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Quasi Radial Basis Functions Applied to Boundary Element Solutions for the Grad-Shafranov Equation

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Abstract

The Grad-Shafranov equation for axisymmetric fusion plasma has been transformed into a boundary integral equation by expanding the inhomogeneous current density term into a set of ‘quasi’ radial basis functions (RBFs). These quasi RBFs are derived in such a way that a particular solution for the Grad-Shafranov equation will be a simple Gaussian function. Two or three stages of eigenvalue iterations are required together with a nonlinear optimization to update the scaling factor of each quasi RBF. It has been found that excellent accuracy of the boundary element solution can be realized with a small number of quasi RBFs when adopting RBF-dependent scaling factors.

Keywords: Nuclear fusion, Plasma, Axisymmetric, Grad-Shafranov equation, Boundary integral equation, Quasi radial basis function, Gaussian function, Scaling factor, Nonlinear optimization

Glossary of symbols related to electromagnetism

B : magnetic field [T]

ψ : magnetic flux [Wb]

F : poloidal current function [T · m]

I_{tot} : total plasma current [MA]

j : plasma current density [MA/m²]

j_ϕ : toroidal component of plasma current density [MA/m²]

μ_0 : magnetic permeability in vacuum

($\mu_0 = 4\pi \times 10^{-7}$ [Wb/MA · m] or $\mu_0 = 4\pi \times 10^{-7}$ [T · m/MA])

p : plasma pressure [MPa]

1. Introduction

The magnetohydrodynamic (MHD) equilibrium in axisymmetric plasma such as a tokamak is described by the Grad-Shafranov equation [1] in terms of the magnetic flux function and the toroidal component of the plasma current. The boundary element method (BEM) [2] was applied to solving this Grad-Shafranov equation [3-6]. In these applications, the domain integral caused by the inhomogeneous plasma current density term is transformed into an equivalent boundary one, by expanding the inhomogeneous term into a 2-D polynomial, using a particular solution corresponding to the polynomial, and applying Green's second identity. Any other approximated function can be also introduced to derive the same type of boundary integral equation, as suggested in Section 2.

A promising alternative to the polynomial is a linear combination of radial basis functions (RBFs) [7] such as the Gaussian function which will be shown in Section 3.1. It is known that in many cases the coefficient matrix when introducing RBFs shows desirable properties, i.e., symmetric, positive definite and diagonal dominant. Unfortunately, however, it is difficult or almost impossible to find a particular solution that satisfies the Grad-Shafranov equation whose inhomogeneous source term is given by such RBFs.

In the present work, conversely, the Gaussian function is assumed to be a particular solution of magnetic flux. When substituting this particular solution into the LHS of the Grad-Shafranov equation, a new function is generated as an inhomogeneous source term on the RHS. The authors name this new function the 'quasi radial basis function'. The mathematical form of this quasi RBF is a little complicated as shown in Section 3.2, so that the above desirable properties of standard RBFs are lost. But still, this quasi RBF provides an accurate approximation of plasma current profile, as will be demonstrated in Section 5.

As long as the scaling factors included in the quasi RBFs are known (even if each RBF has a different scaling factor), it is enough to solve a set of linear equations to determine the unknown weights that form a linear combination of RBFs, as shown in Section 3.2. The scaling factors are also dealt with as RBF-dependent unknowns, as described in Section 3.3, to improve the accuracy of the quasi RBF approximation. In this case, the quasi Newton method [8-11] is introduced to solve the nonlinear optimization problem for estimating both the weights and the scaling factors.

Eigenvalue iteration is required to seek the equilibrium solution of the Grad-Shafranov equation (see

Section 4.1). The drawback of the above nonlinear scheme is that the set of eigenfunctions varies through this eigenvalue iteration. This means that the eigenvalue computation does not converge. Because of this, the eigenvalue iteration should start with the former linear scheme. Once the converged plasma current profile has been obtained, it is approximated using the nonlinear optimization, i.e., even the scaling factors are optimized. Then the eigenvalue iteration restarts with the quasi RBFs having the RBF-dependent scaling factors. The details related to the eigenvalue iteration procedure are given in Section 4.

It will be found in Section 5 that this calculation scheme using the linear scheme together with the nonlinear one gives excellent accuracy of the boundary element solution.

2. The Grad-Shafranov equation and the corresponding boundary integral equation

For an axisymmetric (r, z) system, the differential form of Ampere's law $\mu_0 \mathbf{j} = \nabla \times \mathbf{B}$ can be reduced to a partial differential equation

$$-\Delta^* \psi \equiv - \left\{ r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \right\} \psi = \mu_0 r j_\phi \quad (1)$$

in terms of magnetic flux ψ [1]. Here, j_ϕ denotes the toroidal component of the plasma current, and μ_0 is the permeability of a vacuum. Applying also the equilibrium condition that the plasma pressure is balanced by the magnetic forces, $\mathbf{j} \times \mathbf{B} = \nabla p$, Eq.(1) is rewritten in the form [1]

$$-\Delta^* \psi = \mu_0 r^2 \frac{dp}{d\psi} + \frac{d}{d\psi} \left(\frac{F^2}{2} \right) \equiv \mu_0 r j_\phi, \quad (2)$$

where p is the plasma pressure and F the poloidal current function. Equation (2) is called the Grad-Shafranov equation, however, for convenience the authors call both Eqs.(1) and (2) the Grad-Shafranov equation in the present paper.

One here introduces the fundamental solution ψ^* that satisfies a subsidiary equation

$$-\Delta^* \psi^* = r \delta_i, \quad (3)$$

where Dirac's delta function δ_i means $\delta(r-a)\delta(z-b)$ with the spike at the point i having the

coordinates (a, b) . Physically, Eq.(3) describes the magnetic flux for an arbitrary field point (r, z) caused by a unit toroidal current located at the point (a, b) . The detailed form of the fundamental solution is given by

$$\psi^* = \frac{\sqrt{ar}}{\pi \tilde{k}} \left[\left(1 - \frac{\tilde{k}^2}{2} \right) K(\tilde{k}) - E(\tilde{k}) \right] \quad (4)$$

with

$$\tilde{k}^2 = \frac{4ar}{(r+a)^2 + (z-b)^2}, \quad (5)$$

where $K(\tilde{k})$ and $E(\tilde{k})$ are the complete elliptic integrals of the first and second kinds, respectively.

If one approximates the RHS of Eq.(1) into a linear combination of appropriate basis functions $f_k(r, z)$,

$$\mu_0 r j_\varphi \approx \sum_{k=1}^L w_k f_k(r, z), \quad (6)$$

the Grad-Shafranov equation can be transformed into an equivalent boundary integral equation in terms of the plasma boundary Γ ,

$$c_i \psi_i - \int_{\Gamma} \left(\frac{\psi^*}{r} \frac{\partial \psi}{\partial n} - \frac{\psi}{r} \frac{\partial \psi^*}{\partial n} \right) d\Gamma = \sum_{k=1}^L w_k \left\{ c_i \varphi_i^{(k)} - \int_{\Gamma} \left(\frac{\psi^*}{r} \frac{\partial \varphi^{(k)}}{\partial n} - \frac{\varphi^{(k)}}{r} \frac{\partial \psi^*}{\partial n} \right) d\Gamma \right\}. \quad (7)$$

In Eq.(7), $\varphi^{(k)}$ means a particular solution that satisfies the Grad-Shafranov equation with the basis function $f_k(r, z)$:

$$-\Delta^* \varphi^{(k)} = f_k(r, z). \quad (8)$$

The constant c_i in Eq.(7) depends on the local boundary geometry under consideration: $c_i = 1.0$ for an internal point, while $c_i = 1/2$ on a smooth boundary.

Note here that Eq.(7) does not include any information related to the equilibrium condition, $\mathbf{j} \times \mathbf{B} = \nabla p$, explicitly. In an actual analysis, one needs to add a restriction to consider this equilibrium condition, as will be shown in Sections 4.

3. Quasi radial basis function to approximate plasma current density profile

The radial basis function (RBF) is defined as a function whose value depends only on the distance (radius) from a point called a ‘center’.

3.1 The standard radial basis function

A typical example of RBF is the ‘Gaussian’ RBF. On a 2-D, $r-z$ plane, for instance, it can be written in the form

$$g_k(r, z) = g_k(d_k) = \exp \left\{ -\frac{(r-r_k)^2 + (z-z_k)^2}{\sigma_k^2} \right\}, \quad (9)$$

where σ_k denotes the scaling factor. In Eq.(9), the quantity $d_k = \sqrt{(r-r_k)^2 + (z-z_k)^2}$ means the distance from the center (r_k, z_k) to an arbitrary point (r, z) located on the 2-D plane. With this type of RBF, the quantity related to the plasma current density can be approximated as a linear combination of L radial basis functions distributed on the 2-D plane, i.e.,

$$\mu_0 r j_\varphi(r, z) \approx \sum_{k=1}^L w_k g_k(r, z) \quad (10)$$

with weights w_k . In order to transform the Grad-Shafranov equation whose inhomogeneous source term is given by Eq.(10) (the Gaussian RBF approximation) into the boundary-only integral equation, Eq.(7), one needs to find the detailed mathematical form of a particular solution $\varphi^{(k)}$ that satisfies the equation

$$-\Delta^* \varphi^{(k)} = g_k(r, z) = \exp \left\{ -\frac{(r-r_k)^2 + (z-z_k)^2}{\sigma_k^2} \right\}. \quad (11)$$

Unfortunately, however, it is extremely difficult or almost impossible to derive such a particular solution.

3.2 The quasi radial basis function

Then, one here assumes conversely that the particular solution has the form of a Gaussian function, i.e.,

$$\varphi^{(k)} = \exp \left\{ -\frac{(r-r_k)^2 + (z-z_k)^2}{\sigma_k^2} \right\}. \quad (12)$$

The basis function $f_k(r, z)$ corresponding to this particular solution is generated by substituting Eq.(12) into the LHS of the Grad-Shafranov equation, as,

$$f_k(r, z) \equiv -\Delta^* \varphi^{(k)} = -\left\{ r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} \right\} \exp \left\{ -\frac{(r-r_k)^2 + (z-z_k)^2}{\sigma_k^2} \right\}. \quad (13)$$

One easily finds that the new basis function has the form

$$f_k(r, z) = \frac{2}{\sigma_k^2} \left(1 + \frac{r_k}{r} - 2 \frac{d_k^2}{\sigma_k^2} \right) \exp \left(-\frac{d_k^2}{\sigma_k^2} \right) \quad (14)$$

with $d_k^2 = (r-r_k)^2 + (z-z_k)^2$. This new function is called the ‘quasi radial basis function (quasi RBF)’ in the present paper. Figure 1(a) shows an example of a bird’s eye view of a quasi RBF with parameters: $\sigma_k = 0.30$, $r_k = 3.4$ and $z_k = 0.1$. Figure 1(b) also shows the profiles along the line $z = 0.1$ for $\sigma_k = 0.30, 0.50$ and 0.70 .

Fig.1 Quasi radial basis function

Now, one adopts the quasi RBF given by Eq.(14) to approximate the RHS of the Grad-Shafranov equation, i.e.,

$$\mu_0 r j_\varphi(r, z) \approx \sum_{k=1}^L w_k f_k(r, z). \quad (15)$$

Each quasi RBF, $f_k(r, z)$, is associated with a different center, (r_k, z_k) , and $f_k(r, z)$ is multiplied by a weight w_k . As long as the scaling factor σ_k is not a variable (even if each RBF has a different value), Eq.(15) is linear with respect to the unknown weights w_k . Then in this case all the values of w_k can be determined simply by solving a set of linear equations

$$\begin{bmatrix} f_1(r_1, z_1) & f_2(r_1, z_1) & \cdots & \cdots & f_L(r_1, z_1) \\ f_1(r_2, z_2) & f_2(r_2, z_2) & \cdots & \cdots & f_L(r_2, z_2) \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ f_1(r_M, z_M) & f_2(r_M, z_M) & \cdots & \cdots & f_L(r_M, z_M) \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ \vdots \\ \vdots \\ w_L \end{Bmatrix} = \begin{Bmatrix} (\mu_0 r j_\varphi)_1 \\ (\mu_0 r j_\varphi)_2 \\ \vdots \\ \vdots \\ (\mu_0 r j_\varphi)_M \end{Bmatrix} \quad (16)$$

Here, the total number of quasi RBFs is L , while that of distributed sampling values of $\mu_0 r j_\phi$ is M .

When M is larger than L , the least square technique can be used to determine the unknown weights w_k .

For this purpose, Eq.(16) is solved using the singular value decomposition (SVD) technique [12]. One here rewrites Eq.(16) in the form

$$\mathbf{D}\mathbf{w} = \mathbf{S}. \quad (17)$$

The matrix $\mathbf{D}$ is decomposed as $\mathbf{D} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T$, where $\mathbf{U}$ and $\mathbf{V}^T$ are orthogonal matrices and the symbol T denotes the matrix transpose, while $\mathbf{\Lambda}$ is a diagonal matrix with positive singular values or zero components. Basically the solution of the weights w_k is given by $\mathbf{w} = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{V}^T\mathbf{S}$, however, one can employ the Tikhonov regularization [13] to stabilize the numerical ill-posedness. In this case, the solution is given as

$$\mathbf{w} = \mathbf{U}(\mathbf{\Lambda}^T + \gamma\mathbf{I})^{-1}\mathbf{V}^T\mathbf{S} \quad (18)$$

using the Tikhonov regularization parameter γ .

3.3 Uniform and RBF-dependent scaling factors

The simplest application of the quasi RBF given by Eq.(14) is to assume that the scaling factors σ_k are uniform for all quasi RBFs composing Eq.(15). In Fig.2(a), the shapes of all quasi RBFs (the fine solid lines) are similar in this case; only the magnitude of each quasi RBF is variable according to the values of w_k . The approximated current density profile denoted by the bold solid line does not agree well with the reference (the dashed line).

On the other hand, if one tunes the scaling factor σ_k to a different optimized value for each quasi RBF, the accuracy of the quasi RBF approximation can be highly improved, as shown in Fig.2(b). Even in this case, as long as such RBF-dependent scaling factors have been given beforehand, it is enough to give the set of weights w_k according to Eq.(18).

Fig.2 Current density profile approximated using the quasi RBFs

In most cases, however, the RBF-dependent scaling factors σ_k are unknown. Equation (15) should now be rewritten as

$$\mu_0 r j_\varphi(r, z) \approx \sum_{k=1}^L w_k f_k(r, z; \sigma_k) = \sum_{k=1}^L \left\{ w_k \cdot \frac{2}{\sigma_k^2} \left(1 + \frac{r_k}{r} - 2 \frac{d_k^2}{\sigma_k^2} \right) \exp \left(- \frac{d_k^2}{\sigma_k^2} \right) \right\}. \quad (19)$$

Since the unknown scaling factor σ_k is included in the quasi RBF $f_k(r, z; \sigma_k)$ itself, the problem to seek both w_k and σ_k is not linear any more. The problem will be a ‘nonlinear optimization problem’ to search for the most likely values of both w_k and σ_k for Eq.(19). In more detail, the best solution of a set of w_k and σ_k is that which minimizes the ‘objective function’

$$Q = \left\| \mu_0 r j_\varphi(r, z) - \sum_{k=1}^L w_k f_k(r, z; \sigma_k) \right\|^2. \quad (20)$$

In the present work, the authors use the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method [8-11] as a family of ‘quasi Newton methods’ to solve this nonlinear optimization problem.

4. Eigenvalue iteration and change of quasi RBFs

4.1 Eigenvalue iteration in boundary element analysis

To satisfy the equilibrium condition, $\mathbf{j} \times \mathbf{B} = \nabla p$, the RHS of the Grad-Shafranov equation is often approximated in a simple correlation form as a function of r and ψ , e.g.,

$$\mu_0 r j_\varphi(r, \psi) = c_0 \{ \beta_p r^2 + (1 - \beta_p) R_0^2 \} (1 - X)^{0.6}, \quad (21)$$

where c_0 , β_p and R_0 are constants. The quantity X is the normalized flux defined by

$$X = (\psi - \psi_M) / (\psi_S - \psi_M),$$

where ψ_M and ψ_S are the values of ψ at the magnetic axis and on the plasma boundary, respectively.

Equation (21) is a function of the unknown magnetic flux ψ . Because of this, the Grad-Shafranov equation is usually solved iteratively as an eigenvalue problem.

Now, the Grad-Shafranov equation is rewritten as

$$-\Delta^* \psi^{(i)} = \lambda^{(i)} \cdot \mu_0 r j_\varphi(r, \psi^{(i-1)}) \quad (i \geq 1) \quad (22)$$

with the adjustment parameter $\lambda^{(i)}$ at the i -th iteration. The value of $\mu_0 r j_\varphi(r, \psi^{(i-1)})$ is calculated using a correlation form such as Eq.(21). The total plasma current I_{tot} is introduced as an input data. The adjustment parameter $\lambda^{(i)}$ is updated in such a way that the restriction

$$\lambda^{(i)} \int_{\Omega} j_\varphi(r, \psi^{(i-1)}) d\Omega = I_{tot} \quad (23)$$

is always preserved through the iteration. That is, $\lambda^{(i)}$ is given by [5]

$$\lambda^{(i)} = \lambda^{(i-1)} \frac{I_{tot}}{\int_{\Omega} \lambda^{(i-1)} j_\varphi(r, \psi^{(i-1)}) d\Omega}. \quad (24)$$

A uniform source $\lambda^{(1)} \cdot \mu_0 r j_\varphi = \text{const.}$ and $\lambda^{(1)} = 1.0$ are assumed as the initial estimates. According to Ampere's circuital law, the domain integral in Eq.(24) can be transformed into a circular integral along the plasma boundary, i.e.,

$$\int_{\Omega} \lambda^{(i-1)} j_\varphi(r, \psi^{(i-1)}) d\Omega = \frac{1}{\mu_0} \int_{\Gamma} \frac{1}{r} \frac{\partial \psi^{(i-1)}}{\partial n} d\Gamma. \quad (25)$$

Thus, $\lambda^{(i)}$ is calculated from Eq.(24). Once $\lambda^{(i)}$ has been calculated in this way, using the BEM scheme one obtains the distributions of magnetic flux $\psi^{(i)}$ and $j_\varphi(r, \psi^{(i)})$. Sampling values of $\mu_0 r j_\varphi(r, \psi^{(i)})$ for points in the plasma domain, one determines the weights w_k in Eq.(19) (and also the scaling factors σ_k if one needs them), which will be used next in the computation of $\psi^{(i+1)}$. This procedure is repeated until a given convergence criterion, e.g.,

$$\varepsilon^{(i)} = \left| \left(\lambda^{(i)} - \lambda^{(i-1)} \right) / \lambda^{(i-1)} \right| < 10^{-5} \quad (26)$$

is satisfied.

4.2 Change of scaling factor in quasi RBF

The plasma current density can be expanded into a linear combination of eigenvectors:

$$j_\varphi = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_n \mathbf{v}_n = \sum_j c_j \mathbf{v}_j. \quad (27)$$

The adjustment parameter $\lambda^{(i)}$ described in Section 4.1 converges to the eigenvalue corresponding the

fundamental eigenvector $\mathbf{v}_1$ in Eq.(27) [5]. The set of eigenvectors $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ must be unchanging through the iteration to ensure the convergence. This means that the plasma current density must be expanded into exactly the same set of basis functions through the iteration. In other words, the eigenvalue convergence is not guaranteed if the quasi RBF-dependent scaling factors σ_k in Eq.(19) vary in every iteration. The nonlinear optimization scheme based on Eq.(20) cannot be combined with the eigenvalue iteration.

Considering the above fact, the following procedure is used in the present work to seek the best approximation of plasma current density profile:

- (i) The eigenvalue iteration starts with a set of quasi RBFs, assuming the same known value of σ_k for all RBFs. Only the weights w_k are unknown, which will be given by Eq.(18).
- (ii) Once the convergence criterion of the iteration, Eq.(24), is satisfied, one expands the resultant current density distribution into a set of quasi RBFs using the BFGS quasi Newton method for unknown RBF-dependent scaling factors σ_k and weights w_k .
- (iii) Using the RBF-dependent scaling factors σ_k thus obtained as known parameters, the eigenvalue iteration restarts with the new set of quasi RBFs in which only the weights w_k are unknown. In this case, of course, it is enough to use the set of weights given by Eq.(18). The iteration is continued until the iteration converges.
- (iv) If the quasi RBF approximation of current density profile obtained in this way is accurate enough, terminate the computation. If not, the procedure from (ii) to (iii) can be repeated.

Application of this procedure to an actual computation is illustrated in Section 4.3.

4.3 Examples of eigenvalue iteration process

Figure 3 illustrates the eigenvalue behaviors in the analyses of the ‘hollow’ type plasma (see Section 5) with 61 RBFs. The plots denoted by ‘□’ show the variation in $\varepsilon^{(i)}$ defined by Eq.(26) for the case where the RBF-dependent scaling factors are updated in every iteration using the quasi Newton method. As already predicted in Section 4.2, the eigenvalue does not converge in this case since the set of eigenvectors

$(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n)$ is changing through the iteration.

Fig.3 Behavior of eigenvalue iterations

In contrast, the eigenvalue computation with fixed scaling factors shows a rapid convergence. The values of $\varepsilon^{(i)}$ plotted by the symbol ‘+’ within the range $i \leq 9$ in Fig.3 decreases very rapidly, where the same scaling factor, $\sigma_k = 0.30$, is assumed for all quasi RBFs but only the weights w_k are unknown. Once the eigenvalue converges in this way (the ‘first stage’, (i)), the converged current density distribution is expanded into a new set of quasi RBFs, each of which has a different scaling factor (see (ii)), with the aid of the nonlinear optimization (quasi Newton) scheme. The distributed scaling factors are now fixed through the ‘second stage’ of eigenvalue iteration (see (iii)). This process can be repeated as (iv) to (v) in Fig.4 (the ‘third stage’).

5. Numerical examples

One here considers some problems of modelling the JT-60 tokamak-device. Four types of plasma current shapes, ‘parabolic’, ‘hollow’, ‘peaked’ and ‘broad’ are dealt with. Bird’s eye views of them are shown in Fig.4(a) through Fig.4(d).

Fig.4 Four types of plasma current density profiles

The reference data of plasma boundary, distributions of plasma current density and magnetic flux were provided for each of the above four plasma types. These had been obtained from analyses using a reliable equilibrium code, SELENE, which is based on the finite element method [14]. The quantity $\mu_0 r j_\phi$ is expressed as a function of magnetic flux to satisfy the equilibrium condition, $\mathbf{j} \times \mathbf{B} = \nabla p$. An example of the expression was given by Eq.(21).

5.1 Outline of the boundary element computation

Only the boundary shape among the SELENE computing results was transferred to BEM computations as input data. The same current profile parametrization such as Eq.(21) used in the SELENE computation was again assumed for a BEM computation. The boundary condition $\psi = 0$ was imposed at all nodal points along the boundary.

The discontinuous quadratic boundary elements [5, 6] were adopted for all computations. The calculation conditions in each case are summarized in Table 1. In every case, the procedures described in section 4.2 and 4.3 were used to obtain the converged solution. A uniform scaling factor, $\sigma_k = 0.30$, was used for the first stage calculations in all cases. The Tikhonov regularization parameter in Eq.(18) was set to be $\gamma = 10^{-8}$ in all eigenvalue iterations.

Table 1 Summary of calculation conditions

The authors found through a series of test calculations that a single application of the nonlinear optimization of scaling factors was sufficient. An additional application of this after the second stage did not show a significant effect any more, as will be shown in Tables 2 through 5. Accordingly, the profiles of current density and magnetic flux for the third stage are not shown below. In the nonlinear optimization between the first and the second stages (also between the second and the third stages), for all cases the quasi Newton iteration was repeated until it reached 1,000 counts.

5.2 Results of current density profile

The results of the current density calculations are shown in Fig.5 through Fig.14. The left sides (a) of the figures show contour maps of current density, and the right sides (b) show the maps of error tendency. In the contour maps, the solid lines show the BEM solutions of current density, while the dashed lines denote the reference solutions obtained using the SELENE code. Now one defines the relative error by

$$\varepsilon = ((\text{BEM} - \text{Reference}) / \text{Reference}) \times 100\% ,$$

where ‘BEM’ means the current density calculated using the present quasi RBF based BEM, while ‘Reference’ denotes the one provided from the SELENE computation. In each of the error maps, the total of internal points is categorized into three groups according to the different levels of the relative errors, as also

summarized in Tables 2 through Table 5.

5.2.1 ‘Parabolic’ type

The accuracy of the current density solution, of course, depends on the number of RBFs. In this section, the parabolic type is chosen as an example to compare the results obtained using a large number of RBFs with those obtained using a small number of RBFs.

When using 315 RBFs, the results on the first and the second stages are respectively shown in Figs.5 and 6. In Fig.5(b), over 70% points represent less than 1.0% errors. That is, the first stage with 315 RBFs and the uniform scaling factor, $\sigma_k = 0.30$, already provided sufficiently accurate solutions. Figure 6 shows that the second stage after the nonlinear optimization did not bring a significant improvement of accuracy.

When using only 63 RBFs, in contrast, the error tendencies are quite different between the first and the second stages, as one can compare Fig.8 with Fig.7. Figure 7 shows that the current density solution on the first stage was very dirty. The maximum error of the current density exceeded 224%. At the end of the second stage with distributed scaling factors, however, the accuracy was dramatically improved, as shown in Fig.8, and Table 2 as well. This demonstrates that acceptable accuracy can be realized with a small number of RBFs when using RBF-dependent scaling factors.

Table 2	Errors in ‘parabolic’ current density profile
Fig.5	‘Parabolic’ current density profile with 315 RBFs on the 1st stage
Fig.6	‘Parabolic’ current density profile with 315 RBFs on the 2nd stage
Fig.7	‘Parabolic’ current density profile with 63 RBFs on the 1st stage
Fig.8	‘Parabolic’ current density profile with 63 RBFs on the 2nd stage

5.2.2 Other types

This section reports the results of current density for ‘hollow’, ‘peaked’ and ‘broad’ types, each of which was obtained from a computation using a small number of RBFs, i.e., 61, 64 and 60, respectively.

The ‘hollow’ type profile has a hollow in the middle (see Fig.4(b)). The results for this type are shown in Table 3, Figs.9 and 10. On the second stage, as shown in Fig.10, the accuracy of the current density shows

a dramatic improvement.

The sharp-pointed ‘peaked’ profile (see Fig.4(c)) and the ‘broad’ profile with a flat top and steep side (see Fig.4(d)) are quite challenging to be approximated. Table 4, Figs.11 and 12 show the results for the ‘peaked’ type, while Table 5, Figs.13 and 14 show the results for the ‘broad’ type. Even for these types, it is found on each second stage that the accuracy has been greatly improved. In summary, the RBF-dependent scaling factors have a remarkable effect in reproducing these profiles which are difficult to be well approximated using the uniform RBFs.

Table 3	Errors in ‘hollow’ current density profile with 61 RBFs
Fig.9	‘Hollow’ current density profile with 61 RBFs on the 1st stage
Fig.10	‘Hollow’ current density profile with 61 RBFs on the 2nd stage

Table 4	Errors in ‘peaked’ current density profile with 64 RBFs
Fig.11	‘Peaked’ current density profile with 64 RBFs on the 1st stage
Fig.12	‘Peaked’ current density profile with 64 RBFs on the 2nd stage

Table 5	Errors in ‘broad’ current density profile with 60 RBFs
Fig.13	‘Broad’ current density profile with 60 RBFs on the 1st stage
Fig.14	‘Broad’ current density profile with 60 RBFs on the 2nd stage

5.3 Results for magnetic flux profile

Now the results of the magnetic flux calculations are shown. Only the magnetic flux solutions corresponding to the current density profile approximated with a small number of RBFs are reported in this section. That is, the numbers of RBFs adopted here are 63, 61, 64 and 60 respectively for the ‘parabolic’, ‘hollow’, ‘peaked’ and ‘broad’ type current density profile. Tables 6 through 9 summarize the error tendencies of these four magnetic flux solutions, where the relative error is again defined by

$$\varepsilon = ((\text{BEM} - \text{Reference}) / \text{Reference}) \times 100\%$$

for the magnetic flux. The magnetic flux profiles and the corresponding error maps are shown in Figs. 15 through 22 for the above four cases. The figures with odd numbers show the results on the first stage, while

even numbers show the results on the second stage.

For the ‘parabolic’, ‘hollow’ and ‘peaked’ cases, the relative error of magnetic flux on the second stage is less than 1.0% in the greater part of the plasma region. The accuracy in each case was not largely improved on the third stage (see the portion of ‘3rd stage’ in Tables 6, 7 and 8). These results suggest that two stages of eigenvalue iterations are enough to obtain a converged solution.

The ‘broad’ type is the most challenging. The magnetic flux profile on the 2nd stage shown in Fig.22 looks a little dirty. However, as indicated in Table 9, the maximum and the average errors are not larger than 2.29% and 1.24%, respectively. Table 9 also indicates that the error tendency on the third stage is almost the same as that on the second stage.

As a whole, one finds that the relative error of the magnetic flux in each case is much smaller than that of the current density shown in Section 5.2. This is related to the fact that the magnetic flux is differentiated twice on the LHS of Eq.(1), the Grad-Shafranov equation, while the current density on the RHS is not differentiated.

Table 6	Errors in magnetic flux for ‘parabolic’ type with 63 RBFs
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Fig.15	Magnetic flux solution for ‘parabolic’ case with 63 RBFs on the 1st stage
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Fig.16	Magnetic flux solution for ‘parabolic’ case with 63 RBFs on the 2nd stage
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Table 7	Errors in magnetic flux for ‘hollow’ type with 61 RBFs
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Fig.17	Magnetic flux solution for ‘hollow’ case with 61 RBFs on the 1st stage
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Fig.18	Magnetic flux solution for ‘hollow’ case with 61 RBFs on the 2nd stage
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Table 8	Errors in magnetic flux for ‘peaked’ type with 64 RBFs
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Fig.19	Magnetic flux solution for ‘peaked’ case with 64 RBFs on the 1st stage
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Fig.20	Magnetic flux solution for ‘peaked’ case with 64 RBFs on the 2nd stage
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Table 9	Errors in magnetic flux for ‘broad’ type with 60 RBFs
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Fig.21	Magnetic flux solution for ‘broad’ case with 60 RBFs on the 1st stage
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Fig.22	Magnetic flux solution for ‘broad’ case with 60 RBFs on the 2nd stage
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6. Conclusion

The Grad-Shafranov equation has been transformed into a boundary-only integral equation with the introduction of quasi radial basis functions to expand the plasma current density term. This formulation does not require any computation of domain integral.

In the process of optimizing the set of RBF-dependent scaling factors, a stable convergence of the quasi Newton iteration has been demonstrated. For both types of RBFs, i.e., for the quasi RBFs with a uniform scaling factor and for those with RBF-dependent scaling factors, the eigenvalue iteration to satisfy the MHD equilibrium also shows a rapid and stable convergence. The series of test calculations indicate that one application of quasi Newton iteration between the first and the second stages of eigenvalue iterations is enough to obtain a converged solution.

The current density profile can be well approximated with over 300 quasi RBFs even when using a uniform scaling factor. For the ‘parabolic’, ‘hollow’ and ‘peaked’ cases respectively with 63, 61 and 64 quasi RBFs, the relative error of magnetic flux on the second stage is less than 1.0% in the greater part of the plasma region. Even for the challenging ‘broad’ type with 60 quasi RBFs, the maximum and the average errors still remain about 2.3% and 1.2%. In conclusion, the RBF-dependent scaling factors are highly effective in realizing an acceptable accuracy of current density with a small number of quasi RBFs.

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Fig.22	Magnetic flux solution for ‘broad’ case with 60 RBFs on the 2nd stage

Table 1 Summary of calculation conditions

Plasma type	Modeling			No. of eigenvalue iterations		
	No. of nodes (BEs)	No. of internal points	No. of RBFs	1st stage	2nd stage	3rd stage
Parabolic	132 (44)	1254	315	9	9	9
			63	9	8	9
Hollow	171 (57)	1218	306	9	8	8
			61	9	9	9
Peaked	177 (59)	1304	329	12	12	12
			64	12	12	12
Broad	171 (57)	1199	301	7	7	7
			60	7	7	7

$\sigma = 0.30$ (uniform) for the 1st stage, $\gamma = 10^{-8}$, 1000 iterations for nonlinear optimization

Table 2 Errors in ‘parabolic’ current density profile

Range of error levels	1st stage	2nd stage	3rd stage
315 RBFs			
$\varepsilon < 1.0\%$	880 (70.2%)	907 (72.3%)	903 (72.0%)
$1.0\% < \varepsilon < 10.0\%$	374 (29.8%)	347 (27.7%)	351 (28.0%)
$\varepsilon > 10.0\%$	0 (0.0%)	0 (0.0%)	0 (0.0%)
Total	1254	1254	1254
Max. error	1.8%	1.8%	1.8%
Ave. error	0.7%	0.7%	0.7%
63 RBFs			
Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	91 (7.3%)	752 (60.0%)	716 (57.1%)
$1.0\% < \varepsilon < 10.0\%$	627 (50.0%)	502 (40.0%)	538 (42.9%)
$\varepsilon > 10.0\%$	536 (42.7%)	0 (0.0%)	0 (0.0%)
Total	1254	1254	1254
Max. error	224.3%	8.6%	8.1%
Ave. error	15.1%	0.9%	1.0%

Table 3 Errors in ‘hollow’ current density profile with 61 RBFs

Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	78 (6.4%)	1038 (85.2%)	1045 (85.8%)
$1.0\% < \varepsilon < 10.0\%$	714 (58.6%)	179 (14.7%)	173 (14.2%)
$\varepsilon > 10.0\%$	426 (35.0%)	1 (0.1%)	0 (0.0%)
Total	1218	1218	1218
Max. error	231.0%	10.1%	9.2%
Ave. error	12.2%	0.6%	0.6%

Table 4 Errors in ‘peaked’ current density profile with 64 RBFs

Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	73 (5.6%)	380 (29.1%)	397 (30.4%)
$1.0\% < \varepsilon < 10.0\%$	544 (41.7%)	919 (70.5%)	898 (68.9%)
$\varepsilon > 10.0\%$	687 (52.7%)	5 (0.4%)	9 (0.7)
Total	1304	1304	1304
Max. error	717.6%	23.8%	26.7%
Ave. error	31.7%	1.7%	1.7%

Table 5 Errors in ‘broad’ current density profile with 60 RBFs

Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	71 (5.9%)	383 (31.9%)	401 (33.4%)
$1.0\% < \varepsilon < 10.0\%$	533 (44.5%)	816 (68.1%)	798 (66.6%)
$\varepsilon > 10.0\%$	595 (49.6%)	0 (0.0%)	0 (0.0%)
Total	1199	1199	1199
Max. error	124.7%	7.7%	7.4%
Ave. error	13.8%	1.3%	1.3%

Table 6 Errors in magnetic flux for ‘parabolic’ type with 63 RBFs

Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	20 (1.6%)	1249 (99.6%)	1249 (99.6%)
$1.0\% < \varepsilon < 10.0\%$	1234 (98.4%)	5 (0.4%)	5 (0.4%)
$\varepsilon > 10.0\%$	0 (0.0%)	0 (0.0%)	0 (0.0%)
Total	1254	1254	1254
Max. error	7.8%	1.2%	1.1%
Ave. error	5.4%	0.5%	0.6%

Table 7 Errors in magnetic flux for ‘hollow’ type with 61 RBFs

Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	19 (1.6%)	1218 (100.0%)	1218 (100.0%)
$1.0\% < \varepsilon < 10.0\%$	1199 (98.4%)	0 (0.0%)	0 (0.0%)
$\varepsilon > 10.0\%$	0 (0.0%)	0 (0.0%)	0 (0.0%)
Total	1218	1218	1218
Max. error	7.0%	0.8%	0.9%
Ave. error	4.3%	0.2%	0.2%

Table 8 Errors in magnetic flux for ‘peaked’ type with 64 RBFs

Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	7 (0.5%)	1289 (98.8%)	1283 (98.4%)
$1.0\% < \varepsilon < 10.0\%$	1296 (99.4%)	15 (1.2%)	21 (1.6%)
$\varepsilon > 10.0\%$	1 (0.1%)	0 (0.0%)	0 (0.0%)
Total	1304	1304	1304
Max. error	10.3%	1.2%	1.2%
Ave. error	6.6%	0.5%	0.5%

Table 9 Errors in magnetic flux for ‘broad’ type with 60 RBFs

Range of error levels	1st stage	2nd stage	3rd stage
$\varepsilon < 1.0\%$	15 (1.3%)	66 (5.5%)	112 (9.3%)
$1.0\% < \varepsilon < 10.0\%$	1184 (98.7%)	1133 (94.5%)	1087 (90.7%)
$\varepsilon > 10.0\%$	0 (0.0%)	0 (0.0%)	0 (0.0%)
Total	1199	1199	1199
Max. error	8.7%	2.3%	2.4%
Ave. error	5.5%	1.2%	1.2%

Fig.1 Quasi radial basis function

Fig.2 Current density profile approximated using the quasi RBFs

Fig.3 Behavior of eigenvalue iterations

(a) parabolic

(b) hollow

(c) peaked

(d) broad

Fig.4 Four types of plasma current density profiles

(a) Current density profile

(b) Error map

Fig.5 'Parabolic' current density profile with 315 RBFs on the 1st stage

(a) Current density profile

(b) Error map

Fig.6 'Parabolic' current density profile with 315 RBFs on the 2nd stage

(a) Current density profile

(b) Error map

Fig.7 'Parabolic' current density profile with 63 RBFs on the 1st stage

(a) Current density profile

(b) Error map

Fig.8 'Parabolic' current density profile with 63 RBFs on the 2nd stage

(a) Current density profile

(b) Error map

Fig.9 'Hollow' current density profile with 61 RBFs on the 1st stage

(a) Current density profile

(b) Error map

Fig.10 'Hollow' current density profile with 61 RBFs on the 2nd stage

Fig.11 'Peaked' current density profile with 64 RBFs on the 1st stage

Fig.12 'Peaked' current density profile with 64 RBFs on the 2nd stage

(a) Current density profile

(b) Error map

Fig.13 'Broad' current density profile with 60 RBFs on the 1st stage

(a) Current density profile

(b) Error map

Fig.14 'Broad' current density profile with 60 RBFs on the 2nd stage

Fig.15 Magnetic flux solution for 'parabolic' case with 63 RBFs on the 1st stage

Fig.16 Magnetic flux solution for 'parabolic' case with 63 RBFs on the 2nd stage

(a) Magnetic flux profile

(b) Error map

Fig.17 Magnetic flux solution for 'hollow' case with 61 RBFs on the 1st stage

(a) Magnetic flux profile

(b) Error map

Fig.18 Magnetic flux solution for 'hollow' case with 61 RBFs on the 2nd stage

(a) Magnetic flux profile

(b) Error map

Fig.19 Magnetic flux solution for 'peaked' case with 64 RBFs on the 1st stage

(a) Magnetic flux profile

(b) Error map

Fig.20 Magnetic flux solution for 'peaked' case with 64 RBFs on the 2nd stage

Fig.21 Magnetic flux solution for 'broad' case with 60 RBFs on the 1st stage

Fig.22 Magnetic flux solution for 'broad' case with 60 RBFs on the 2nd stage