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Author(s)	Pei, Soo-Chang; Ding, Jian-Jiun
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Uncertainty Principle of the 2-D Affine Generalized Fractional Fourier Transform

Soo-Chang Pei* and Jian-Jiun Ding†

Department of Electrical Engineering, National Taiwan University, Taipei, Taiwan, 10617

E-mail: pei@cc.ee.ntu.edu.tw*, djj@cc.ee.ntu.edu.tw†, TEL: 886-2-33663700-321

Abstract— The uncertainty principles of the 1-D fractional Fourier transform and the 1-D linear canonical transform have been derived. We extend the previous works and discuss the uncertainty principle for the two-dimensional affine generalized Fourier transform (2-D AGFFT). We find that derived uncertainty principle of the 2-D AGFFT can also be used for determining the uncertainty principles of many 2-D operations, such as the 2-D fractional Fourier transform, the 2-D linear canonical transform, and the 2-D Fresnel transform. These uncertainty principles are useful for time-frequency analysis and signal analysis. Moreover, we find that the rotation and the chirp multiplication of the 2-D Gaussian function can satisfy the lower bound of the uncertainty principle of the 2-D AGFFT.

I. INTRODUCTION

The well-known Heisenberg uncertainty principle states that, if $X(\omega)$ is the 1-D Fourier transform (FT) of $x(t)$

$$\text{FT: } X(\omega) = FT[x(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad (1)$$

and the 2nd moments of time and frequency are

$$\Delta_t^2 = \int_{-\infty}^{\infty} t^2 |x(t)|^2 dt / \int_{-\infty}^{\infty} |x(t)|^2 dt, \quad (2)$$

$$\Delta_\omega^2 = \int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 d\omega / \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega, \quad (3)$$

when $\int_{-\infty}^{\infty} |x(t)|^2 dt = 1$, the following inequality is satisfied [1]

$$\Delta_t^2 \Delta_\omega^2 \geq \frac{1}{4}. \quad (4)$$

Then, in [2], the uncertainty principle was generalized into the case of the 1-D fractional Fourier transform (FRFT) [3]:

$$\text{FRFT: } X_\alpha(u) = \sqrt{\frac{1-j\cot\alpha}{2\pi}} e^{ju^2 \frac{\cot\alpha}{2}} \int_{-\infty}^{\infty} e^{-j\omega t \csc\alpha} e^{jt^2 \frac{\cot\alpha}{2}} x(t) dt. \quad (5)$$

If

$$\Delta_u^2 = \int_{-\infty}^{\infty} u^2 |X_\alpha(u)|^2 du / \int_{-\infty}^{\infty} |X_\alpha(u)|^2 du, \quad (6)$$

then

$$\Delta_t^2 \Delta_u^2 \geq \frac{\sin^2 \alpha}{4}. \quad (7)$$

Recently, the uncertainty principle was generalized into the case of the 1-D linear canonical transform (LCT) [4][5]. If

$$\text{LCT: } X_{(a,b,c,d)}(u) = \sqrt{\frac{1}{j2\pi b}} e^{ju^2 \frac{d}{2b}} \int_{-\infty}^{\infty} e^{-j\frac{ut}{b}} e^{jt^2 \frac{a}{2b}} x(t) dt, \quad (8)$$

then

$$\Delta_t^2 \Delta_u^2 \geq \frac{b^2}{4}, \quad (9)$$

where

$$\Delta_u^2 = \int_{-\infty}^{\infty} u^2 |X_{(a,b,c,d)}(u)|^2 du / \int_{-\infty}^{\infty} |X_{(a,b,c,d)}(u)|^2 du. \quad (10)$$

The uncertainty principle of the 1-D case has been discussed a lot. In this paper, we extend the previous works and derive the uncertainty principle for the two dimensional affine generalized fractional Fourier transform (2-D AGFFT). **The derived uncertainty principle is shown in Theorem 2.** As Heisenberg's uncertainty principle, the derived uncertainty principle will be useful in signal processing applications, such as time-frequency analysis, signal synthesis, communication, sampling theory, and filter design.

Moreover, since many 2-D operations are the special cases of the 2-D AGFFT (such as the 2-D FRFT and the 2-D Fresnel transform), we can use the derived uncertainty principle to find the uncertainty principles for these operations.

II. TWO-DIMENSIONAL AFFINE GENERALIZED FRACTIONAL FOURIER TRANSFORM

The two-dimensional affine generalized fractional Fourier transform (**2-D AGFFT**) is defined as [6][7]

$$G_{(A,B,C,D)}(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{(A,B,C,D)}(u,v,x,y) \cdot g(x,y) \cdot dx dy, \quad (11)$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \quad (12)$$

represents the 16 parameters of 2-D AGFFT, and

$$K_{(A,B,C,D)}(u,v,x,y) = \frac{1}{2\pi\sqrt{-\det(\mathbf{B})}} e^{\frac{j}{2\det(\mathbf{B})}(k_1 u^2 + k_2 u \cdot v + k_3 v^2)} e^{\frac{j}{\det(\mathbf{B})}((-b_{22}u + b_{12}v)x + (b_{21}u - b_{11}v)y)} e^{\frac{j}{2\det(\mathbf{B})}(p_1 x^2 + p_2 x \cdot y + p_3 y^2)}, \quad (13)$$

where $k_1 = d_{11}b_{22} - d_{12}b_{21}$, $k_2 = 2(-d_{11}b_{12} + d_{12}b_{11})$,

$$k_3 = -d_{21}b_{12} + d_{22}b_{11}, \quad p_1 = a_{11}b_{22} - a_{21}b_{12},$$

$$p_2 = 2(a_{12}b_{22} - a_{22}b_{12}), \quad p_3 = -a_{12}b_{21} + a_{22}b_{11}. \quad (14)$$

Moreover, the following constraints should be satisfied [6][7]:

$$\mathbf{A}^T \mathbf{C} = \mathbf{C}^T \mathbf{A}, \quad \mathbf{B}^T \mathbf{D} = \mathbf{D}^T \mathbf{B}, \quad \mathbf{A}^T \mathbf{D} - \mathbf{C}^T \mathbf{B} = \mathbf{I}. \quad (15)$$

The 2-D AGFFT is useful for filter design, signal analysis, data compression, communication, optics, and image processing [6]. It is a generalization of many 2-D operations. For example, the 2-D FT is a special case of the AGFFT where

$$b_{11} = b_{22} = 1, \quad c_{11} = c_{22} = -1, \quad a_{11} = a_{12} = a_{21} = a_{22} = 0,$$

$$b_{12} = b_{21} = c_{12} = c_{21} = d_{11} = d_{12} = d_{21} = d_{22} = 0. \quad (16)$$

The 2-D fractional Fourier transform (2-D FRFT) [3] is:

$$\mathbf{2-D FRFT: } G_{\alpha,\beta}(u,v) = \frac{\sqrt{(1-j\cot\alpha)(1-j\cot\beta)}}{2\pi} e^{\frac{j}{2}(u^2\cot\alpha+v^2\cot\beta)} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(ux\csc\alpha+vy\csc\beta)} e^{\frac{j}{2}(x^2\cot\alpha+y^2\cot\beta)} g(x,y) dx dy. \quad (17)$$

It is a special case of the 2-D AGFFT where

$$a_{11}=d_{11}=\cos\alpha, b_{11}=-c_{11}=\sin\alpha, a_{22}=d_{22}=\cos\beta, b_{22}=-c_{22}=\sin\beta, a_{21}=a_{12}=b_{12}=b_{21}=c_{12}=c_{21}=d_{12}=d_{21}=0. \quad (18)$$

The 2-D linear canonical transform (LCT) is defined as

$$\mathbf{2-D LCT: } G_{(a,b,c,d,a_1,b_1,c_1,d_1)}(u,v) = \frac{1}{2\pi\sqrt{-bb_1}} e^{j(\frac{d}{2b}u^2+\frac{d_1}{2b_1}v^2)} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(\frac{ax}{b}+\frac{vy}{b_1})} e^{j(\frac{a}{2b}x^2+\frac{a_1}{2b_1}y^2)} g(x,y) dx dy. \quad (19)$$

It is a special case of the 2-D AGFFT where

$$a_{11}=a, a_{22}=a_1, b_{11}=b, b_{22}=b_1, c_{11}=c, c_{22}=c_1, d_{11}=d, d_{22}=d_1, a_{21}=a_{12}=b_{12}=b_{21}=c_{12}=c_{21}=d_{12}=d_{21}=0. \quad (20)$$

The 2-D Fresnel transform is:

$$G_{(\lambda,z)}(u,v) = -j \frac{e^{jkz}}{\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j\frac{\pi}{\lambda z}[(u-x)^2+(v-y)^2]} g(x,y) dx dy. \quad (21)$$

It describes the light propagation in the free space. If the constant phase is ignored, the 2-D Fresnel transform can be viewed as the special case of the AGFFT where

$$a_{11}=a_{22}=d_{11}=d_{22}=1, b_{11}=b_{22}=\lambda z/2\pi, \quad (22)$$

$$a_{21}=a_{12}=b_{12}=b_{21}=c_{11}=c_{12}=c_{21}=c_{22}=d_{12}=d_{21}=0. \quad (23)$$

III. UNCERTAINTY PRINCIPLE OF THE 2-D AGFFT

As the 1-D case, in this paper, we always suppose that the signal $g(x, y)$ is normalized

$$\int_{-\infty}^{\infty} |g(x, y)|^2 dx dy = 1. \quad (24)$$

We will try to find the lower bound of $\Delta_{x,y}^2 \Delta_{u,v}^2$, where

$$\Delta_{x,y}^2 = \int_{-\infty}^{\infty} (x^2 + y^2) |g(x, y)|^2 dx dy, \quad (25)$$

$$\Delta_{u,v}^2 = \int_{-\infty}^{\infty} (u^2 + v^2) |G_{(A,B,C,D)}(u, v)|^2 dudv. \quad (26)$$

and $G_{(A,B,C,D)}(u, v)$ is the 2-D AGFFT (defined in (11)-(15)) of $g(x, y)$. The formula of the 2-D AGFFT is very complicated. It has 16 parameters. We should use some ways to simplify the derivation of the uncertainty principle.

[Lemma 1] First, note that, if

$$g_0(x, y) = e^{\frac{j}{2\det(\mathbf{B})}(p_1 \cdot x^2 + p_2 \cdot x \cdot y + p_3 \cdot y^2)} g(x, y), \quad (27)$$

$$H(u, v) = e^{\frac{-j}{2\det(\mathbf{B})}(k_1 \cdot u^2 + k_2 \cdot u \cdot v + k_3 \cdot v^2)} G_{(A,B,C,D)}(u, v), \quad (28)$$

then, since $|g_0(x, y)| = |g(x, y)|$ and $|H(u, v)| = |G_{(A,B,C,D)}(u, v)|$,

$$\Delta_{x,y}^2 = \int_{-\infty}^{\infty} (x^2 + y^2) |g_0(x, y)|^2 dx dy, \quad (29)$$

$$\Delta_{u,v}^2 = \int_{-\infty}^{\infty} (u^2 + v^2) |H(u, v)|^2 dudv. \quad (30)$$

Note that

$$H(u, v) = \frac{1}{2\pi\sqrt{-\det(\mathbf{B})}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{j}{\det(\mathbf{B})}((-b_{22}u+b_{21}v)x+(b_{21}u-b_{11}v)y)} g_0(x, y) dx dy. \quad (31)$$

[Lemma 2] Moreover, the rotation operation does not affect the 2nd order moment. That is, if

$$g_1(x, y) = g_0(x \cos\theta + y \sin\theta, -x \sin\theta + y \cos\theta), \quad (32)$$

$$H_1(u, v) = H(u \cos\phi + v \sin\phi, -u \sin\phi + v \cos\phi), \quad (33)$$

then

$$\int_{-\infty}^{\infty} (x^2 + y^2) |g_1(x, y)|^2 dx dy = \int_{-\infty}^{\infty} (x^2 + y^2) |g_0(x, y)|^2 dx dy, \quad (34)$$

$$\int_{-\infty}^{\infty} (u^2 + v^2) |H_1(u, v)|^2 dudv = \int_{-\infty}^{\infty} (u^2 + v^2) |H(u, v)|^2 dudv. \quad (35)$$

Substituting (32) and (33) into (31), we obtain

$$H_1(u, v) = \frac{1}{2\pi\sqrt{-\det(\mathbf{B})}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j((\eta_1 u + \eta_2 v)x + (\eta_3 u + \eta_4 v)y)} g_1(x, y) dx dy, \quad (36)$$

where η_1, η_2, η_3 , and η_4 can be calculated from:

$$\begin{bmatrix} \eta_1 & \eta_2 \\ \eta_3 & \eta_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \frac{b_{22}}{\det(\mathbf{B})} & \frac{-b_{12}}{\det(\mathbf{B})} \\ \frac{-b_{21}}{\det(\mathbf{B})} & \frac{b_{11}}{\det(\mathbf{B})} \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}. \quad (37)$$

Note that, if η_2 and η_3 are zero, the relation between $H_1(u, v)$ and $g_1(x, y)$ in (36) will be simplified into the 2-D scaled FT. The uncertainty principle of the 2-D scaled FT is easier to find. To make $\eta_2 = \eta_3 = 0$, θ and ϕ should satisfy

$$\begin{bmatrix} \frac{b_{22}}{\det(\mathbf{B})} & \frac{-b_{12}}{\det(\mathbf{B})} \\ \frac{-b_{21}}{\det(\mathbf{B})} & \frac{b_{11}}{\det(\mathbf{B})} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \eta_1 & 0 \\ 0 & \eta_4 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}, \quad (38)$$

$$b_{11} + b_{22} = (\eta_1^{-1} + \eta_4^{-1}) \cos(\phi - \theta), \quad b_{11} - b_{22} = (\eta_1^{-1} - \eta_4^{-1}) \cos(\phi + \theta),$$

$$b_{12} + b_{21} = (\eta_4^{-1} - \eta_1^{-1}) \sin(\phi + \theta), \quad b_{12} - b_{21} = (\eta_4^{-1} + \eta_1^{-1}) \sin(\phi - \theta). \quad (39)$$

Therefore,

$$\sqrt{(b_{11} + b_{22})^2 + (b_{12} - b_{21})^2} = |\eta_1^{-1} + \eta_4^{-1}|, \quad (40)$$

$$\sqrt{(b_{11} - b_{22})^2 + (b_{21} + b_{12})^2} = |\eta_1^{-1} - \eta_4^{-1}|. \quad (41)$$

Thus, we can choose

$$\eta_1 = 2 / (\sqrt{(b_{11} + b_{22})^2 + (b_{12} - b_{21})^2} + \sqrt{(b_{11} - b_{22})^2 + (b_{12} + b_{21})^2})$$

$$\eta_4 = 2 / (\sqrt{(b_{11} + b_{22})^2 + (b_{12} - b_{21})^2} - \sqrt{(b_{11} - b_{22})^2 + (b_{12} + b_{21})^2})$$

$$\text{if } (b_{11} + b_{22})^2 + (b_{12} - b_{21})^2 > (b_{11} - b_{22})^2 + (b_{12} + b_{21})^2 \quad (42)$$

and

$$\eta_1 = 2 / (\sqrt{(b_{11} + b_{22})^2 + (b_{12} - b_{21})^2} - \sqrt{(b_{11} - b_{22})^2 + (b_{12} + b_{21})^2})$$

$$\eta_4 = 2 / (\sqrt{(b_{11} + b_{22})^2 + (b_{12} - b_{21})^2} + \sqrt{(b_{11} - b_{22})^2 + (b_{12} + b_{21})^2})$$

$$\text{if } (b_{11} + b_{22})^2 + (b_{12} - b_{21})^2 < (b_{11} - b_{22})^2 + (b_{12} + b_{21})^2. \quad (43)$$

Then, from (39),

$$\phi = (\psi_1 + \psi_2)/2, \quad \theta = (\psi_1 - \psi_2)/2, \quad (44)$$

$$\text{where } \psi_1 = \cos^{-1} \frac{b_{11} - b_{22}}{\eta_1^{-1} - \eta_4^{-1}} = \sin^{-1} \frac{b_{12} + b_{21}}{\eta_1^{-1} - \eta_4^{-1}},$$

$$\psi_2 = \cos^{-1} \frac{b_{11} + b_{22}}{\eta_1^{-1} + \eta_4^{-1}} = \sin^{-1} \frac{b_{12} - b_{21}}{\eta_1^{-1} + \eta_4^{-1}}. \quad (45)$$

If we choose $\eta_1, \eta_2, \eta_3, \eta_4, \phi$, and θ as (42) (or (43)) and (44), $\eta_2 = \eta_3 = 0$ and the relation between $H_1(u, v)$ and $g_1(x, y)$ in (36) becomes the 2-D scaled FT.

$$H_1(u, v) = \frac{\sqrt{-\eta_1 \eta_4}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(\eta_1 u x + \eta_4 v y)} g_1(x, y) dx dy. \quad (46)$$

[Theorem 1] For the 2-D scaled Fourier transform:

$$G_{SF}(f, h) = \frac{\sqrt{-\sigma_1 \sigma_2}}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-j(\sigma_1 f x + \sigma_2 h y)} g(x, y) dx dy. \quad (47)$$

$$\begin{aligned} \text{If } \Delta_{x,y}^2 &= \int_{-\infty}^{\infty} (x^2 + y^2) |g(x, y)|^2 dx dy, \\ \Delta_{SF}^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (f^2 + h^2) |G_{SF}(f, h)|^2 df dh, \end{aligned} \quad (48)$$

then

$$\Delta_{x,y}^2 \Delta_{SF}^2 \geq \frac{1}{4} (|\sigma_1^{-1}| + |\sigma_2^{-1}|)^2. \quad (49)$$

(Proof): Since $G_{SF}(f, h) = \sqrt{\sigma_1 \sigma_2} G(\sigma_1 f, \sigma_2 h)$, where $G(f, h)$ is the FT of $g(x, y)$, if we set $f_1 = \sigma_1 f$ and $h_1 = \sigma_2 h$, then $df dh = df_1 dh_1 / |\sigma_1 \sigma_2|$ and (47) becomes

$$\Delta_{SF}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p^2 f_1^2 + q^2 h_1^2) |G(f_1, h_1)|^2 df_1 dh_1. \quad (50)$$

where $p = 1/\sigma_1$ and $q = 1/\sigma_2$. Then since

$$\begin{aligned} &(p^2 f^2 + q^2 h^2) |G(f, h)|^2 \\ &= (-|p|f + |q|h) G(f, h) (-|p|f - |q|h) G^*(f, h), \end{aligned} \quad (51)$$

$$IFT[(-|p|f + |q|h) G(f, h)] = [j|p| \frac{\partial}{\partial x} + |q| \frac{\partial}{\partial y}] g(x, y), \quad (52)$$

from Parseval's Theorem of the 2-D FT:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x, y)|^2 dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |G(f, h)|^2 df dh, \quad (53)$$

if $G(f, h) = FT[g(x, y)]$, (51) can be rewritten as

$$\begin{aligned} \Delta_{SF}^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [j|p| \frac{\partial}{\partial x} + |q| \frac{\partial}{\partial y}] g(x, y) [-j|p| \frac{\partial}{\partial x} + |q| \frac{\partial}{\partial y}] \\ &g^*(x, y) dx dy. \end{aligned} \quad (54)$$

Furthermore, in (48),

$$(x^2 + y^2) |g(x, y)|^2 = (jx + y) g(x, y) (-jx + y) g^*(x, y). \quad (55)$$

Therefore,

$$\Delta_{x,y}^2 \Delta_{SF}^2 = \|(jx + y) g(x, y)\|^2 \|[j|p| \frac{\partial}{\partial x} + |q| \frac{\partial}{\partial y}] g(x, y)\|^2, \quad (56)$$

Then, from Cauchy-Schwartz inequality,

$$\|f(x, y)\|^2 \|g(x, y)\|^2 \geq |\langle f(x, y), g(x, y) \rangle|^2, \quad (57)$$

$$\|f(x, y)\|^2 \|g(x, y)\|^2 \geq [|\langle f(x, y), g(x, y) \rangle|^2 + |\langle f^*(x, y), g^*(x, y) \rangle|^2] / 2, \quad (58)$$

(56) can be rewritten as:

$$\begin{aligned} \Delta_{x,y}^2 \Delta_{SF}^2 &\geq \\ &\frac{1}{2} \left\langle jxg, j|p| \frac{\partial}{\partial x} g \right\rangle + \left\langle yg, j|p| \frac{\partial}{\partial x} g \right\rangle + \left\langle jxg, |q| \frac{\partial}{\partial y} g \right\rangle + \left\langle yg, |q| \frac{\partial}{\partial y} g \right\rangle \Bigg|^2 + \\ &\frac{1}{2} \left\langle j|p| \frac{\partial}{\partial x} g, jxg \right\rangle + \left\langle j|p| \frac{\partial}{\partial x} g, yg \right\rangle + \left\langle |q| \frac{\partial}{\partial y} g, jxg \right\rangle + \left\langle |q| \frac{\partial}{\partial y} g, yg \right\rangle \Bigg|^2. \end{aligned} \quad (59)$$

Note that (56) can also be expressed as

$$\Delta_{x,y}^2 \Delta_{SF}^2 = \|(jx - y) g(x, y)\|^2 \|[j|p| \frac{\partial}{\partial x} - |q| \frac{\partial}{\partial y}] g(x, y)\|^2. \quad (60)$$

From the similar process, we obtain

$$\begin{aligned} \Delta_{x,y}^2 \Delta_{SF}^2 &\geq \\ &\frac{1}{2} \left\langle jxg, j|p| \frac{\partial}{\partial x} g \right\rangle - \left\langle yg, j|p| \frac{\partial}{\partial x} g \right\rangle - \left\langle jxg, |q| \frac{\partial}{\partial y} g \right\rangle + \left\langle yg, |q| \frac{\partial}{\partial y} g \right\rangle \Bigg|^2 + \\ &\frac{1}{2} \left\langle j|p| \frac{\partial}{\partial x} g, jxg \right\rangle - \left\langle j|p| \frac{\partial}{\partial x} g, yg \right\rangle - \left\langle |q| \frac{\partial}{\partial y} g, jxg \right\rangle + \left\langle |q| \frac{\partial}{\partial y} g, yg \right\rangle \Bigg|^2. \end{aligned} \quad (61)$$

Adding (61) by (63) and using the fact that

$$|a|^2 + |b|^2 + |c|^2 + |d|^2 \geq 4 \left| \frac{a+b+c+d}{4} \right|^2, \quad (62)$$

we obtain

$$\begin{aligned} \Delta_{x,y}^2 \Delta_{SF}^2 &\geq \frac{1}{4} \left\langle jxg, j|p| \frac{\partial}{\partial x} g \right\rangle + \left\langle j|p| \frac{\partial}{\partial x} g, jxg \right\rangle + \\ &\left\langle yg, |q| \frac{\partial}{\partial y} g \right\rangle + \left\langle |q| \frac{\partial}{\partial y} g, yg \right\rangle \Bigg|^2. \end{aligned} \quad (63)$$

Then,

$$\begin{aligned} &\left\langle jxg, j|p| \frac{\partial}{\partial x} g \right\rangle + \left\langle j|p| \frac{\partial}{\partial x} g, jxg \right\rangle \\ &= |p| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \frac{\partial}{\partial x} [g(x, y) g^*(x, y)] dx dy \\ &= |p| \int_{-\infty}^{\infty} \left[xg(x, y) g^*(x, y) \Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty} g(x, y) g^*(x, y) dx \right] dy \\ &= -|p| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |g(x, y)|^2 dx dy = -|p|. \end{aligned} \quad (64)$$

Similarly,

$$\left\langle yg, |q| \frac{\partial}{\partial y} g \right\rangle + \left\langle |q| \frac{\partial}{\partial y} g, yg \right\rangle = -|q|. \quad (65)$$

Therefore,

$$\Delta_{x,y}^2 \Delta_{pf, qh}^2 \geq \frac{1}{4} (|p| + |q|)^2 = \frac{1}{4} (|\sigma_1^{-1}| + |\sigma_2^{-1}|)^2. \quad \#$$

Lemmas 1 and 2 and Theorem 1 much simplify the derivation of the uncertainty principle. From (29), (30), (34), (35), (46), and (49),

$$\Delta_{x,y}^2 \Delta_{u,v}^2 \geq \frac{1}{4} (|\eta_1^{-1}| + |\eta_4^{-1}|)^2. \quad (66)$$

Moreover, from (42) and (43)

$$\begin{aligned} &|\eta_1^{-1}| + |\eta_4^{-1}| = \\ &\max \left(\sqrt{(b_{11} + b_{22})^2 + (b_{12} - b_{21})^2}, \sqrt{(b_{11} - b_{22})^2 + (b_{12} + b_{21})^2} \right). \end{aligned} \quad (67)$$

Thus, we obtain:

[Theorem 2] Uncertainty Principle of the 2-D AGFFT:

If $\Delta_{x,y}^2$ and $\Delta_{u,v}^2$ are the 2nd order moments of $g(x, y)$ and the 2-D AGFFT of $g(x, y)$, as in (25) and (26), respectively, then

$$\begin{aligned} \Delta_{x,y}^2 \Delta_{u,v}^2 &\geq \\ &\frac{1}{4} \max \left((b_{11} + b_{22})^2 + (b_{12} - b_{21})^2, (b_{11} - b_{22})^2 + (b_{12} + b_{21})^2 \right). \end{aligned} \quad (68)$$

IV. THE RELATED PRINCIPLES

[Remark 1] More generally, if $G_{(A_1, B_1, C_1, D_1)}(u, v)$ and $G_{(A, B, C, D)}(u, v)$ are the 2-D AGFFTs of $g(x, y)$ with parameters $\{\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D}_1\}$ and $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$, respectively, and

$$\begin{aligned}\Delta_{u_1, v_1}^2 &= \int_{-\infty}^{\infty} (u_1^2 + v_1^2) |G_{(A_1, B_1, C_1, D_1)}(u, v)|^2 dudv, \\ \Delta_{u, v}^2 &= \int_{-\infty}^{\infty} (u^2 + v^2) |G_{(A, B, C, D)}(u, v)|^2 dudv,\end{aligned}\quad (69)$$

then

$$\begin{aligned}\Delta_{x, y}^2 \Delta_{u, v}^2 &\geq \frac{1}{4} \max\left((q_{11} + q_{22})^2 + (q_{12} - q_{21})^2, \right. \\ &\quad \left. (q_{11} - q_{22})^2 + (q_{12} + q_{21})^2\right),\end{aligned}\quad (70)$$

where $\begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D}_1 \end{bmatrix}^{-1}$, $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$. (71)

This can be proven from the fact that $G_{(A, B, C, D)}(u, v)$ is the 2-D AGFFT of $G_{(A_1, B_1, C_1, D_1)}(u, v)$ with parameters $\{\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S}\}$.

[Theorem 3] It is known that, for the 1-D FT, the 1-D Gaussian function can satisfy the lower bound of inequality of the uncertainty principle [1]. For the 2-D AGFFT, the **chirp multiplication and rotation of the 2-D Gaussian** will satisfy the lower bound of inequality of the uncertainty principle. If

$$\begin{aligned}g(x, y) &= \sqrt{\pi^{-1}} e^{\frac{-j}{2\det(\mathbf{B})}(p_1 \cdot x^2 + p_2 \cdot x \cdot y + p_3 \cdot y^2)} \times \\ &\quad e^{-\frac{1}{2}((x \cos \theta - y \sin \theta)^2 |\eta_1| + (x \sin \theta + y \cos \theta)^2 |\eta_4|)},\end{aligned}\quad (72)$$

where p_1, p_2 , and p_3 are defined in (14), η_1 and η_4 are calculated from (42) or (43), and θ is determined from (44), then the 2-D AGFFT of $g(x, y)$ is

$$\begin{aligned}G_{(A, B, C, D)}(u, v) &= \sqrt{(-\pi)^{-1}} e^{\frac{j}{2\det(\mathbf{B})}(k_1 \cdot u^2 + k_2 \cdot u \cdot v + k_3 \cdot v^2)} \times \\ &\quad e^{-\frac{1}{2}((u \cos \phi + v \sin \phi)^2 |\eta_1| + (-u \sin \phi + v \cos \phi)^2 |\eta_4|)},\end{aligned}\quad (73)$$

where ϕ is calculated from (44). Then,

$$\Delta_{x, y}^2 = \Delta_{u, v}^2 = \left(\sqrt{1/|\eta_1|} + \sqrt{1/|\eta_4|}\right) / 2, \quad (74)$$

$$\begin{aligned}\Delta_{x, y}^2 \Delta_{u, v}^2 &= \frac{1}{4} \max\left((b_{11} + b_{22})^2 + (b_{12} - b_{21})^2, \right. \\ &\quad \left. (b_{11} - b_{22})^2 + (b_{12} + b_{21})^2\right).\end{aligned}\quad (75)$$

Thus, the function in (74) satisfies the **lower bound** of inequality of the uncertainty principle for the 2-D AGFFT.

V. SOME IMPORTANT SPECIAL CASES

Since the 2-D AGFFT is the generalization of many operations, we can use (68) to find the uncertainty principle of these operations.

[Corollary 1] Uncertainty Principle of the 2-D FRFT:

From (68) and (18), if

$$\Delta_{u, v}^2 = \int_{-\infty}^{\infty} (u^2 + v^2) |G_{\alpha, \beta}(u, v)|^2 dudv, \quad (76)$$

where $G_{\alpha, \beta}(u, v)$ is the 2-D FRFT of $g(x, y)$, as in (7), then

$$\Delta_{x, y}^2 \Delta_{u, v}^2 \geq \frac{1}{4} (|\sin \alpha| + |\sin \beta|)^2. \quad (77)$$

Moreover, when

$$g(x, y) = \sqrt{\pi^{-1}} e^{\frac{-j}{2}(x^2 \cot \alpha + p_3 \cdot y^2 \cot \beta)} e^{-\frac{1}{2}(x^2 |\csc \alpha| + y^2 |\csc \beta|)}, \quad (78)$$

the equality that $\Delta_{x, y}^2 \Delta_{u, v}^2 = (|\sin \alpha| + |\sin \beta|)^2 / 4$ is satisfied.

[Corollary 2] Uncertainty Principle of the 2-D LCT:

From (68) and (20), if

$$\Delta_{u, v}^2 = \int_{-\infty}^{\infty} (u^2 + v^2) |G_{(a, b, c, d, a_1, b_1, c_1, d_1)}(u, v)|^2 dudv, \quad (79)$$

then

$$\Delta_{x, y}^2 \Delta_{u, v}^2 \geq \frac{1}{4} (|b| + |b_1|)^2. \quad (80)$$

More, the equality is satisfied when

$$g(x, y) = \sqrt{\pi^{-1}} e^{-j\left(\frac{a}{2b}x^2 + \frac{a_1}{2b_1}y^2\right)} e^{-\frac{1}{2}(x^2/|b| + y^2/|b_1|)}, \quad (81)$$

[Corollary 3] Uncertainty Principle of the 2-D Fresnel Transform:

$$\Delta_{x, y}^2 \Delta_{u, v}^2 \geq \frac{\lambda^2 z^2}{4\pi^2}. \quad (82)$$

The equality that $\Delta_{x, y}^2 \Delta_{u, v}^2 = \lambda^2 z^2 / 4\pi^2$ is satisfied when

$$g(x, y) = \sqrt{\pi^{-1}} e^{-j\frac{\pi}{\lambda z}(x^2 + y^2)} e^{-\frac{\pi}{\lambda z}(x^2 + y^2)}. \quad (83)$$

VI. CONCLUSIONS

In this paper, we derived the uncertainty principle of the 2-D AGFFT (See Theorem 2.) We also showed that the lower bound can be achieved by the 2-D Gaussian function with rotation and chirp multiplication (See Theorem 3.) The uncertainty principle of the 2-D AGFFT would be very useful in time-frequency analysis, developing sampling theory in 2-D case, filter design, signal synthesis, and optics.

VII. REFERENCES

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