and hence this results in the increase of $\sqrt{\kappa}$. Moreover N_{CG} and N_{ICCG} increases consistently with $\sqrt{\kappa}$. These numerical results are consistent with the prediction given in the previous section.

VI. CONCLUSIONS

In this paper the observed deceleration in convergence of the CG and ICCG methods applied to magnetostatic problems by the tree–cotree gauging is explained on the basis of the eigenvalue separation property. The contents are summarized as follows:

1) The convergence of the CG and also ICCG methods are not influenced from zero eigenvalues of a singular matrix. Those convergence can be characterized by means of the condition number, which is defined as the ratio of the minimum nonzero eigenvalue to the maximum one.
2) The tree–cotree gauging can be regarded as the successive elimination of degree of freedom in the spanning tree. Hence the generated eigenvalues during the elimination process obey the eigenvalue separation property.
3) The eigenvalue separation property states that neither minimum nonzero eigenvalue nor maximum one increase through the elimination of degree of freedom. Hence the condition number tends to grow by its definition. Moreover the maximum eigenvalue tends to keep constant whereas the minimum nonzero eigenvalue reduces. This property also makes the condition number larger.

The logic used in this paper does not owe to the special property of the curl–curl matrix except its singularity. Hence the results are not valid only for the curl–curl matrix in magnetostatics, but also other singular matrices. One can expect in general that the elimination of degree of freedom from a singular system results in the worse condition number and resultant deceleration in the CG methods. In fact such phenomena have been observed in the eddy current analysis with the edge finite elements. In this case convergence in the ICCG becomes worse when the redundant terms coming from the divergence-free condition of eddy currents are eliminated. The detailed analysis of this phenomena will be reported elsewhere.

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