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SURFACE SYMMETRIES, HOMOLOGY REPRESENTATIONS, AND GROUP COHOMOLOGY

TOSHIYUKI AKITA

Given a finite group $G$ of automorphisms of a compact Riemann surface, we discuss a relation between Mumford-Morita-Miller classes of odd indices and the homology representation of $G$. Since most participants were group theorists rather than topologists, I separate the algebraic and the topological ingredients and explain the former in detail.

1. SURFACE SYMMETRIES

1.1. The Grieder group of a finite group. Let $G$ be a finite group and $\gamma$ the conjugacy class of $\gamma \in G$. We denote by $\langle \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_q \rangle$ an unordered $q$-tuple ($q \geq 0$) of conjugacy classes of nontrivial elements of $G$ satisfying $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$, and $\mathcal{M}_G$ the set of all such $q$-tuples. We can define an abelian monoid structure on $\mathcal{M}_G$ by

$$\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle + \langle \hat{\gamma}_q+1, \ldots, \hat{\gamma}_r \rangle = \langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q, \hat{\gamma}_q+1, \ldots, \hat{\gamma}_r \rangle.$$  

The identity element is the empty tuple $\langle \rangle$. We call $\mathcal{M}_G$ the Grieder monoid of $G$. Now let $\mathcal{M}_G'$ be the submonoid generated by $\langle \hat{\gamma}, \hat{\gamma}^{-1} \rangle$ ($\gamma \in G$) and set $\mathcal{A}_G := \mathcal{M}_G / \mathcal{M}_G'$. The quotient $\mathcal{A}_G$ is an abelian group. The inverse element is given by

$$-\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle = \langle \hat{\gamma}_1^{-1}, \ldots, \hat{\gamma}_q^{-1} \rangle \text{ in } \mathcal{A}_G.$$  

We call $\mathcal{A}_G$ the Grieder group of $G$. As the names suggest, $\mathcal{M}_G$ and $\mathcal{A}_G$ were introduced and studied by Grieder [5, 6] to study surface symmetries. First of all, $\mathcal{A}_G$ is finitely generated:

**Proposition 1 ([5]).** $\mathcal{A}_G \cong \mathbb{Z}^m \oplus \mathbb{Z}_2^n$ for some $m, n \geq 0$.

A homomorphism $f : H \to G$ of groups induces a homomorphism $f_* : \mathcal{A}_H \to \mathcal{A}_G$ of abelian groups by $f_* (\langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle) = \langle f(\hat{\gamma}_1), \ldots, f(\hat{\gamma}_q) \rangle$ so that the assignment $G \mapsto \mathcal{A}_G$ is a covariant functor. In addition, for an inclusion $i : H \hookrightarrow G$, one can also define the restriction $i^* : \mathcal{A}_G \to \mathcal{A}_H$ via surface symmetries. Grieder [5] verified the double coset formula and hence proved the following proposition:
Proposition 2. The assignment $G \mapsto \mathcal{M}_G$ is a Mackey functor.

1.2. Ramification data. By a surface symmetry we mean a pair $(G, C)$, where $C$ is a compact Riemann surface of genus $g \geq 2$, and $G$ is a finite group of automorphisms of $C$. For each $x \in C$, let $G_x$ be the isotropy subgroup at $x$. Note that $G_x$ is necessary cyclic. Set $S = \{x \in C \mid G_x \neq 1\}$, and let $S / G = \{x_1, x_2, \ldots, x_q\}$ be a set of representatives of $G$-orbits of elements of $S$. For each $x_i \in S / G$, choose a generator $\gamma_i$ of $G_{x_i}$ such that $\gamma_i$ acts on the holomorphic tangent space $T_{x_i}C$ at $x_i$. The ramification data of $(G, C)$, abbreviated by $\delta(G, C)$, is the unordered $q$-tuple $\langle \gamma_1, \gamma_2, \cdots, \gamma_q \rangle$. It satisfies $\gamma_1 \gamma_2 \cdots \gamma_q \in [G, G]$, and hence $\delta(G, C)$ is an element of the Grieder monoid $\mathcal{M}_G$. Conversely, we have the following proposition.

Proposition 3 (see [5]). For any element $\mu \in \mathcal{M}_G$, there exists a surface symmetry $(G, C)$ whose ramification data coincides with $\mu$.

2. Group cohomology

2.1. The first Chern class. Let $\langle \gamma \rangle$ be a cyclic group of order $m$ generated by $\gamma$ and $\rho_\gamma : \langle \gamma \rangle \to \mathbb{C}^\times$ a linear character defined by $\gamma \mapsto \exp(2\pi i / m)$. For any finite group $G$, we have natural isomorphisms

$$\text{Hom}(G, \mathbb{C}^\times) \cong H^1(G, \mathbb{C}^\times) \cong H^2(G, \mathbb{Z}).$$

Here, the latter isomorphism is the connecting homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 0$. Define $c(\gamma) \in H^2(\langle \gamma \rangle, \mathbb{Z})$ to be the image of $\rho_\gamma$ under the isomorphism $\text{Hom}(\langle \gamma \rangle, \mathbb{C}^\times) \cong H^2(\langle \gamma \rangle, \mathbb{Z})$. The cohomology class $c(\gamma)$ is sometimes called the first Chern class of $\rho_\gamma$.

2.2. MMM classes (algebra). For each element $\mu = \langle \hat{\gamma}_1, \ldots, \hat{\gamma}_q \rangle$ of $\mathcal{M}_G$, define a series of cohomology classes $e_k(\mu) \in H^{2k}(G, \mathbb{Z})$ ($k \geq 1$) by

$$e_k(\mu) := \sum_{i=1}^q \text{Tr}_{\langle \hat{\gamma}_i \rangle}^G (c(\gamma_i)^k) \in H^{2k}(G, \mathbb{Z}),$$

where $\text{Tr}_{\langle \gamma \rangle}^G : H^*(\langle \gamma \rangle, \mathbb{Z}) \to H^*(G, \mathbb{Z})$ is the transfer. We call $e_k(\mu)$ the $k$-th Mumford-Morita-Miller class of $\mu$ (MMM class in short). The definition of $e_k(\mu)$ is motivated by topology, as will be explained in the next subsection. Observe that the assignment $\mu \mapsto e_k(\mu)$ defines a well-defined homomorphism $\mathcal{M}_G \to H^{2k}(G, \mathbb{Z})$ of abelian monoids. For $k$ odd, it induces a well-defined homomorphism $\mathcal{A}_G \to H^{2k}(G, \mathbb{Z})$ of abelian groups, for we have $c(\gamma^{-1}) = -c(\gamma)$. In addition, we can prove the following proposition:
Proposition 4. For odd \( k \geq 1 \), the homomorphism \( A_G \to H^{2k}(G, \mathbb{Z}) \) is a natural transformation of Mackey functors.

2.3. MMM classes (topology). The definition of \( e_k(\mu) \) is inspired by a result of Kawazumi and Uemura [8] concerning of characteristic classes of oriented surface bundles. Let \( \Sigma_g \) be the closed oriented surface of genus \( g \geq 2 \). Let \( \pi : E \to B \) an oriented \( \Sigma_g \)-bundle, \( T^vE \) the tangent bundle along the fiber of \( \pi \), and \( e \in H^2(E; \mathbb{Z}) \) the Euler class of \( T^vE \). Define \( e_k^{\text{top}}(\pi) \in H^{2k}(B; \mathbb{Z}) \) by \( e_k^{\text{top}}(\pi) := \pi_!(e^{k+1}) \) where \( \pi_!: H^*(E; \mathbb{Z}) \to H^{*-2}(B; \mathbb{Z}) \) is the Gysin homomorphism (the superscript "top" stands for "topology"). \( e_k^{\text{top}}(\pi) \) is called the \( k \)-th Mumford-Morita-Miller class of \( \pi \), as it was introduced in [11, 10, 9].

Now let \( (G, C) \) be a surface symmetry as in Section 1.2. Associated with \( (G, C) \), there is an oriented surface bundle \( \pi : EG \times_G C \to BG \) called the Borel construction, where \( EG \to BG \) is the universal \( G \)-bundle. We denote by \( e_k^{\text{top}}(G, C) \in H^{2k}(G, \mathbb{Z}) \) the \( k \)-th MMM class of the Borel construction \( \pi \). A result of Kawazumi and Uemura [8] implies the following result:

Theorem 5. We have \( e_k^{\text{top}}(G, C) = e_k(\delta(G, C)) \) where \( \delta(G, C) \) is the ramification data of \( (G, C) \).

3. Homology representations

3.1. Algebra. In what follows, we denote by \( R(G) \) the complex representation ring (or the character ring) of a finite group \( G \). Let \( \langle \gamma \rangle \) be a cyclic group of order \( m \) generated by \( \gamma \) and \( \rho_\gamma : \langle \gamma \rangle \to \mathbb{C}^\times \) a linear character as in Section 2.1. Define \( \Delta_\gamma \in R(\langle \gamma \rangle) \otimes \mathbb{Q} \) by

\[
\Delta_\gamma := 2 \sum_{k=1}^{m-1} \left( \frac{k}{m} - \frac{1}{2} \right) \rho_\gamma^\otimes k = \frac{2}{m} \sum_{k=1}^{m-1} k\rho_\gamma^\otimes k - r(\gamma) + 1(\gamma),
\]

where \( r(\gamma) \) is the regular representation and \( 1(\gamma) \) is the trivial 1-dimensional representation of \( \langle \gamma \rangle \). Now, for each element \( \mu = \langle \gamma_1, \ldots, \gamma_q \rangle \) of \( \mathcal{M}_G \), define the \( G \)-signature \( \sigma(\mu) \) of \( \mu \) by

\[
\sigma(\mu) := \sum_{k=1}^{q} \text{Ind}_{\langle \gamma_k \rangle}^G(\Delta_\gamma) \in R(G) \otimes \mathbb{Q}.
\]

Proposition 6. \( \sigma(\mu) \in R(G) \) for every \( G \) and \( \mu \in \mathcal{M}_G \).

See the next section for the proof. Note that, in case \( \mu \in \mathcal{M}_G \) consists of a single conjugacy class (\( \mu = \langle \gamma \rangle \) for \( \gamma \in [G, G] \)), Proposition 6 was proved by T. Yoshida [13].
The assignment \( \mu \mapsto \sigma(\mu) \) yields a homomorphism \( \mathcal{M}_G \to R(G) \) of monoids, which induces a well-defined homomorphism \( \mathcal{A}_G \to R(G) \) of abelian groups. In addition, we can prove the following proposition:

**Proposition 7.** \( \mathcal{A}_G \to R(G) \) is a natural transformation of Mackey functors.

### 3.2. Topology

Let \((G, C)\) be a surface symmetry, and \( H_C \) the space of holomorphic 1-forms on \( C \). Note that \( \dim_C H_C = g \) where \( g \) is the genus of the Riemann surface \( C \). Then \( G \) acts on \( H_C \) and hence \( H_C \) is a complex representation of \( G \). A virtual representation \( \sigma^{\text{top}}(G, C) := H_C - \overline{H}_C \in R(G) \) is called the \( G \)-signature of \((G, C)\), where \( \overline{H}_C \) is the complex conjugate.

**Proposition 8.** We have \( \sigma^{\text{top}}(G, C) = \sigma(\delta(G, C)) \) where \( \delta(G, C) \) is the ramification data of \((G, C)\).

The character of \( \sigma^{\text{top}}(G, C) \) is given by the Eichler trace formula (see [4] for instance). The proposition can be verified by comparing characters of \( \sigma^{\text{top}}(G, C) \) and \( \sigma(\delta(G, C)) \). An alternative proof was given by N. Kawazumi (unpublished manuscript). Since every \( \mu \in \mathcal{M}_G \) can be realized as a ramification data of a surface symmetry, Proposition 6 follows from the last proposition. The following fact is an easy consequence of Proposition 8.

**Corollary 9.** If all the complex characters of \( G \) are \( \mathbb{R} \)-valued, then \( \sigma(\mu) = 0 \) for all \( \mu \in \mathcal{M}_G \).

**Proof.** Choose a surface symmetry \((G, C)\) with \( \delta(G, C) = \mu \). Then we have \( \sigma(\mu) = \sigma^{\text{top}}(G, C) = H_C - \overline{H}_C = 0 \) since \( H_C = \overline{H}_C \) by the assumption.

### 4. A RELATION OF \( e_k(\mu) \) AND \( \sigma(\mu) \)

**Theorem 10.** Let \( G \) be a finite group and \( \mu, \nu \in \mathcal{M}_G \).

1. If \( \sigma(\mu) = \sigma(\nu) \) then \( e_k(\mu) = e_k(\nu) \) for all odd \( k \geq 1 \).
2. If \( \sigma(\mu) = 0 \) then \( e_k(\mu) = 0 \) for all odd \( k \geq 1 \).

Since \( R(G) \) is free as an abelian group, the homomorphism \( \mathcal{A}_G \to R(G) \) in Section 3.1 induces \( \phi_1 : \mathcal{A}_G/\text{Tor}(\mathcal{A}_G) \to R(G) \), where \( \text{Tor}(\mathcal{A}_G) \) is the torsion subgroup of \( \mathcal{A}_G \). For odd \( k \geq 1 \), let \( \phi_2 : \text{Tor}(\mathcal{A}_G) \to H^{2k}(G, \mathbb{Z}) \) be the restriction of the homomorphism \( \mathcal{A}_G \to H^{2k}(G, \mathbb{Z}) \) in Section 2.2. The proof of Theorem 10 is based on the following two facts:

**Theorem 11.** For any finite group \( G \) and any odd \( k \geq 1 \),

1. The homomorphism \( \phi_1 : \mathcal{A}_G/\text{Tor}(\mathcal{A}_G) \to R(G) \) is injective.
2. The homomorphism \( \phi_2 : \text{Tor}(\mathcal{A}_G) \to H^{2k}(G, \mathbb{Z}) \) is trivial.
The first statement is proved by using a result of Edmonds and Ewing [3], while the second statement is proved by considering the cohomology of metacyclic 2-groups. The detail will appear elsewhere. Theorem 10 and Corollary 9 imply the following corollary:

**Corollary 12.** If all the complex characters of \( G \) are \( \mathbb{R} \)-valued, then \( e_k(\mu) = 0 \) for all \( \mu \in \mathcal{M}_G \) and odd \( k \geq 1 \).

Define \( \mathcal{R}_G \) to be the image of \( \phi_1 : \mathcal{A}_G \to \mathbb{R}(\mathcal{A}_G) \to R(G) \). In view of Theorem 11, there exists a series of homomorphisms \( \Phi_k : \mathcal{R}_G \to H^{2k}(G, \mathbb{Z}) \) (\( k \) odd) which assigns \( e_k(\mu) \) to \( \sigma(\mu) \). Let \( c : \text{Hom}(G, \mathbb{C}^\times) \to H^2(G, \mathbb{Z}) \) be the natural isomorphism as in Section 2.1 and \( \det : R(G) \to \text{Hom}(G, \mathbb{C}^\times) \) the determinant homomorphism (see [13] for precise). Then the homomorphism \( \Phi_1 \) is determined by the following proposition:

**Proposition 13.** \( e_1(\mu) = 6 \cdot c(\det(\sigma(\mu))) \) for all \( \mu \in \mathcal{M}_G \).

The proposition follows from the Grothendieck-Riemann-Roch theorem and a result of Harer [7] (see also [1, Proposition 6]). Proposition 13 can be generalized to larger \( k \), provided \( G \) is cyclic. Recall that, for every finite group \( G \), there is a series of homomorphisms \( s_k : R(G) \to H^{2k}(G, \mathbb{Z}) \) (\( k \geq 0 \)) of abelian groups, which satisfies the following properties:

(1) \( s_1(\rho) = c(\det(\rho)) \) for all \( \rho \in R(G) \).

(2) If \( \rho \) is a linear character, then \( s_k(\rho) = c(\rho)^k \).

\( s_k(\rho) \) is called the \( k \)-th Newton class of \( \rho \in R(G) \). See [12] for further details. Let \( B_{2k} \) be the \( 2k \)-th Bernoulli number and \( N_{2k}, D_{2k} \) coprime integers satisfying \( B_{2k}/k = N_k/D_k \). Then a result of the author and Kawazumi [2] implies the following result:

**Theorem 14.** If \( G \) is cyclic, then \( N_{2k} \cdot e_{2k-1}(\mu) = D_{2k} \cdot s_{2k-1}(\sigma(\mu)) \) holds for all \( \mu \in \mathcal{M}_G \) and \( k \geq 1 \).

Now let \( G \) be a cyclic group of order \( m \), and suppose that \( N_{2k} \) is prime to \( m \). Choose an integer \( N_{2k}^* \) satisfying \( N_{2k} \cdot N_{2k}^* \equiv 1 \) (mod \( m \)). Under these assumptions, we have

\[
e_{2k-1}(\mu) = N_{2k}^* D_{2k} \cdot s_{2k-1}(\sigma(\mu))
\]

for all \( \mu \in \mathcal{M}_G \), and hence determining \( \Phi_{2k-1} \) for these cases. In particular, we have \( e_1(\mu) = 6 \cdot s_1(\sigma(\mu)) \), \( e_3(\mu) = -60 \cdot s_3(\sigma(\mu)) \), \( e_5(\mu) = 126 \cdot s_5(\sigma(\mu)) \), \( e_7(\mu) = -120 \cdot s_7(\sigma(\mu)) \) for any cyclic group \( G \) and \( \mu \in \mathcal{M}_G \), since \( N_{2k} = 1 \) for \( 1 \leq k \leq 4 \).
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