Entropy production at weak Gibbs measures and a generalized variational principle

Michiko Yuri
Department of Mathematics, Graduate School of Science
Hokkaido University, Kita 10, Nishi 8, Kita-ku
Sapporo 060-0810, Japan
(e-mail : yuri@math.sci.hokudai.ac.jp)

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Abstract

We shall consider piecewise invertible systems exhibiting intermittency and establish a generalized variational principle adapted to non-stationary process in the following sense ; the supremum is attained by nonsingular (not necessarily invariant) probability measures and if the system exhibits hyperbolicity then it reduces to the usual variational principle for the pressure. Our method relies on Ruelle's program in the study of nonequilibrium statistical mechanics to analyze dissipative phenomena. We show nonpositivity of entropy production at weak Gibbs measures and clarify when it indeed vanishes. We also discuss a generalized variational principle in the context of σ-finite invariant measures.

1 Introduction

In the study of non-equilibrium statistical mechanics, Ruelle introduced the concept of entropy production to explain irreversibility on the basis of microscopic dynamics and to give quantitative prediction for dissipative phenomena. In his program, invertible time evolution that does not preserve any smooth measure was considered ([R1],[R2],[GR]). In this paper, we shall take the first step towards placing these approaches in a more general framework. We shall concern with dissipative phenomena observed in complex systems exhibiting *intermittency*. More specifically, we shall consider non-stationary non-invertible process of which statistical laws are determined by either (bi-)nonsingular probability measures or σ-finite infinite invariant measures. Let (X, d) be a complete separable metric space and let $\mathcal{F}$ be the σ-algebra of subsets of X generated by the collection of open sets. We shall consider piecewise invertible systems

$(T, X, Q = \{X_i\}_{i\in I})$ (see the definition in §3) and (bi-)nonsingular probability measures m on $(X, \mathcal{F})$ with respect to T. Let $\mathcal{L}_m$ be the transfer operator associated with m. Then we define

$$e_T(m) := \limsup_{n\to\infty} \frac{1}{n} \int_X \log \mathcal{L}_m^n 1 \circ T^n(x) dm(x),$$

which is called the *asymptotic averaged entropy production of* T *at* m.

The main purpose of this article is to show the following facts.

(1) If $e_T(m)$ at m is nonzero, then there is a gap between the next two generalized entropies :

$$\overline{h}_m(T, Q) := \limsup_{n\to\infty} \frac{1}{n} H_m(\vee_{k=0}^{n-1} T^{-k} Q)$$

and

$$\hat{h}_m(T, Q) := \limsup_{n\to\infty} \frac{1}{n} \int_X I_m(\vee_{k=0}^{n-1} T^{-k} Q | \vee_{k=n}^{\infty} T^{-k} Q)(x) dm(x).$$

(2) $e_T(m)$ at a weak Gibbs measure m is nonpositive. We shall give a sufficient condition for $e_T(m)$ being zero.

(3) For a given potential ϕ, we shall introduce a generalized pressure $\mathcal{GP}_T(\phi)$ in the context of nonsingular probability measures. Then we shall establish a generalized variational principle for $\mathcal{GP}_T(\phi)$. Moreover, we shall clarify when it can be reduced to the usual one for the pressure $\mathcal{P}_T(\phi)$ in the frame work of invariant probability measures. This allows us to obtain naturally a weaker notion of equilibrium state for ϕ. More specifically, we shall answer to the following questions :

Question (A) When does $\mathcal{P}_T(\phi) \le \mathcal{GP}_T(\phi)$ hold ?

Question (B) When does the equality $\mathcal{P}_T(\phi) = \mathcal{GP}_T(\phi)$ hold ? In this case, does the usual equilibrium state for ϕ attain $\mathcal{GP}_T(\phi)$?

We should remark that $e_T(m) = 0$ if m is T-invariant. Even if m is not T-invariant, $e_T(m)$ vanishes if $\{\log \mathcal{L}_m^n 1(x)\}_{n\ge1}$ is uniformly bounded. This property fails typically in case that the potential $\phi = \log \frac{dm}{d(m\circ T)}$ admits an indifferent periodic point. As we will see in §5, our results can apply to various types of intermittent systems. In particular, we have a complete answer to the questions (A) and (B) if the second moment of the stopping time over a hyperbolic region is finite. We should remark that this phenomena is observed for another type of non-hyperbolic maps (unimodal and multimodal maps) for which the usual variational principle can be established (see [PS1],[PS2], [PZ])). Finally, we can verify that as long as we restrict our attention to uniformly expanding Markov systems and potentials of summable variations there is no difference between our generalized variational principle and the usual variational principle for the pressure.

The paper is organized as follows. In §2, we introduce definitions of generalized entropies for nonsingular transformations. In §3, we collect fundamental results for piecewise invertible sofic systems which play important roles in establishing our main results in §4. In particular, we give sufficient conditions for (bi-)nonsingular probability measures satisfying the weak Gibbs property ([Y4]) and a formula of conditional probabilities that allows one to establish the Dobrushin-Lanford-Ruelle equations ([Y8].[MRTMV]). In §5, we apply our results to intermittent systems and establish a generalized variational principle in the context of σ-finite invariant measures. All proofs of results in §3-5 are postponed to §6.

2 Generalized entropies for nonsingular transformations

Let (X,d) be a complete separable metric space and let $\mathcal{F}$ be the σ-algebra of subsets of X generated by the collection of open sets. $(X,\mathcal{F})$ is called a *standard Borel space*. Let m be a Borel probability measure on the standard Borel space $(X,\mathcal{F})$. We call $(X,\mathcal{F},m)$ a *standard probability space*. We shall consider a (bi-)nonsingular transformation T of the standard probability space $(X,\mathcal{F},m)$, (i.e., $m\circ T^{-1}\sim m$). Suppose that $Q=\{X_i\}_{i\in I}$ is a measurable disjoint countable partition of X. The *information function of* Q is defined by $I_m(Q)(x):=-\sum_{i\in I}\log m(X_i)1_{X_i}(x)$ (where $0\log 0:=0$) and the *entropy of the partition* Q is defined by $H_m(Q):=\int_X I_m(Q)(x)dm(x)$. We denote $X_{i_1\ldots i_n}:=\bigcap_{k=0}^{n-1}T^{-k}X_{i_{k+1}}\in\vee_{k=0}^{n-1}T^{-k}Q$ which is called a cylinder of rank n (with respect to T).

Definition. The *generalized entropy of* T *on* $(X,\mathcal{F},m)$ *with respect to* $Q=\{X_i\}_{i\in I}$ is defined by

$$\overline{h}_m(T,Q):=\limsup_{n\to\infty}\frac{1}{n}H_m(\vee_{k=0}^{n-1}T^{-k}Q)$$

$$\left(=\limsup_{n\to\infty}\int_X\left(-\frac{1}{n}\log m(X_{i_1\ldots i_n}(x))\right)dm(x)\right),$$

where $X_{i_1\ldots i_n}(x)$ denotes the unique cylinder of rank n containing x.

When m is T-invariant, $\overline{h}_m(T,Q)$ just coincides with the entropy $h_m(T,Q)$ of T with respect to Q. We also introduce another description of the entropy of a (bi-)nonsingular transformation in terms of the conditional informations. In order to simplify the notation, if $\mathcal{P}$ is a sub-σ-algebra of $\mathcal{F}$ generated by elements of a partition P then $I_m(Q|P)$ denotes the *conditional information of Q given $\mathcal{P}$* defined by

$$I_m(Q|\mathcal{P})(x) = -\Sigma_{i\in I}\log m(X_i|\mathcal{P})(x)1_{X_i}(x).$$

Definition The *generalized conditional entropy of T on $(X,\mathcal{F},m)$ with respect to $Q=\{X_i\}_{i\in I}$* is defined by

$$\hat{h}_m(T,Q) := \limsup_{n\to\infty}\frac{1}{n}\int_X I_m(\vee_{k=0}^{n-1}T^{-k}Q|\vee_{k=n}^{\infty}T^{-k}Q)(x)dm(x)$$

$$\left(=\limsup_{n\to\infty}\frac{1}{n}\int_X\sum_{k=0}^{n-1}I_m(T^{-k}Q|\vee_{l=k+1}^{\infty}T^{-l}Q)dm(x)\right).$$

In particular, if m is T-invariant and satisfies $H_m(Q)<\infty$ then $\hat{h}_m(T,Q)$ coincides with the next description of $h_m(T,Q)$ in terms of conditional informations,

$$H_m(Q|\vee_{k=1}^{\infty}T^{-k}Q) = \int_X I_m(Q|\vee_{k=1}^{\infty}T^{-k}Q)(x)dm(x).$$

3 Weak Gibbs measures for piecewise invertible systems

We shall consider a measurable countable disjoint partition $Q=\{X_i\}_{i\in I}$ of X which satisfies $cl(\bigcup_{i\in I}\mathrm{int}X_i)=X$ and cl int $X_i\supset X_i$ if int $X_i\neq\emptyset$. Let T be a noninvertible transformation of X satisfying the next conditions.

(C1) $\bigcup_{i\in I}\mathrm{int}X_i\subset T(\bigcup_{i\in I}\mathrm{int}X_i)$ and $T(\bigcup_{\mathrm{int}X_i=\emptyset}X_i)\subset\bigcup_{\mathrm{int}X_i=\emptyset}X_i$.

(C2) (**Local invertibility**) $T|_{X_i}$ is one to one for every $X_i\in Q$, and for $X_i\in Q$ with int $X_i\neq\emptyset$, $T|_{\mathrm{int}X_i}:\mathrm{int}X_i\to T(\mathrm{int}X_i)$ is a homeomorphism. Moreover $(T|_{\mathrm{int}X_i})^{-1}$ is extended to a homeomorphism $v_i:\mathrm{cl}T(intX_i)\to$ cl int X_i.

(C3) (**T-generator condition**) $\vee_{n=0}^{\infty}T^{-n}Q=\epsilon$ (the partition into points).

We call (T,X,Q) a *piecewise invertible system.* We remark that for every $X_{i_1\ldots i_n}:=\bigcap_{k=0}^{n-1}T^{-k}X_{i_{k+1}}$ with int $X_{i_1\ldots i_n}\neq\emptyset$, $T^n|_{X_{i_1\ldots i_n}}:X_{i_1\ldots i_n}\to T^nX_{i_1\ldots i_n}$ is one to one. Moreover, $T^n|_{\mathrm{int}X_{i_1\ldots i_n}}:\mathrm{int}X_{i_1\ldots i_n}\to T^n(\mathrm{int}X_{i_1\ldots i_n})$ is a homeomorphism and $(T^n|_{\mathrm{int}X_{i_1\ldots i_n}})^{-1}$ is extended to a homeomorphism $v_{i_1\ldots i_n}:=v_{i_1}\circ v_{i_2}\circ\ldots\circ v_{i_n}:\mathrm{cl}T^n(\mathrm{int}X_{i_1\ldots i_n})\to$ cl int$X_{i_1\ldots i_n}$. For every $n\geq 1$ we define $\mathcal{A}_n:=\{(i_1\ldots i_n)\in I^n|\ intX_{i_1\ldots i_n}\neq\emptyset\}$ and $\mathcal{U}^{(n)}:=\{T^n(\mathrm{int}X_{i_1\ldots i_n})|\forall(i_1\ldots i_n)\in\mathcal{A}_n\}$. We denote $\mathcal{U}:=\bigcup_{n\geq 1}\mathcal{U}^{(n)}$. Then we have

Lemma 3.1 $\bigcup_{U\in\mathcal{U}^{(1)}}U=\bigcup_{U\in\mathcal{U}}U$.

Suppose that the next condition is satisfied.

(Finite range structure FRS) $\mathcal{U}$ is a finite set.

Then (T,X,Q) provides nice (countable) symbolic dynamics similar to sofic shifts (see [Y1],[Y2]). Therefore, we call (T,X,Q) satisfying the FRS condition a *sofic system.* If $T\mathrm{int}X_i \cap \mathrm{int}X_j \neq \emptyset$ implies $T\mathrm{int}X_i \supset \mathrm{int}X_j$, then $\mathcal{U} = \mathcal{U}^{(1)}$ and the (T,X,Q) is called a *Markov system.* In particular, if $\mathcal{U}$ consists of a single element then (T,X,Q) is called a *Bernoulli system.* We also call $X_i \in Q$ with $T(\mathrm{int}X_i) = \bigcup_{U\in\mathcal{U}} U$ a *Bernoulli cylinder.* $\mathcal{N}_T(X)$ denotes the set of all (bi-)nonsingular probability measures with respect to T and $\mathcal{M}_T(X)$ denotes the set of all T- invariant probability measures. Then $\mathcal{M}_T(X) \subset \mathcal{N}_T(X)$. We recall that a (bi-)nonsingular transformation T of the standard probability space $(X,\mathcal{F},m)$ is locally invertible (i.e., $\exists \mathcal{P} = \{Y_j\}_{j\in J}$ a disjoint partition of X s.t. $m(X\setminus\bigcup_{j\in J} Y_j) = 0$ and T is invertible on each $Y_j \in \mathcal{P}$) iff $T^{-1}\{x\}$ is countable for m-a.e. $x \in X$ (c.f.[Aa]). Since the condition (C2) implies that T is countable to one, if $m \in \mathcal{N}_T(X)$ then $m\circ T \le m$ so that there exists a measurable function $\phi : X \to \mathbf{R}$ satisfying $\phi(x) = \log \frac{dm}{d(m\circ T)}(x)$ (m-a.e. $x \in X$). Then the transfer operator $\mathcal{L}_m : L^1(m) \to L^1(m)$ associated with m is defined by ; $\forall f \in L^1(m)$

$$\mathcal{L}_m f(x) = \sum_{y\in T^{-1}\{x\}} \exp[\phi(y)]f(y) \ (m\text{-a.e.}x \in X).$$

We note that $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ satisfies $m(\bigcup_{i\in I} \mathrm{int}X_i) = 1$. In this case, m-a.e. $x \in X, \mathcal{L}_m f(x) = \mathcal{L}_\phi f(x)$, where $\mathcal{L}_\phi$ is the so-called *Perron-Frobenius operator associated with* ϕ defined by

$$\mathcal{L}_\phi f(x) := \sum_{i\in I} \exp[\phi(v_i(x)]f(v_i(x))1_{\mathrm{cl}(T(\mathrm{int}X_i))}(x). \text{ (C.f.[}Y7\text{].)}$$

Since $m(\bigcup_{U\in\mathcal{U}} U) = 1$ gives $m(\bigcup_{(i_1\ldots i_n)\in\mathcal{A}_n} \mathrm{int}X_{i_1\ldots i_n}) = 1(\forall n \ge 1)$, we see that for every $n \ge 1$ and m-a.e. $x \in X$

$$\mathcal{L}_m^n f(x) = \mathcal{L}_\phi{}^n f(x)$$

$$= \sum_{(i_1\ldots i_n)\in\mathcal{A}_n} \exp[\sum_{k=0}^{n-1} \phi\circ T^k(v_{i_1\ldots i_n}(x))]f(v_{i_1\ldots i_n}(x))1_{\mathrm{cl}(T^n(\mathrm{int}X_{i_1\ldots i_n}))}(x).$$

In particular, if (T,X,Q) is a Bernoulli system and $\{\exp[\phi\circ v_i] : X \to \mathbf{R}\}_{i\in I}$ is an equi-continuous family satisfying $\sup_{x\in X}\sum_{i\in I}\exp[\phi\circ v_i(x)] < \infty$, then $\mathcal{L}_\phi$ preserves $C(X)$.

We introduce a weaker notion of Bowen's Gibbs measure (c.f.[Y4]).

Definition A Borel probability measure m on $(X,\mathcal{F})$ is called a *weak Gibbs (WG) measure for* ϕ *with a constant* P if there exists a sequence of positive numbers $\{K_n\}_{n\geq 1}$ with $\lim_{n\to\infty}\frac{1}{n}\log K_n = 0$ such that $\forall n\geq 1, \forall X_{i_1\ldots i_n}\in \vee_{k=0}^{n-1}T^{-k}Q$ with $m(X_{i_1\ldots i_n})>0$ and for m-a.e. $x\in X_{i_1\ldots i_n}$

$$K_n^{-1}\leq \frac{m(X_{i_1\ldots i_n})}{\exp[\sum_{k=0}^{n-1}\phi T^k(x)+nP]}\leq K_n.$$

Proposition 3.1 *Let m be a T-invariant weak Gibbs measure for ϕ with 0. If either $H_m(Q)<\infty$ or $\int_X \phi dm > -\infty$ is satisfied, then $h_m(T,Q)+\int_X \phi dm = h_m(T)+\int_X \phi dm = 0$.*

We should remark that if $H_m(Q)<\infty$ then $h_m(T,Q)=h_m(T)$ because of (C3) ! Next we introduce another notion of Gibbs measures which is more closely related to the Gibbsian states in statistical mechanics.

Definition A Borel probability measure m on $(X,\mathcal{F})$ is called a (WG-1) measure for ϕ if $\forall n\geq 1, \forall X_{i_1\ldots i_n}\ \vee_{k=0}^{n-1}T^{-k}Q$ with $m(X_{i_1\ldots i_n})>0$, and for m-a.e. $x\in T^{-n}T^nX_{i_1\ldots i_n}$,

$$m(X_{i_1\ldots i_n}|T^{-n}(\vee_{k=0}^{\infty}T^{-k}Q))(x)=\left(\frac{{\mathcal{L}_\phi}^n 1_{X_{i_1\ldots i_n}}}{{\mathcal{L}_\phi}^n 1}\right)\circ T^n(x).$$

Indeed, the (WG-1) property allows one to establish the Dobrushin-Lanford-Ruelle (DLR) equations (see [Y8] and [MRTMV] for more details).

Definition We say that $\phi : X\to \mathbf{R}$ is a potential of *weak bounded variation* (WBV) if there exists a sequence of positive numbers $\{C_n\}_{n\geq 1}$ satisfying $\lim_{n\to\infty}\frac{1}{n}\log C_n = 0$ and $\forall n\geq 1$,

$$\sup_{X_{i_1\ldots i_n}\in\vee_{k=0}^{n-1}T^{-k}Q}\frac{\sup_{x\in X_{i_1\ldots i_n}}\exp[\sum_{k=0}^{n-1}\phi(T^kx)]}{\inf_{x\in X_{i_1\ldots i_n}}\exp[\sum_{k=0}^{n-1}\phi(T^kx)]}\leq C_n.$$

Remark (A) If $Var_n(T,\phi):=\sup_{Y\in\vee_{k=0}^{n-1}T^{-k}Q}\sup_{x,y\in Y}|\phi(x)-\phi(y)|\to 0$ as $n\to\infty$, then ϕ satisfies the WBV property. If $\{\phi\circ v_i : \mathrm{cl}(T\mathrm{int}X_i)\to\mathbf{R}\}_{i\in I}$ is a family of partially defined equi-continuous functions and $\sigma(n):=\sup_{Y\in\vee_{k=0}^{n-1}T^{-k}Q}\mathrm{diam}Y\to 0$ as $n\to\infty$, then $Var_n(T,\phi)\to 0 (n\to\infty)$.

Theorem 3.1 *Let $(X,\mathcal{F})$ be a standard Borel space and let $(T,X,Q=\{X_i\}_{i\in I})$ be a piecewise invertible system. Suppose that $m\in\mathcal{N}_T(X)$ admits a function $\phi : X\to\mathbf{R}$ satisfying $\phi=\log\frac{dm}{d(m\circ T)}$. Then we have the followings :*

(i) *m is a (WG-1)-measure for ϕ.*

(ii) *If ϕ satisfies the WBV property and $\inf_{U\in\mathcal{U}}m(U)>0$, then m is a weak Gibbs measure for ϕ with 0.*

We refer the reader to [Y3, Y7] which contains sufficient conditions for ϕ possessing $m\in\mathcal{N}_T(X)$ with $\phi=\log\frac{dm}{d(m\circ T)}$.

4 The asymptotic averaged entropy production and a generalized variational principle

Now we come to discuss properties of the asymptotic averaged entropy production of T associated with (bi)nonsingular probability measures.

Lemma 4.1 *Let $(X, \mathcal{F})$ be a standard Borel space and let (T, X, Q) be a piecewise invertible system. If $m \in \mathcal{N}_T(X)$, then $\forall n \geq 1$ and for m-a.e. $x \in X$*

$$I_m(\vee_{k=0}^{n-1} T^{-k}Q|T^{-n}\epsilon)(x) + \sum_{k=0}^{n-1} \log \frac{dm}{d(m \circ T)} \circ T^k(x) = \log \mathcal{L}_m^n 1 \circ T^n(x).$$

As we have already remarked in §1, our main interest is in case that the WBV sequence $\{C_n\}_{n\geq 1}$ for $\phi := \log \frac{dm}{d(m\circ T)}$ diverges as $n \to \infty$.

Definition A periodic point x_0 with $T^q x_0 = x_0$ is called *indifferent with respect to m* if $\sum_{k=0}^{q-1} \phi \circ T^k(x_0) = 0$ for $\phi : X \to \mathbf{R}$ with $\exp[\phi] = \frac{dm}{d(m\circ T)}$.

Neither summability of variations of ϕ nor the uniformly bounded distortion property for $\exp \phi$ holds under the existence of an indifferent periodic point with respect to m (see Lemma 6.1 in [Y5]). The next result insists that $e_T(m)$ at a weak Gibbs measure m is nonpositive and it becomes indeed zero under certain conditions.

Theorem 4.1 *Let $(X, \mathcal{F})$ be a standard Borel space and let (T, X, Q) be a piecewise invertible sofic system. Suppose that $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ admits a potential ϕ of WBV satisfying $\exp[\phi] = \frac{dm}{d(m\circ T)}$. Then m is a weak Gibbs measure for ϕ (with 0) and $e_T(m) \leq 0$. Assume further that either of the following conditions is satisfied.*

(i) *(T, X, Q) is a Bernoulli system.*

(ii) *m possesses an indifferent periodic point x_0 with period q which is contained in a Bernoulli cylinder of rank q.*

Then we have

$$e_T(m) = \lim_{n\to\infty} \frac{1}{n} \int_X \log \mathcal{L}_m^n 1 \circ T^n(x) dm(x) = \hat{h}_m(T, Q) - \overline{h}_m(T, Q) = 0.$$

Lemma 4.2 *Under the assumptions in the theorem 4.1,*

$$\lim_{n\to\infty} \left\| \frac{1}{n} \log \mathcal{L}_m^n 1 \right\|_{L^\infty(m)} = 0$$

and the decay rate is subexponential.

Definition. For a given potential $\phi : X \to \mathbf{R}$ we define

$$\mathcal{N}_T(X,\phi) := \{\mu \in \mathcal{N}_T(X) | I_\mu(T^{-k}Q|T^{-(k+1)}\epsilon) + \phi \circ T^k,$$

$$I_\mu(T^{-k}Q|T^{-(k+1)}\epsilon) + \log \frac{d\mu}{d(\mu T)} \circ T^k \in L^1(\mu) \ (\forall k \geq 0)\}.$$

For every $\mu \in \mathcal{N}_T(X,\phi)$, we denote

$$\mathcal{F}(\phi,\mu) := \hat{h}_\mu(T,Q) + \liminf_{n\to\infty} \frac{\int_X \sum_{k=0}^{n-1} \phi T^k(x)}{n} d\mu(x) - e_T(\mu) - \limsup_{n\to\infty} \frac{1}{n} \min\{A_n, B_n\},$$

where

$$A_n := \int_X \sum_{k=0}^{n-1} \left(\mathcal{L}_\mu^k 1(x) \frac{\exp[\phi]}{d\mu/d(\mu T)}(x) - 1 \right) d\mu(x)$$

and

$$B_n := \int_X \left(\mathcal{L}_\phi{}^n 1(x) - 1\right) d\mu(x).$$

Theorem 4.2 *Let $(X,\mathcal{F})$ be a standard Borel space and let $(T, X, Q = \{X_i\}_{i\in I})$ be a piecewise invertible sofic system. If $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U\in\mathcal{U}} U) = 1$ admits a function ϕ satisfying $\exp[\phi] = \frac{dm}{d(m\circ T)}$ and $\sup_{x\in X} \mathcal{L}_\phi 1(x) < \infty$, then the next two properties hold.*

(i) *If ϕ satisfies the WBV property and m possesses an indifferent periodic point with period q in a Bernoulli cylinder of rank q, then*

$$e_T(m) = \hat{h}_m(T,Q) + \liminf_{n\to\infty} \int_X \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x) dm(x) = 0.$$

(ii) *For every $\mu \in \mathcal{N}_T(X,\phi)$*

$$\limsup_{n\to\infty} \frac{1}{n} \int_X \left(I_\mu(\vee_{k=0}^{n-1} T^{-k}Q|T^{-n}\epsilon)(x) + \sum_{k=0}^{n-1} \phi \circ T^k(x) \right) d\mu(x)$$

$$\leq e_T(\mu) + \limsup_{n\to\infty} \frac{1}{n} \min\{A_n, B_n\} < \infty.$$

Definition We define a *generalized pressure $\mathcal{GP}_T(\phi)$ of ϕ* by

$$\mathcal{GP}_T(\phi) := \sup\{\mathcal{F}(\phi,\mu) \mid \mu \in \mathcal{N}_T(X,\phi)\}.$$

We say that $\mu \in \mathcal{N}_T(X,\phi)$ is a *weak equilibrium state for ϕ* if $\mathcal{F}(\phi,\mu) = \mathcal{GP}_T(\phi)$. $\mathcal{WE}_T(\phi)$ denotes the set of all weak equilibrium states for ϕ.

Theorem 4.3 (A generalized variational principle) *Let ϕ be a potential of WBV with $\sup_{x \in X} \mathcal{L}_\phi 1(x) < \infty$. Assume that there exists $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U \in \mathcal{U}} U) = 1$ satisfying $\exp[\phi] = \frac{dm}{d(m \circ T)}$ and possesses an indifferent periodic point x_0 with $T^q x_0 = x_0$ in a Bernoulli cylinder of rank q. Then $\mathcal{GP}_T(\phi) = 0$ and we have the followings.*

(i) *Every $\mu = hm$ with $\frac{h}{hT} \equiv 1$ (m-a.e.) satisfies $\mu \in \mathcal{WE}_T(\phi)$.*

(ii) *If the indifferent periodic point x_0 satisfies $\frac{1}{q}\sum_{k=0}^{q-1} \exp[\phi(T^k x_0)] = 1$, then $\frac{1}{q}\sum_{k=0}^{q-1} \delta_{T^k x_0} \in \mathcal{WE}_T(\phi)$.*

Remark (B) If m is not T-invariant, then $\mu = hm$ with $\frac{h}{hT} \equiv 1$ is also not T-invariant. If m is a WG measure for ϕ, then the supremum is attained by every WG measure for the common potential ϕ.

Since m itself and $\frac{1}{q}\sum_{k=0}^{q-1} \delta_{T^k x_0}$ are weak equilibrium states for ϕ, we have the next result.

Corollary 4.1 *$\mathcal{WE}_T(\phi)$ consists of more than two elements.*

We shall observe further details for (ii) in Theorem 4.2 in case when $\mu \leq m$ with $d\mu/dm = h$. First, we note that $A_n := \int_X \sum_{k=0}^{n-1} \left(\frac{hT}{h}(T^k x) - 1\right) d\mu(x)$ and $B_n := \int_X \left(\frac{hT^n}{h}(x) - 1\right) d\mu(x)$. Then the followings are obtained:

(a) In case when $\mu \in \mathcal{M}_T(X)$, we see that

$$A_n = n(\int_X hT(x)dm(x) - 1),\ B_n = \int_X hT^n(x)dm(x) - 1$$

and

$$\mathcal{F}(\phi, \mu) = \int_X \log \frac{hT}{h}(x)d\mu(x) - \min\{\int_X hT(x)dm(x) - 1,\ \limsup_{n\to\infty} \int_X \frac{hT^n(x)}{n}dm(x)\}.$$

(Here, we use $h_\mu(T, Q) + \int_X \phi(x)d\mu(x) + \int_X \log \frac{h}{hT}(x)d\mu(x) = 0$ which follows from Lemma 4.1).

(b) In case when $m \in \mathcal{M}_T(X)$, $\mathcal{L}_m^n 1(x) = \mathcal{L}_\phi{}^n 1(x) = 1$ (μ-a.e. $x \in X$) so that

$$\mathcal{F}(\phi, \mu) = \hat{h}_\mu(T, Q) + \liminf_{n\to\infty} \frac{\int_X \sum_{k=0}^{n-1} \phi T^k(x)}{n} d\mu(x) - e_T(\mu) - \limsup_{n\to\infty} \frac{1}{n} \min\{A_n, 0\}$$

.

Therefore,

(c) if both μ and m are T-invariant then $\mathcal{F}(\phi, \mu) = \int_X \log \frac{hT}{h} d\mu$.

Definition For given $\phi : X \to \mathbf{R}$, we define

$$\mathcal{M}_T(X,\phi) := \{\mu \in \mathcal{M}_T(X) |\ I_\mu(Q|T^{-1}\epsilon) + \phi \in L^1(\mu),\ \text{either } h_\mu(T) < \infty$$

$$\text{or } \int_X \phi d\mu > -\infty \text{ with } h_\mu(T) = \int_X I_\mu(Q|T^{-1}\epsilon) d\mu \text{ is satisfied}\}.$$

The *pressure of* ϕ is defined by

$$\mathcal{P}_T(\phi) := \sup\{h_\mu(T) + \int_X \phi d\mu \mid \mu \in \mathcal{M}_T(X,\phi)\}.$$

If $\mu \in \mathcal{M}_T(X,\phi)$ satisfies $h_\mu(T) + \int_X \phi d\mu = \mathcal{P}_T(\phi)$, then μ is called an *equilibrium state for* ϕ. $\mathcal{E}_T(\phi)$ denotes the set of all equilibrium states for ϕ.

The next result gives an answer to the question (A) in §1.

Lemma 4.3 *If* $\mathcal{P}_T(\phi)$ *coincides with*

$$\hat{\mathcal{P}}_T(\phi) := \sup\{h_\mu(T) + \int_X \phi d\mu \mid \mu = hm \in \mathcal{M}_T(X,\phi),$$

$$\int_X \log \frac{hT}{h} d\mu = 0 \text{ and } \limsup_{n\to\infty} \int_X \frac{hT^n}{n} dm = 0\},$$

then $\mathcal{P}_T(\phi) \le \mathcal{G}\mathcal{P}_T(\phi)$.

We have the next answer to the question (B) in §1.

Theorem 4.4 *Let* (T, X, Q) *be a piecewise invertible sofic system and let* ϕ *be a potential of WBV with* $sup_{x\in X}\mathcal{L}_\phi 1(x) < \infty$. *Assume that there exists* $m \in \mathcal{N}_T(X)$ *with* $m(\bigcup_{U\in\mathcal{U}} U) = 1$ *satisfying* $\exp[\phi] = \frac{dm}{d(m\circ T)}$ *and possesses an indifferent periodic point with period* q *in a Bernoulli cylinder of rank* q. *If there exists an absolutely continuous weak Gibbs measure* $\mu \in M_T(X,\phi)$ *for* ϕ *(with* 0*) with respect to* m *and satisfies* $\lim_{n\to\infty} \frac{1}{n}\int_X \mathcal{L}_m^n 1 d\mu = 0$, *then we have*

(i) $\mathcal{F}(\phi,\mu) = 0 = h_\mu(T) + \int_X \phi d\mu$ *and* $\mu \in \mathcal{W}\mathcal{E}_T(\phi)$.

(ii) $\mu \in \mathcal{E}_T(\phi)$ *iff* $\mathcal{P}_T(\phi) = \hat{\mathcal{P}}_T(\phi)$ *and in this case* $P_T(\phi) = \mathcal{G}\mathcal{P}_T(\phi)$.

(iii) *If* $\sup_{x\in X} \mathcal{L}_\phi^n 1(x) = o(n)$, *then* $\mathcal{E}_T(\phi) \subset \mathcal{W}\mathcal{E}_T(\phi)$.

We will see in §5 that all examples admitting indifferent periodic points $x_0$satisfy that $\mu, \frac{1}{q}\sum_{k=0}^{n-1} \delta_{T^k x_0} \in \mathcal{E}_T(\phi)$ and $\frac{1}{q}\sum_{k=0}^{n-1} \delta_{T^k x_0}, m \in \mathcal{W}\mathcal{E}_T(\phi)$. In order to clarify whether $\mu \in \mathcal{W}\mathcal{E}_T(\phi)$ or not, we need to verify finiteness of the second moment of the stopping time over hyperbolic regions (see Proposition 5.2).

Corollary 4.2 (The uniqueness of weak equilibrium states) *If the weak Gibbs measure $\mu = hm$ for ϕ satisfies $\mu \sim m$ and*

$$||\mathcal{L}_m^n 1 - h||_{L^\infty(m)} \longrightarrow 0 \ (n \to \infty),$$

then

$$\sharp \mathcal{E}_T(\phi) = \sharp \mathcal{WE}_T(\phi) = 1.$$

Indeed, the assumption in Corollary 4.2 implies that μ and m are exact and hence ergodic ([Aa]). Therefore, the condition $\frac{h}{hT} \equiv 1$ forces h to be m-a.e. constant.

Remark (C) If $m \in \mathcal{N}_T(X)$ admits an indifferent periodic point x_0, then the uniform convergence of $\mathcal{L}_m^n 1$ typically fails. Even if the $L^1(m)$ convergence is valid, the invariant density h may be unbounded.

5 Applications to Intermittent Systems

In this section, we shall apply our results in previous sections to intermittent phenomena caused by indifferent periodic points. Let (T, X, Q) be a piecewise invertible sofic system and let ϕ be a potential of WBV with $\sup_{x \in X} \mathcal{L}_\phi 1(x) < \infty$. Assume that there exists $m \in \mathcal{N}_T(X)$ with $m(\bigcup_{U \in \mathcal{U}} U) = 1$ satisfying $\exp[\phi] = \frac{dm}{d(m \circ T)}$ and possesses an indifferent periodic point with period q in a Bernoulli cylinder of rank q. $\mathcal{A}$ denotes the set of all admissible cylinders $\bigcup_{n=1}^\infty \{X_{i_1 \dots i_n} | (i_1 \dots i_n) \in \mathcal{A}_n\}$. For $\mathcal{R} \subset \mathcal{A}$, we define the stopping time over $\mathcal{R}$, $R : X \to \mathbf{N} \cup \{\infty\}$ by $R(x) = \inf\{n \in \mathbf{N} : X_{i_1 \dots i_n}(x) \in \mathcal{R}\}$. Then for every $n \geq 1$, we define $D_n = \{x \in X | R(x) > n\}$ and $B_n = \{x \in X | R(x) = n\}$. Put $D_0 = X$. Define $I^* = \bigcup_{n=1}^\infty \{(i_1 \dots i_n) : X_{i_1 \dots i_n} \subset B_n\}$. Then $Q^* = \{X_\alpha\}_{\alpha \in I^*}$ is a countable partition of $\bigcup_{i=1}^\infty B_i$. Now we define Schweiger's jump transformation $T^* : \bigcup_{i=1}^\infty B_i \to X$ by $T^* x = T^{R(x)} x$. Put $X^* = X \backslash (\bigcup_{m=0}^\infty T^{*-m} (\bigcap_{n \geq 0} D_n)$. Then (T^*, X^*, Q^*) is a piecewise invertible sofic system. We impose the next two conditions on $\mathcal{R}$.

(1) $X_{i_1 \dots i_n j_1 \dots j_m} \in \mathcal{R}$ whenever $X_{j_1 \dots j_m} \in \mathcal{R}$(*the strong playback property*).

(2) $\exists 0 < \Gamma < \infty, \gamma^* > 1$ such that for every $(\alpha_1 \dots \alpha_n) \in I^{*n}$ with $X_{\alpha_1 \dots \alpha_n} \in \mathcal{A}$ and for all $n \geq 1$,

$$d(v_{\alpha_1 \dots \alpha_n} x, v_{\alpha_1 \dots \alpha_n} y) \leq \Gamma \gamma^{*-n} d(x, y) \ (\forall x, y \in T^{*n} X_{\alpha_1 \dots \alpha_n}).$$

If the induced potential $\phi^* := \sum_{k=0}^{R(.)-1} \phi T^k$ satisfies equi-Hölder continuity of $\{\phi^* v_\alpha\}_{\alpha \in I^*}$ then the measure theoretical bounded distortion is valid for $\exp \phi^* \ (= \frac{dm}{d(mT^*)})$. Then we have a T^*-invariant measure μ^* of which density is bounded and piecewise Hölder continuous with respect to a finite partition generated by $\mathcal{U}$. Furthermore, if each $U \in \mathcal{U}$ contains a full cylinder in

$\mathcal{R}$ then the density is bounded away from zero and μ^* is a Gibbs measure for ϕ^* which is exact and exponential mixing . Moreover, if $\lim_{n\to\infty} m(D_n) = 0$, then $m(X) = m(X^*)$ and the following formula gives a T-invariant σ-finite conservative measure $\mu \sim m$, which is exact ([Y1], c.f.[Sch]).

$$\mu(E) = \sum_{n=0}^{\infty} \mu^*(D_n \cap T^{-n}E)(\forall E \in \mathcal{F}).$$

Such μ is unique up to constant. In particular, if $\sum_{n=0}^{\infty} m(D_n) < \infty$ then μ is finite. We should recall that the distortion of $\frac{dm}{d(mT^n)}$ over cylinders of rank n touching indifferent periodic points diverges as $n \to \infty$ (c.f. Lemma 6.1 in [Y5]). Therefore, all indifferent periodic points with respect to m are contained in $\bigcap_{n\geq 0} D_n$. In particular, $\bigcap_{n\geq 0} D_n$ consists of only indifferent periodic points in our examples below. We have the following results.

Proposition 5.1 *(Theorem 3.2 in [Y4]). Suppose that*

(i) $\exists 0 < r_1 < r_2 < \infty$ *and* $\exists \alpha > 1$ *such that*

$$r_1 n^{-\alpha} \leq m(D_n) \leq r_2 n^{-\alpha},$$

(ii) $\exists 0 < G_1 < \infty$ *such that* $\forall X_{d_1\ldots d_n} \in \mathcal{D}_n, m(D_n) \leq G_1 m(X_{d_1\ldots d_n})$.

Then μ *is a weak Gibbs measure for* ϕ *with* 0.

Definition A cylinder $X_{i_1\ldots i_n}$ is called a *Markov cylinder* if for every $U \in \mathcal{U}$ with $U \cap \mathrm{int} X_{i_1\ldots i_n} \neq \emptyset$ it holds that $\mathrm{int}\, X_{i_1\ldots i_n} \subset U$.

If (T, X, Q) is a piecewise invertible sofic system, then it follows from Theorem 3.1 in [Y2] that there exists a Markov cylinder. We have the next result

Proposition 5.2 *If* B_1 *consists of Markov cylinders and* $\int_X R^2 dm < \infty$*, then* μ *satisfies*

$$\lim_{n\to\infty} \frac{1}{n} \int_X \mathcal{L}_m^n 1 d\mu = 0.$$

Now we can apply all our results in §3 - §4 and Propositions 5.1-2 to the following intermittent systems preserving absolutely continuous probability measures.

Example 1. (**A one-parameter family of maps on the interval** $[0,1]$**.**)

Let $X = [0,1]$ and let m be the normalized Lebesgue measure of $[0,1]$. For $\beta > 0$ define

$$T_\beta(x) = \begin{cases} \frac{x}{(1-x^\beta)^{1/\beta}} & \text{on } X_0 = [0, (1/2)^{1/\beta}) \\ \frac{x}{(1/2)^{1/\beta}-1} + \frac{1}{1-(1/2)^{1/\beta}} & \text{on } X_1 = [(1/2)^{1/\beta}, 1] \end{cases}$$

The map T_β has an indifferent fixed point 0. It is similar in its properties to the more familiar Manneville-Pomeau maps : $x \to x + x^{1+\beta}$ (mod 1), in that it also has intermittent behavior. $(T, X, Q = \{X_0, X_1\})$ is a piecewise invertible Bernoulli system and $\phi = \log |T'|$ satisfies the WBV property. Therefore, m is a weak Gibbs measure for ϕ with 0. If $\beta < 1$ then T_β admits an invariant weak Gibbs equilibrium measure $\mu \sim m$. We see that $e_T(m) = e_T(\mu) = 0$ and $\mu, \delta_0 \in \mathcal{E}_T(\phi)$. We also know that $m, \delta_0 \in \mathcal{WE}_T(\phi)$. In particular, if $\beta < \frac{1}{2}$ then $\mu \in \mathcal{WE}_T(\phi)$, too.

Example 2. (**Inhomogeneous Diophantine approximations [Y2, Y4-Y8,]**) We define $X = \{(x, y) \in \mathbf{R}^2 : 0 \le y \le 1, -y \le x < -y + 1\}$ and $T : X \to X$ by

$$T(x, y) = \left(\frac{1}{x} - \left[\frac{1-y}{x}\right] + \left[-\frac{y}{x}\right], -\left[-\frac{y}{x}\right] - \frac{y}{x}\right),$$

where $[x] = \max\{n \in \mathbf{Z} | n \le x\} (x \in \mathbf{N})$ and $[x] = \max\{n \in \mathbf{Z} | n < x\} (x \in \mathbf{Z} \backslash \mathbf{N})$. This map admits indifferent periodic points $(1, 0)$ and $(-1, 1)$ with period 2, i.e., $|\det DT^2(1, 0)| = |\det DT^2(-1, 1)| = 1$. Let $a(x, y) = [\frac{(1-y)}{x}] - [\frac{-y}{x}]$ and $b(x, y) = -[-\frac{y}{x}]$. We can introduce an index set

$$I := \{(a, b) \in \mathbf{Z^2} : a > b > 0, \text{or } a < b < 0\}$$

and a partition $Q := \{X_{(a,b) \in I}\}$, where $X_{(a,b)} = \{(x, y) \in X : a(x, y) = a, b(x, y) = b\}$. Let m be the normalized Lebesgue measure of X. (T, X, Q) is a piecewise invertible Bernoulli system and $\phi = \log \frac{dm}{d(mT)}$ satisfies the WBV property. Therefore, m is a weak Gibbs measure for ϕ with 0. There exists a T-invariant weak Gibbs equilibrium measure $\mu \sim m$. We see that $e_T(m) = e_T(\mu) = 0$ and $\mu, 2^{-1}(\delta_{(1,0)} + \delta_{(-1,1)}) \in \mathcal{E}_T(\phi)$. We also know that $m, 2^{-1}(\delta_{(1,0)} + \delta_{(-1,1)}) \in \mathcal{WE}_T(\phi)$.

Example 3. (**A complex continued fraction [Y6, Y7]**) We can define a complex continued fraction transformation $T : X \to X$ on the diamond shaped region $X = \{z = x_1\alpha + x_2\overline{\alpha} : -1/2 \le x_1, x_2 \le 1/2\}$, where $\alpha = 1+i$, by $T(z) = 1/z - [1/z]_1$. Here $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$, where z is written in the form $z = x_1\alpha + x_2\overline{\alpha}$, $[x] = \max\{n \in \mathbf{Z} | n \le x\} (x \in \mathbf{N})$ and $[x] = \max\{n \in \mathbf{Z} | n < x\} (x \in \mathbf{Z} - \mathbf{N})$. This transformation has an indifferent periodic orbit $\{1, -1\}$ of period 2 and two indifferent fixed points at i and $-i$. For each $n\alpha + m\overline{\alpha} \in I := \{m\alpha + n\overline{\alpha} : (m, n) \in \mathbf{Z}^2 - (0, 0)\}$, we define $X_{n\alpha + m\overline{\alpha}} := \{z \in X : [1/z]_1 = n\alpha + m\overline{\alpha}\}$. Then we have a countable Markov partition $Q = \{X_a\}_{a \in I}$ of X and (T, X, Q) is a transitive sofic system. For $\phi(z) = -\log |T'(z)| (= -2 \log |z|)$, we can verify the WBV property of ϕ so that m is a weak Gibbs measure for 2ϕ with 0. There exists a T-invariant ergodic probability measure $\mu \sim m$ for 2ϕ, and we see that $e_T(m) = e_T(\mu) = 0$. Moreover, $\mu, \delta_i, \delta_{-i}, 2^{-1}(\delta_1 + \delta_{(-1)}) \in \mathcal{E}_T(\phi)$ and $m, \delta_i, \delta_{-i}, 2^{-1}(\delta_1 + \delta_{(-1)}) \in \mathcal{WE}_T(\phi)$.

In case when $\sum_{n=0}^{\infty} m(D_n) = \infty$, Theorem 4.4 does not apply so that we need further observations. Let $\mathcal{M}_T^\infty(X)$ be the set of all σ-finite invariant measures on $(X, \mathcal{F})$ and define $\mathcal{E}_T^\infty(X) := \{v \in \mathcal{M}_T^\infty(X) | \nu \text{ is conservative and ergodic }\}$. For every $\nu \in \mathcal{E}_T^\infty(X)$, we can define the induced transformation T_A over $A \in \mathcal{F}$ with $0 < \nu(A) < \infty$ by $T_A(x) = T^{R_A(x)}(x)$, where $R_A(x) := \inf\{n \in \mathbf{N} | T^n(x) \in A\}$. Then $\nu_A := \frac{\nu|_A}{\nu(A)}$ is T_A-invariant and ergodic. ν can be represented by ν_A via the next Kac' formula:

$$\int_X 1_E(x) d(x) = \int_A \sum_{k=0}^{R_A(x)-1} 1_E(T^k x) d\nu_A(x) \ (\forall E \in \mathcal{F}).$$

Moreover, if $A_i \in \mathcal{F}$ satisfy $0 < \nu(A_i) < \infty$ $(i = 1, 2)$ then $T_{A_1} = (T_{A_1 \cup A_2})_{A_1}$ so that

$$\frac{h_{\nu_1 \cup \nu_2}(T_{A_1 \cup A_2})}{\nu_{A_1 \cup A_2}(A_1)} = h_{(\nu_1 \cup \nu_2)_{A_1}}((T_{A_1 \cup A_2})_{A_1}).$$

Hence we have

$$\nu(A_1) h_{\nu_1}(T_{A_1}) = \nu(A_1 \cup A_2) h_{\nu_1 \cup \nu_2}(T_{A_1 \cup A_2}) = \nu(A_2) h_{\nu_2}(T_{A_2}).$$

This number is used as the entropy $h_\nu(T)$ of T with respect to ν (c.f.[T]). For a function $f : X \to \mathbf{R}$, define $f_A(x) := \sum_{k=0}^{R_A(x)-1} f T^k(x)$. If $A_i \in \mathcal{F}$ with $0 < \nu(A_i) < \infty$ $(i = 1, 2)$, then the above Kac' formula gives

$$\int_{A_1} f_{A_1}(x) d\nu_{A_1}(x) = \int_{A_1 \cup A_2} f_{A_1 \cup A_2}(x) d\nu_{A_1 \cup A_2}(x) = \int_{A_2} f_{A_2}(x) d\nu_{A_2}(x)$$

whenever the integrals are well-defined. Let $A \in Q$ be a Markov cylinder and let Q_A be a countable disjoint partition of $\{x \in A | R_A(x) < \infty\}$ consisting of all cylinders $X_{i_1 \dots i_n} \subset \{x \in A | R_A(x) = n - 1\}$ with $n \geq 2$. We define

$$\mathcal{E}_T^\infty(X, \phi, A) := \{\nu \in \mathcal{E}_T^\infty(X) \mid 0 < \nu(A) < \infty,\ I_{\nu_A}(Q_A | T_A^{-1}\epsilon) + \phi_A \in L^1(\nu_A),$$

$$\text{either } h_{\nu_A}(T_A) < \infty \text{ or } \int_A \phi_A d\nu_A > -\infty \text{ with } h_{\nu_A}(T_A) = \int_X I_{\nu_A}(Q_A | T_A^{-1}\epsilon) d\nu_A\}$$

and

$$P^\infty(T, \phi, A) := \sup_{\nu \in \mathcal{E}_T^\infty(X, \phi, A)} \{\nu(A)(h_{\nu_A}(T_A) + \int_A \phi_A d\nu_A)\}$$

$$(= \sup_{\nu \in \mathcal{E}_T^\infty(X, \phi, A)} \{h_\nu(T) + \nu(A) \int_A \phi_A d\nu_A\}).$$

By using a similar argument in [Y1: Lemma 7.1] (c.f.[T]), we can prove the next result.

Theorem 5.1 *Let B_1 be a Markov cylinder of rank 1. If $\phi_{B_1} \in L^1(\mu_{B_1})$, then $P^\infty(T, \phi, B_1) = h_\mu(T) + \int_X \phi d\mu = 0$.*

The next example gives a countable non-Markovian sofic system preserving an absolutely continuous invariant measures to which Theorem 5.1 can apply.

Example 4 (**A two dimensional map related to a negative continued fraction**) Let $X = \{(x,y) : 0 < x \leq 1, 0 \leq y \leq 1\}$ and T is defined by

$$T(x,y) = (-\frac{1}{x} - [-\frac{1}{x}], \frac{y}{x} - [\frac{y}{x}]).$$

The first component of T is known as a map related to negative continued fraction which preserves a σ-finite infinite ergodic invariant measure equivalent to the normalized Lebesgue measure of $[0,1]$. We define

$$I = \{(a,b) \in \mathbf{N} \times (\mathbf{N} \cup \{0\}) : a \geq 2, a > b\}$$

and a countable disjoint partition $Q = \{X_{(a,b)}\}_{(a,b)\in I}$, where $X_{(a,b)}$ is defined by :

$$(x,y) \in X_{(a,b)} \ \text{ iff } \ -[-1/x] = a \ \text{ and } \ [y/x] = b.$$

Then (T,X,Q) is a piecewise invertible non-Markovian sofic system. Indeed, we see that $\mathcal{U}$ consists of $U_0 = X$ and $U_1 = \{(x,y) \in X : x+y \leq 1\}$. Let m be the normalized Lebesgue measure of X. We can verify that $\phi = \log \frac{dm}{d(mT)}$ is a potential of WBV and m is a weak Gibbs measure for ϕ with 0. Let D_n be the union of cylinders of rank n touching $\{1\} \times [0,1]$ which consists of indifferent fixed points with respect to m. Then we see that $\sum_{n\geq 0} m(D_n) = \infty$ and T preserves a σ-finite infinite conservative ergodic measure $\mu \sim m$. The invariant density is given by :

$$\frac{d\mu}{dm}(x,y) = \begin{cases} \frac{2-x}{2(1-x)^2} & \text{if } x+y<1 \\ \frac{1}{2(1-x)} & \text{if } x+y>1 \end{cases}$$

and there is no finite absolutely continuous invariant measure with respect to m ([Y1,Y2]).

6 Proofs

Proof of Lemma 3.1. First we show that $\bigcup_{U\in\mathcal{U}^{(n)}} U \subseteq \bigcup_{U\in\mathcal{U}^{(n-1)}} U$. Indeed,

$$\bigcup_{U\in\mathcal{U}^{(n)}} U = T^{n-1}\left(\bigcup_{(i_1\ldots i_n)\in\mathcal{A}_n} (T\mathrm{int}X_{i_1} \cap \mathrm{int}X_{i_2\ldots i_n})\right)$$

$$= T^{n-1}\left(\bigcup_{(i_2\ldots i_n)\in\mathcal{A}_{n-1}}\left(\bigcup_{i_1\in I;(i_1 i_2\ldots i_n)\in\mathcal{A}_n} (T\mathrm{int}X_{i_1} \cap \mathrm{int}X_{i_2\ldots i_n})\right)\right)$$

$$\subseteq T^{n-1}\left(\bigcup_{(i_2\dots i_n)\in\mathcal{A}_{n-1}} \mathrm{int}X_{i_2\dots i_n}\cap\bigcup_{U\in\mathcal{U}^{(1)}} U\right)\subseteq\bigcup_{U\in\mathcal{U}^{(n-1)}} U.$$

Next we assume $x\in\bigcup_{U\in\mathcal{U}^{(n-1)}} U$. Then $\exists(i_1\dots i_n)\in\mathcal{A}_n$ such that $x\in T^{n-1}(\mathrm{int}X_{i_1\dots i_{n-1}})$. Therefore by (C1) we have $v_{i_1\dots i_{n-1}}(x)\in\mathrm{int}X_{i_1\dots i_{n-1}}\subset\bigcup_{U'\in\mathcal{U}^{1)}} U'$. This implies that $\exists X_j\in Q$ with $\mathrm{int}X_j\neq\emptyset$ satisfying

$$v_{i_1\dots i_{n-1}}(x)\in\mathrm{int}X_{i_1\dots i_{n-1}}\cap T(\mathrm{int}X_j)=T(\mathrm{int}X_{ji_1\dots i_{n-1}}).$$

Since $x\in T^n(\mathrm{int}X_{ji_1\dots i_{n-1}})\in\mathcal{U}^{(n)}$, we complete the proof. □

Proof of Proposition 3.1. First we note that the weak Gibbs property of m gives

$$-\frac{1}{n}\log K_n\leq\frac{1}{n}\int_X\left(\log m(X_{i_1\dots i_n}(x))-\sum_{k=0}^{n-1}\phi\circ T^k(x)\right)dm(x)\leq\frac{1}{n}\log K_n.$$

Therefore, if m is T-invariant then for every $n\geq 1$

$$-\frac{1}{n}\log K_n\leq\frac{1}{n}\sum_{(i_1\dots i_n)\in\mathcal{A}_n} m(X_{i_1\dots i_n})\log m(X_{i_1\dots i_n})-\int_X\phi(x)dm(x)\leq\frac{1}{n}\log K_n.$$

From the above inequalities, we conclude $h_m(T,Q)+\int_X\phi dm=0$.□

Proof of Theorem 3.1. First we note that $\sigma(T^{-n}\epsilon)=T^{-n}\mathcal{F}$. Therefore $\forall A\in\sigma(T^{-n}\epsilon),\exists E\in\mathcal{F}$ such that $A=T^{-n}E$. It follows from the property of $\mathcal{L}_\phi$ that

$$\int_A\left(\frac{\mathcal{L}_\phi{}^n 1_{X_{i_1\dots i_n}}}{\mathcal{L}_\phi{}^n 1}\right)\circ T^n(x)dm(x)=\int_X\mathcal{L}_\phi{}^n 1(x)\left(\frac{\mathcal{L}_\phi{}^n 1_{X_{i_1\dots i_n}}(x)}{\mathcal{L}_\phi{}^n 1(x)}\right)1_E(x)dm(x)$$
$$=m(A\cap X_{i_1\dots i_n}).$$

Since $\left(\frac{\mathcal{L}_\phi{}^n 1_{X_{i_1\dots i_n}}}{\mathcal{L}_\phi{}^n 1}\right)\circ T^n$ gives a $\sigma(T^{-n}\epsilon)$-measurable function, we complete the proof of (i). Next we note that $\forall X_{i_1\dots i_n}$ with $(i_1\dots i_n)\in\mathcal{A}_n$

$$m(X_{i_1\dots i_n})=\int_{T^nX_{i_1\dots i_n}}\frac{dm}{d(mT^n)}(v_{i_1\dots i_n}y)dm(y)=\int_{T^nX_{i_1\dots i_n}}\exp[\sum_{k=0}^{n-1}\phi\circ T^k(y)]dm(y).$$

Then (ii) follows from the WBV property of ϕ. □

Proof of Lemma 4.1. It follows from (ii) in Theorem 3.1 that m satisfies the (WG-1) property for every $\phi: X\to\mathbf{R}$ satisfying $\exp\phi=\frac{dm}{d(mT)}$. Therefore we obtain for the unique cylinder $X_{i_1\dots i_n}(x)$ of rank n containing x

$$m(X_{i_1\dots i_n}(x)|T^{-n}\epsilon)(x)=\frac{\frac{dm}{d(mT^n)}(x)}{\mathcal{L}_m^n 1\circ T^n(x)}$$

and this equality gives the desired result. □

Proof of Lemma 4.2. Let x_0 be a fixed point with $x_0 \in X_{i_0}$. First we note that $i_0(n) := \underbrace{i_0 \dots i_0}_{n} \in \mathcal{A}_n (\forall n \geq 1)$ and $T^n \mathrm{int} X_{i_0(n)} = \bigcup_{U \in \mathcal{U}} U$. Since $m(\bigcup_{U \in \mathcal{U}} U) = 1$, we have for m-a.e. $x \in X$

$$\mathcal{L}_m^n 1(x) = \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} \frac{dm}{d(m \circ T^n)}(v_{i_1 \dots i_n}(x)) \; 1_{T^n X_{i_1 \dots i_n}}(x)$$

$$\geq \frac{dm}{d(m \circ T^n)}(v_{i_0 \dots i_0}(x)) = \exp[\sum_{k=0}^{n-1} \phi \circ T^k (v_{i_0 \dots i_0}(x))]$$

$$= \frac{\exp[\sum_{k=0}^{n-1} \phi \circ T^k (v_{i_0 \dots i_0}(x))]}{\exp[\sum_{k=0}^{n-1} \phi \circ T^k (v_{i_0 \dots i_0}(x_0))]} \geq {C_n}^{-1}.$$

On the other hand, we see that

$$(*) \; \mathcal{L}_m^n 1(x) \leq C_n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} \inf_{y \in T^n X_{i_1 \dots i_n}} \frac{dm}{d(m \circ T^n)}(v_{i_1 \dots i_n}(y))$$

$$\leq C_n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} m(X_{i_1 \dots i_n}) \leq C_n.$$

If all cylinders are Bernoulli cylinders, then we have that

$$\mathcal{L}_m^n 1(x) \geq {C_n}^{-1} \sum_{(i_1 \dots i_n) \in \mathcal{A}_n} m(X_{i_1 \dots i_n}) \geq {C_n}^{-1}.$$

These observations give us $||\frac{1}{n} \log \mathcal{L}_m^n 1||_{L^\infty(m)} \leq \frac{1}{n} \log C_n$ when either (i) or (ii) with $q = 1$ is satisfied. Moreover, the decay rate $\frac{1}{n} \log C_n$ is subexponential. Let x_0 be a periodic point with period q in a Bernoulli cylinder of rank q. Then, as we have observed in the above, for every $n = sq + r (0 \leq r \leq q-1)$ we have $C_{sq}^{-1} \leq \mathcal{L}_\phi^{sq} 1(x) \leq C_{sq}$ which allows one to see that

$$\mathcal{L}_\phi^n 1(x) \leq C_{sq} \mathcal{L}_\phi^r 1(x) \leq C_{sq} C_r$$

and

$$\mathcal{L}_\phi^n 1(x) \geq C_{sq}^{-1} \mathcal{L}_\phi^r 1(x) \geq C_{sq}^{-1} C_r^{-1} \sum_{(i_1 \dots i_r), x \in T^r X_{i_1 \dots i_r}} m(X_{i_1 \dots i_r}).$$

We complete the proof. □

Proof of Theorem 4.1. The first assertion follows from the WBV property for ϕ directly. Indeed we can take the WBV sequence $\{C_n\}_{n \geq 1}$ for ϕ as the weak Gibbs sequence $\{K_n\}_{n \geq 1}$ for m. As we have observed the inequalities (*) in the proof of Lemma 4.2, $\frac{1}{n} \int_X \mathcal{L}_m^n 1(x) \log \mathcal{L}_m^n 1(x) dm(x) \leq \frac{1}{n} \log C_n$ holds for all

$n \geq 1$. Therefore, $\limsup_{n\to\infty} \frac{1}{n}\int_X \mathcal{L}_m^n 1(x) \log \mathcal{L}_m^n 1(x) dm(x) \leq 0$. The second assertion is proved. It follows from Lemma 4.2 that

$$0 = e_T(m) = \lim_{n\to\infty} \frac{1}{n} \int_X \log \mathcal{L}_m^n 1 \circ T^n(x) dm(x)$$

$$= \limsup_{n\to\infty} \frac{1}{n} \int_X I_m(\vee_{k-0}^{n-1} T^{-k} Q | T^{-n}\epsilon)(x) dm(x) + \liminf_{n\to\infty} \frac{1}{n} \int_X \log \frac{dm}{d(mT^n)}(x) dm(x).$$

Hence we have

$$\hat{h}_m(T, Q) = -\liminf_{n\to\infty} \frac{1}{n} \int_X \log \frac{dm}{d(mT^n)}(x) dm(x)$$

$$= \limsup_{n\to\infty} \frac{1}{n} \int_X \left(-\log \frac{dm}{d(mT^n)}(x) \right) dm(x).$$

On the other hand, the WG property of m for ϕ allows us to see that

$$\limsup_{n\to\infty} \frac{1}{n} \int_X (-\log m(X_{i_1\dots i_n}(x)) dm(x) = \limsup_{n\to\infty} \frac{1}{n} \int_X (-\log \frac{dm}{d(mT^n)}(x)) dm(x).$$

We complete the proof. □

Proof of Theorem 4.2. As we have already observed in the proof of Theorem 4.1, the WG property allows us to have

$$\hat{h}_m(T, Q) = \limsup_{n\to\infty} \int_X \frac{1}{n} \sum_{k=0}^{n-1} (-\phi \circ T^k(x)) dm(x) = -\liminf_{n\to\infty} \int_X \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x) dm(x).$$

Thus (i) is proved. For the proof of (ii), first we note the next equality.

$$\int_X \left(I_\mu(\vee_{k=0}^{n-1} T^{-k} Q | T^{-n}\epsilon)(x) + \sum_{k=0}^{n-1} \phi \circ T^k(x) \right) d\mu(x)$$

$$= \int_X \left(I_\mu(\vee_{k=0}^{n-1} T^{-k} Q | T^{-n}\epsilon)(x) + \log \frac{d\mu}{d(\mu T^n)}(x) + \sum_{k=0}^{n-1} \log \frac{\exp[\phi]}{d\mu/d(\mu T)} \circ T^k(x) \right) d\mu(x).$$

Then by using the property that $\log x \leq x - 1 (\forall x > 0)$ we have

$$\int_X \left(I_\mu(\vee_{k=0}^{n-1} T^{-k} Q | T^{-n}\epsilon)(x) + \sum_{k=0}^{n-1} \phi \circ T^k(x) \right) d\mu(x)$$

$$\leq \int_X \left(I_\mu(\vee_{k=0}^{n-1} T^{-k} Q | T^{-n}\epsilon)(x) + \log \frac{d\mu}{d(\mu T^n)}(x) \right) d\mu(x) + \min\{A_n, B_n\},$$

where $A_n := \int_X \sum_{k=0}^{n-1}(\frac{\exp[\phi]}{d\mu/d(\mu T)}\circ T^k - 1)d\mu$, and $B_n := \int_X \left(\frac{\exp[\sum_{k=0}^{n-1}\phi\circ T^k]}{d\mu/d(\mu T^n)} - 1\right) d\mu(x)$. Moreover, we can write

$$B_n = \int_X \mathcal{L}_\mu^n \left(\frac{\exp[\sum_{k=0}^{n-1}\phi\circ T^k(x)]}{d\mu/d(\mu T^n)(x)} - 1\right) d\mu(x)$$

$$= \int_X \sum_{y\in T^{-n}x} \frac{d\mu}{d(\mu T^n)}(y) \left(\frac{\exp[\sum_{k=0}^{n-1}\phi\circ T^k(y)]}{d\mu/d(\mu T^n)(y)} - 1\right) d\mu(x)$$

$$= \int_X \sum_{y\in T^{-n}x} \left(\exp[\sum_{k=0}^{n-1}\phi\circ T^k(y)] - \frac{d\mu}{d(\mu T^n)}(y)\right) d\mu(x) = \int_X (\mathcal{L}_\phi{}^n 1(x) - 1) d\mu(x).$$

Then we obtain the desired inequality immediately. □

Proof of Theorem 4.3. $\mathcal{GP}_T(\phi) = 0$ follows from Theorem 4.2 directly. By (i) in Theorem 3.1, we see that $\mu = hm$ with $\frac{h}{hT} \equiv 1$ is a (WG-1) measure for the common potential ϕ. Therefore, $\forall n \geq 1$ and μ-a.e. $x \in X$

$$(**)\ I_\mu(\vee_{k=0}^{n-1}T^{-k}Q|T^{-n}\epsilon)(x) + \sum_{k=0}^{n-1}\phi\circ T^k(x) = \log \mathcal{L}_\mu^n 1\circ T^n(x)$$

because of Lemma 4.1, and for $\mu \in \mathcal{N}_T(X,\phi)$ we have

$$\frac{1}{n}\int_X I_\mu(\vee_{k=0}^{n-1}T^{-k}Q|T^{-n}\epsilon)(x)d\mu(x) + \frac{1}{n}\int_X \sum_{k=0}^{n-1}\phi\circ T^k(x)d\mu(x)$$

$$= \frac{1}{n}\int_X \log \mathcal{L}_\mu^n 1\circ T^n(x) d\mu(x).$$

Since $\mu \leq m$, $\mu(\bigcup_{U\in\mathcal{U}} U) = 1$ holds so that $\mathcal{F}(\phi,\mu) = 0$ follows from Theorem 4.1. (i) is proved. For proving (ii), WOLG we assume x_0 is a fixed point. Then $\mu = \delta_{x_0} \in M_T(X)$ satisfies $\frac{d\mu}{d\mu T} = 1$ and $\mathcal{L}_\mu f(x) = f(x_0)$ so that $A_n = 0$. Also we have $B_n \geq 0$ because x_0 is an indifferent fixed point for m. We complete the proof. □.

Proof of Lemma 4.3. First we note the following inequalities;

$$\mathcal{GP}_T(\phi) \geq \sup\{\mathcal{F}(\phi,\mu) | \mu \in \mathcal{M}_T(X,\phi)\}$$

$$= \sup\{h_\mu(T) + \int_X \phi d\mu - \limsup \frac{1}{n}\min\{A_n, B_n\} \mid \mu \in \mathcal{M}_T(X,\phi)\}$$

$$\geq \sup\{h_\mu(T) + \int_X \phi d\mu - \limsup \frac{1}{n}\min\{A_n, B_n\} \mid \mu \in \mathcal{M}_T(X,\phi), \mu \leq m\}$$

$$= \sup\{h_\mu(T) + \int_X \phi d\mu - \min\{\int_X hT dm - 1, \limsup_{n\to\infty}\int_X \frac{hT^n}{n} dm\} \mid \mu = hm \in \mathcal{M}_T(X,\phi)\}$$

$$\geq \sup\{h_\mu(T)+\int_X \phi d\mu - \min\{\int_X hTdm-1, \limsup_{n\to\infty}\int_X \frac{hT^n}{n}dm\} \mid \mu = hm \in \mathcal{M}_T(X,\phi),$$
$$\text{and } \int_X \log\frac{hT}{h}d\mu = 0\}.$$

As $\int_X \log\frac{hT}{h}d\mu \leq \int_X \left(\frac{hT}{h}-1\right)d\mu$ holds, we have

$$\mathcal{GP}_T(\phi) \geq \sup\{h_\mu(T)+\int_X \phi d\mu \mid \mu = hm \in \mathcal{M}_T(X,\phi), \int_X \log\frac{hT}{h}d\mu = 0,$$
$$\text{and } \limsup_{n\to\infty}\int_X \frac{hT^n}{n}dm = 0\} = \hat{\mathcal{P}}_T(\phi).\square$$

Proof of Theorem 4.4. We first show that $\int_X \log\frac{h}{hT}d\mu = 0$. By (i) in Theorem 3.1, μ is a (WG-1) measure for $\hat{\phi} : X \to \mathbf{R}$ with $\hat{\phi} := \log\frac{d\mu}{d(\mu T)} = \log\frac{dm}{d(mT)} + \log\frac{h}{hT}$. Therefore, the equality (**) in the proof of Theorem 4.3 holds for $\hat{\phi}$ so that

$$\int_X \left(I_\mu(Q|T^{-1}\epsilon)(x)d\mu(x) + \hat{\phi}(x)\right)d\mu(x) = 0.$$

On the other hand, since the WG property of μ for ϕ gives $H_\mu(Q) < \infty$, from Proposition 3.1 we see that

$$\int_X \left(I_\mu(Q|T^{-1}\epsilon)(x)d\mu(x) + \phi(x)\right)d\mu(x)$$
$$= H_\mu(Q|T^{-1}\epsilon) + \int_X \phi(x)d\mu(x) = h_\mu(T) + \int_X \phi(x)d\mu(x) = 0.$$

This implies $\int_X \log\frac{h}{hT}d\mu = 0$. The observation (a) and the inequality $\int_X \log\frac{hT}{h}d\mu \leq \int_X \left(\frac{hT}{h}-1\right)d\mu$ allow us to have

$$\limsup_{n\to\infty}\frac{1}{n}\min\{A_n, B_n\} = \min\{\int_X hTdm - 1, 0\} = 0.$$

Hence we have $\mathcal{F}(\phi,\mu) = 0 = h_\mu(T) + \int_X \phi(x)d\mu(x)$. On the other hand, from Theorem 4.3 we know $\mathcal{GP}_T(\phi) = 0$. We proved (i). If $\hat{\mathcal{P}}_T(\phi) = \mathcal{P}_T(\phi)$, then $\mathcal{P}_T(\phi) \leq 0$ which implies $\mu \in \mathcal{E}_T(\phi)$. Conversely, if $\mu \in \mathcal{E}_T(\phi)$ then $\mathcal{P}_T(\phi) = 0 \geq \mathcal{GP}_T(\phi)$. Since μ attains $\hat{\mathcal{P}}_T(\phi)$, we have $\hat{\mathcal{P}}_T(\phi) = \mathcal{P}_T(\phi) = \mathcal{GP}_T(\phi)$. We proved (ii). Let $\nu \in \mathcal{E}_T(\phi)$. Then we see that

$$\frac{A_n}{n} = \int_X \left(\frac{\exp\phi}{d\nu/d(\nu T)} - 1\right)d\nu \geq \int_X \log\frac{\exp\phi}{d\nu/d(\nu T)}d\nu = h_\nu(T) + \int_X \phi d\nu = 0.$$

Hence we have $\mathcal{F}(\phi,\nu) = 0$. We complete the proof. $\square$

Proof of Proposition 5.2. By Theorem 3.1 in [Y4], $\exists 0 < H < \infty$ so that

$$\int_X \mathcal{L}_\phi^n 1(x) d\mu(x) = \sum_{k=1}^{\infty} \int_{B_k} \mathcal{L}_\phi^n 1(x) h(x) dm(x) \le H \sum_{k=1}^{\infty} k \int_{B_k} \mathcal{L}_\phi^n 1(x) dm(x)$$

$$= H \sum_{k=1}^{\infty} k \int_X 1_{B_k} \circ T^n(x) dm(x) = H \sum_{k=1}^{\infty} k \times m(T^{-n} B_k).$$

Let $\mathcal{V}$ be a finite disjoint partition of X generated by $\mathcal{U}$. Then we can write

$$B_k = \bigcup_{V \in \mathcal{V}} B_k^V, \text{ where } B_k^V := \bigcup_{X_{b_1 \dots b_k} \subset B_k \cap V} X_{b_1 \dots b_k}.$$

Define $\mathcal{A}_n^V := \{(i_1 \dots i_n) \in \mathcal{A}_n | T^n X_{i_1 \dots i_n} \supset V\}$. For every $X_{b_1 \dots b_k} \subset B_k \cap V$, we can define a finite disjoint partition of $\mathcal{A}_n^V$ by $\mathcal{A}_n^V = \bigcup_{l=0}^n \mathcal{A}_n^{V,l}(b_1 \dots b_k)$, where

$$\mathcal{A}_n^{V,l}(b_1 \dots b_k) := \{(i_1 \dots i_n) \in \mathcal{A}_n^V | X_{i_1 \dots i_{n-l}} \in \mathcal{R}, X_{i_{n-l+1} \dots i_n b_1 \dots b_k} \subset B_{k+l}\}.$$

Therefore, we have

$$m(T^{-n} B_k) \le \sum_{V \in \mathcal{V}} \sum_{X_{b_1 \dots b_k} \subset B_k \cap V} m\Big(\bigcup_{(i_1 \dots i_n) \in \mathcal{A}_n^V} X_{i_1 \dots i_n b_1 \dots b_k} \Big)$$

$$= \sum_{V \in \mathcal{V}} \sum_{X_{b_1 \dots b_k} \subset B_k \cap V} \left(\sum_{l=0}^n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n^{V,l}(b_1 \dots b_k)} m(X_{i_1 \dots i_{n-l}} \cap T^{-(n-l)} X_{i_{n-l+1} \dots i_n b_1 \dots b_k}) \right).$$

Since $\frac{dm}{d(mT^*)}$ satisfies the uniformly bounded distortion property, $\exists 1 \le C < \infty$ such that

$$m(X_{i_1 \dots i_n b_1 \dots b_k}) \le C m(X_{i_1 \dots i_{n-l}}) m(X_{i_{n-l+1} \dots i_n b_1 \dots b_k})).$$

Hence we have

$$m(T^{-n} B_k) \le C \sum_{V \in \mathcal{V}} \sum_{X_{b_1 \dots b_k} \subset B_k \cap V} \left(\sum_{l=0}^n \sum_{(i_1 \dots i_n) \in \mathcal{A}_n^{V,l}(b_1 \dots b_k)} m(X_{i_1 \dots i_{n-l}}) m(X_{i_{n-l+1} \dots i_n b_1 \dots b_k})) \right).$$

The RHS is bounded from above by

$$C \sum_{V \in \mathcal{V}} \sum_{l=0}^n \sum_{\substack{X_{i_1 \dots i_{n-l}} \\ \subset D_{n-l}^c}} m(X_{i_1 \dots i_{n-l}}) \left(\sum_{\substack{X_{b_1 \dots b_k} \\ \subset B_k \cap V}} \left(\sum_{\substack{(i_{n-l+1} \dots i_n): \\ (i_1 \dots i_n) \in \mathcal{A}_n, \\ X_{i_{n-l+1} \dots i_n b_1 \dots b_k} \subset B_{k+l}}} m(X_{i_{n-l+1} \dots i_n b_1 \dots b_k}) \right) \right)$$

$$= C \sum_{V \in \mathcal{V}} \sum_{l=0}^{n} \left(\sum_{X_{i_1 \dots i_{n-l}} \subset D_{n-l}^c} m(X_{i_1 \dots i_{n-l}}) \right) m(B_{k+l}).$$

Therefore, we have $m(T^{-n}B_k) \leq C\sharp\mathcal{V} \sum_{l=0}^{n} m(B_{k+l})$. Since $\bigcup_{l=0}^{n} B_{k+l} \subset D_{k-1}$, $n^{-1} \int_X \mathcal{L}_\phi^n 1 d\mu$ is bounded from above by $HC\sharp\mathcal{V} \sum_{k=1}^{\infty} k \left(m(D_{k-1}) \right)$. We remark that $\exists 1 \leq r_1' < r_2' < \infty$ such that $\forall n \geq 1$

$$r_1' n^{-(\alpha+1)} \leq m(B_n) \leq r_2' n^{-(\alpha+1)}.$$

Finally we see that $\int_X R^2 dm = \sum_{k=1}^{\infty} k^2 m(B_k) < \infty$ implies $\sum_{k=1}^{\infty} km(D_k) < \infty$. We complete the proof. $\square$

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