Intermittent switching for three repulsively coupled oscillators

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We study intermittent switching behaviors in a system with three identical oscillators coupled diffusively and repulsively, to clarify a bifurcation scenario which generates such intermittent switching behaviors. We use the Stuart-Landau oscillator, which is a general form of Hopf bifurcation, and can describe both cases: limit cycle and inactive (i.e., non-self-oscillatory) cases. From a numerical study of the bifurcation structure, two different routes to chaos which have $S_3$ symmetry were found. One is the sudden appearance of chaos as Pomeau-Manneville intermittency, which is found for the inactive case. In this case a trajectory shows switching among three mutually symmetric tori when a parameter exceeds critical value. The other route, which appears for the limit cycle case, consists of two parts: First, chaos with lower symmetry appears through period doubling, and after the two successive attractor-merging crises, chaos which has $S_3$ symmetry appears. At each crisis, the attractor changes its symmetry.

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I. INTRODUCTION

Coupled nonlinear oscillators show a rich variety of collective phenomena such as synchronization [1,2], clustering [3] and chaos [4], and are widely studied not only theoretically, but also experimentally [5–8]. In this regard, one area of interest is in spontaneous transitions among several states. For example, some periodic oscillating patterns which have spatiotemporal symmetry have been observed in the three coupled biological oscillators comprising the plasmodial slime mold [5,6]. In this system, slime mold shows an oscillating pattern for a long time, however, after a while it shows switching to different oscillating patterns. Similar switching behavior is also observed in a system with the three coupled oscillators in Belousov-Zhabotinsky (BZ) reaction systems [7].

One explanation of such a switching behavior is chaotic itinerancy [9]. Chaotic itinerancy is a concept to describe a dynamical behavior which shows chaotic transitions among low dimensional ordered states. The switching behavior often appears for a system with coupled chaotic elements and a system with a large number of coupled periodic elements. However, the switching behavior is less studied for a small number of coupled periodic elements. In the present paper, we propose a model with three coupled oscillators, which shows intermittent switching behaviors. We employ an idealized amplitude model for each oscillator, to investigate the relation between the appearance of switching behavior and the characteristic property for the oscillator itself. We clarify the bifurcation scenarios leading to the switching behaviors in the system by changing the intensity of amplitude dependency of the oscillators.

In most studies, collective behavior of globally coupled oscillators has been studied by using the phase model [10], because the phase model is a good approximation of coupled limit cycle oscillators for weak coupling. It is known that systems with three weakly coupled oscillators [11–13] show a rich variety of solutions. If the phase oscillators are identical [12], heteroclinic bifurcations may occur. However, the system does not show any chaotic motion if the coupling term depends only on phase difference. On the other hand, switching behaviors have been observed in three coupled biological oscillators comprised of the plasmodial slime mold [5,6] and the three coupled oscillators in BZ reaction systems [7]. In those experiments, each oscillator is periodic without coupling. Thus strong coupling and/or amplitude dependency seems to be necessary to obtain switching behavior for coupled identical limit cycle oscillators. We employ the idealized amplitude model, the Stuart-Landau oscillator, for each element because the Stuart-Landau equation has an amplitude dependency on phase velocity, and the intensity of the amplitude dependency can be controlled by a parameter $\beta$. Collective behaviors for $N$ identical Stuart-Landau oscillators diffusively coupled have been studied [4,14]. For $N=3$, characteristic frequencies were investigated in a certain parameter range with a complex coupling constant. Transitions from periodic motion to doubly periodic motion and doubly periodic motion to chaotic motion are reported [14]. However, a detailed bifurcation structure has not been studied, and a switching behavior near the bifurcation point has also not been previously reported.

We study a system with three oscillators coupled repulsively. We call the coupling attractive if a pair of coupled oscillators prefer to have the same state variable, and repulsive if they prefer to have different state variables. Purely repulsive coupling has been less studied than attractive coupling since entrainment of oscillators is one of the main interests in the field of coupled oscillators. Repulsive coupling becomes important when the sum of the state variables associated with the individual oscillators tends to keep a constant value, for example, saline oscillators [17] and slime mold oscillators [5].

To study the bifurcation structure, we follow branches of solutions by numerical continuation using AUTO [18], a software for continuation and bifurcation problems. Both for the inactive (i.e., non-self-oscillatory) case and for the limit...
cycle case, we find the chaos which has $S_3$ symmetry, however, the routes to chaos are different. For the inactive case, chaos appears suddenly as Pomeau-Manneville intermittency, and a trajectory shows switching among three mutually symmetric tori when a parameter exceeds the critical point. On the other hand, the appearance of $S_3$ symmetric chaos for the limit cycle case consists of two parts: first, chaos with lower symmetry appears through period-doubling bifurcations, and two successive attractor-merging crises gives the chaos with $S_3$ symmetry. After the second attractor merging crisis, a trajectory shows switching among three chaotic regions. This phenomenon can be explained by crisis-induced intermittency [19]. Crisis-induced intermittency is that intermittently switching among chaotic states after interior crisis or attractor-merging crisis [19]. For crisis-induced intermittency an orbit stays near one of the old attractors which existed before the crisis for a long time, after which it abruptly switches to a region of another ruin of the old attractors.

We investigate the relation between the switching among three tori and the switching among three chaotic regions. The switching among three tori can be explained as follows. Switching among three chaotic regions appears at any parameter for $S_3$ symmetric chaos, and the orbit often passes through the neighborhoods of the three tori. Resident time for each chaotic region becomes short if the parameter is far from the attractor-merging crisis. On the other hand, resident time for the neighborhood of the torus becomes longer if the linear instability of the torus is small, and becomes dominant if the linear instability is sufficiently small. This situation corresponds to switching among three tori.

This paper is organized as follows. In Sec. II we show our model equations: three repulsively coupled Stuart-Landau oscillators. In Sec. III the bifurcation structure for the inactive case is investigated. Intermittent switching among the $S_2$ tori is also discussed. In Sec. IV we investigate the bifurcation structure of the $S_2$ torus solution for the limit cycle case. Period doubling routes to chaos and attractor merging crises are shown. Section V discusses the relation between the switching behaviors observed in Secs. III and IV.

II. MODEL

We consider the following coupled Stuart-Landau oscillators [10]:

\[ \dot{z}_j = (\alpha + i)z_j - (1 - i\beta)|z_j|^2z_j + \kappa \sum_{n=1}^{N} (z_n - z_j), \]

where $N=3$. The variable $z_j$, which describes the state of the $j$th element, is a complex number, $\alpha$ is a parameter specifying the distance from the Hopf bifurcation, $\beta$ represents the amplitude dependency of phase velocity, and $\kappa < 0$ is the coupling strength.

In this model each oscillator is coupled to every other oscillator. Note that due to the symmetry, if one oscillating solution exists, then other oscillating solutions, obtained by permuting $(z_1, z_2, z_3)$, also exist. We define amplitude $r_j$ and phase $\theta_j$ as $r_j = |z_j|$ and $\theta_j = \arg z_j$, respectively.

Without coupling, each element $z_j$ converges to the limit cycle whose amplitude is $\sqrt{\alpha}$ if $\alpha > 0$, and settles down to the fixed point $z_j = 0$ if $\alpha = 0$. For attractive coupling $\kappa > 0$, the complete synchronized state $z_1 = z_2 = z_3$ is stable. The dynamics of oscillators are more complicated for repulsive coupling where $\kappa < 0$. Without loss of generality, we examine $\alpha = -1$ and $\alpha = 1$, both with $\beta \geq 0$.

If one of $z_1$, $z_2$, and $z_3$ is not 0, we can reduce the system to the one with lower dimension. Equation (1) can be rewritten as the following equation:

\[ w_j = (\alpha - |w_j|^2 + i\beta |w_j|^2 - |w_3|^2)w_j + \kappa \sum_{n=1}^{3} \left( w_n - w_j - i\omega \frac{\text{Im} w_n}{w_3} \right), \]

where $w_j \equiv z_j e^{-i\theta_3} = r_j e^{i\phi_j}$ and $\phi_j \equiv \theta_j - \theta_3$. This equation consists of five real variables, because the imaginary part of $\omega_3$ becomes 0.

Due to a shift invariance of the form $(\theta_1, \theta_2, \theta_3) \rightarrow (\theta_1 + c, \theta_2 + c, \theta_3 + c)$, steadily rotating solutions in Eq. (1) correspond to fixed point solutions of Eq. (2), and quasiperiodic solutions of Eq. (1), which arise from the secondary Hopf (Neimark-Sacker) bifurcation of steadily rotating solutions, correspond to periodic solutions of Eq. (2). Hence we follow a branch of the periodic solution of Eq. (2) instead of the corresponding quasiperiodic solution of Eq. (1). In the following sections, the Hopf and secondary Hopf bifurcations are discussed, however, we omit “secondary” if there is no misunderstanding.

III. INACTIVE CASE $\alpha = -1$

Let us first consider the case of $\alpha = -1$. In this case, when there is no coupling, each element does not have a limit cycle.

A. Periodic solutions

First, we show several periodic solutions, and how they bifurcate. Figure 1 shows a phase diagram of Eq. (1) obtained by AUTO. We found roughly five types of solutions in $(\kappa, \beta)$ parameter space and refer to these as trivial fixed point, rotating, partial antiphase, quasiperiodic, and chaotic. Even finer structure exists around a boundary of region II in Fig. 1 but we did not study this. A trivial fixed point $z_1 = z_2 = z_3 = 0$ is stable for $\kappa > -1/3$. For $\kappa < -1/3$, the trivial fixed point becomes unstable from the stability analysis of Eq. (1), and three types of periodic solutions, namely, stable rotating, unstable partial antiphase, and unstable partial in-phase, emerge from the bifurcation point of the trivial fixed point at $\kappa = -1/3$.

For general globally coupled identical elements, Theorem 4.1 from Chap. XVIII in Ref. [20] proves that rotating, partial antiphase, and partial in-phase solutions appear if the in-phase solution $(z_1 = z_2 = z_3)$ does not bifurcate from the Hopf bifurcation point of the trivial fixed point.

A rotating solution exists for $\kappa < -1/3$ and is stable in region II of Fig. 1. The rotating solution is obtained analytically from Eq. (1), and is written as
Numbered regions correspond to the following stable solutions. I: trivial fixed point; II: rotating; III: partial antiphase; IV: torus bifurcated from partial antiphase; V: chaotic. An intermittent switching solution is observed above the boundary between IV and V. The chaotic behavior is also observed in the white color region between II and IV. Where the meshed region II overlaps III and IV, both stable solutions are possible. Stable torus bifurcated from the rotating solution exists in a narrow region above the boundary of region II when \( \kappa > -0.39 \), but this is not illustrated here.

\[
(z_1(t),z_2(t),z_3(t)) = (f(t),f(t)e^{-2\pi i/3},f(t)e^{-4\pi i/3}),
\]

where \( f(t) = \text{Re}(e^{i(1+2\pi\kappa)t+\theta_0}) \). The rotating solution becomes unstable through a second Hopf bifurcation. The Hopf bifurcation line is illustrated as the upper boundary of region II in Fig. 1. An unstable quasiperiodic solution emerges from the Hopf bifurcation point. An inset of Fig. 1 shows that the rotating solution is stable if \( \kappa \) is sufficiently near the Hopf bifurcation point \( \kappa = -1/3 \). Clearly the effect of amplitude dependency will be very small for the rotating solutions of Eq. (3) in this region since \( R \) will be small.

A partial antiphase solution can be written as

\[
(z_1(t),z_2(t),z_3(t)) = (0,f(t),-f(t)).
\]

A partial antiphase solution is stable in region III. Note that region III is not in contact with region I in Fig. 1. A partial antiphase solution is unstable if \( \kappa \) is sufficiently near \(-1/3\) as shown in the inset of Fig. 1. The boundary of region III corresponds to secondary Hopf bifurcation points of the partial antiphase solution.

A partial in-phase solution is a periodic solution in which two elements have the same position, i.e., \( z_i = z_j \). A partial in-phase solution exists if \( \kappa < -1/3 \) and is always unstable.

**B. First route to \( S_3 \) chaos: From \( S_2 \) torus to \( S_3 \) chaos**

A quasiperiodic solution appears in region IV in Fig. 1. The boundary line between region III and region IV is the secondary Hopf bifurcation point of the partial antiphase solution. Decreasing \( \kappa \) from region III to region IV, a new oscillation mode appears. A time series of the quasiperiodic solution near the Hopf bifurcation point is shown in Figs. 2(a) and 2(b). Figure 2(a) shows that two elements oscillate with large amplitude with time period \( T \). The orbit of the solution on a coordinate which is rotating with \( \Omega_2 \) is shown in Fig. 2(c), where \( \Omega_2 \) is the mean phase velocity of \( \theta \). Figure 2(c) shows that the phase difference between them is around \( \pi \). It can also be seen from Fig. 2(a) that the other element with smaller amplitude oscillates with a different frequency. The mean phase velocities for elements obey the following relations: \( \Omega_1 = \Omega_3 - 2\pi/3 \) and \( \Omega_2 = \Omega_3 \). The periods \( T \) and \( 2\pi/\Omega_3 \) are the characteristic periods for the quasiperiodic solution.

Although this solution is quasiperiodic in fixed coordinates \( (z_1,z_2,z_3) \), Fig. 2 shows that amplitude \( r_j \) and phase difference \( \phi_{ij} \) of the quasiperiodic solution oscillate periodically. This is related to the fact that the partial antiphase solution corresponds to a fixed point in Eq. (2), as mentioned in the previous section. The quasiperiodic solution corresponds to a periodic solution of Eq. (2), and the period of the this solution is same as the one of \( (r_1,r_2,r_3) \) [Fig. 2(b)]. From a viewpoint of symmetry, Fig. 2(b) shows that \( r_2 \) oscillate at a time period of \( T \) with a half-period time lag, and
factor $K$ and $\phi$; with $\alpha=-1$, $\kappa=-3$, and $\beta=9.8$. The vertical line represents the start and end time of each laminar state. A schematic illustration of each state is shown above the time series. The two elements oscillating with largest amplitude are almost antiphase. The switching among the three $S_2$ tori appears to be random.

$r_1$ oscillates at a period of $T/2$. The attractor is invariant under permutation $(z_1, z_3)$ and so we refer to the attractor as an $S_2$ torus for convenience. $S_n$ represents the symmetric group consisting of all permutations of $n$ elements. The $S_2$ torus solution is stable in region IV and becomes unstable in region V as $\beta$ passes through the boundary between regions IV and V due to a subcritical pitchfork bifurcation at the boundary.

A chaotic solution emerges in the region illustrated as region V in Fig. 1. A time series of the chaotic solution is illustrated in Fig. 3. Figure 3 shows intermittent switching among three states, each of which corresponds to an unstable $S_2$ torus. The switching behavior is as follows. One element oscillates in a small amplitude orbit for a long time. At the end of this time the element leaves the small amplitude orbit and another element takes its place. In other words, the trajectory exhibits intermittent switching among three $S_2$ tori in phase space. The trajectory often visits the same $S_2$ torus in succession, however, the order in which the $S_2$ tori are visited seems random.

C. Average laminar length

We study the average laminar length of the switching solution close to the critical point $\beta_c$. The average laminar length $\langle l \rangle$ of this intermittency scales as follows [21]:

$$\langle l \rangle \propto (\beta - \beta_c)^{-\gamma}$$

for $\beta$ close to the critical bifurcation point $\beta_c$. The scaling factor $\gamma$ is called a critical exponent. We adopt a surface $r_2$ as a Poincaré section and plot the point $(w_1, w_2, w_3)$ when the trajectory crosses it with $r_1 - r_2 > 0$. We call that the system is in the laminar state when it is in a certain neighborhood of the unstable $S_2$ torus. The result of numerical simulation depicted in Fig. 3 shows $\gamma=1$. In general, intermittency was distinguished by the three types of bifurcation points [22]: type I, saddle node bifurcation; type II, subcritical Hopf bifurcation; and type III, subcritical period-doubling bifurcation. These types of intermittency are known as Pomeau-Manneville intermittency. The critical exponent $\gamma$ is 0.5 for type I intermittency and 1 for type II and type III intermittencies. In this case the critical bifurcation point is a subcritical pitchfork bifurcation of the $S_2$ torus solution. The intermittency after the pitchfork bifurcation does not belong to type I, II, or III. However, we can apply the analysis of type III intermittency to our system. In general, if period-doubling bifurcation takes place for a one-dimensional map $M$, the pitchfork bifurcation takes place for the two times iterated map $M^2$. Therefore the probability distribution for the laminar length can be estimated by the same procedure as for type III intermittency [23], which gives $\gamma=1$. This value of $\gamma$ matches the result of our numerical experiment. From the viewpoint of symmetry, a chaotic attractor which corresponds to the switching trajectory has $S_3$ symmetry; invariant from permutation $(z_1, z_2, z_3)$. We call the $S_3$ symmetric chaotic attractor $S_3$ chaos for convenience. In this section we observed the following route to chaos: trivial fixed point $\rightarrow$ partial antiphase $\rightarrow S_2$ torus $\rightarrow S_3$ chaos. In the next section we discuss the structure of $S_3$ chaos for $\alpha=1$.

IV. LIMIT CYCLE OSCILLATOR CASE $\alpha=1$

In this section we consider the case of $\alpha=1$. For $\alpha=1$, each element has a limit cycle when there is no coupling term. We will discuss other switching behaviors which show switching among chaotic states. We define the $S_1$ torus as a torus which is not invariant for any permutation of $(z_1, z_2, z_3)$.
FIG. 5. A two-parameter bifurcation diagram of the $S_2$ torus solution for $\alpha=1$. A stable $S_2$ torus exists in region IV which is located between solid lines representing pitchfork bifurcation points. The gray region represents a region where stable $S_1$ tori exist. Stable $S_2$ tori and $S_1$ tori coexist in the overlapped region. The bifurcation points are denoted as follows: PF, pitchfork bifurcation of $S_2$ torus; SN, saddle-node bifurcation of $S_1$ torus; PD, period-doubling bifurcation of $S_1$ torus; HC, heteroclinic bifurcation. There is a region near HC with large $\beta$, on whose boundary the $S_2$ torus solution bifurcates. However, the region is not illustrated here because we did not study it in any detail.

except the identity permutation. The $S_1$ torus is bifurcated from the symmetry breaking pitchfork bifurcation of the $S_2$ torus. To show the second route to the switching behavior, we focus on bifurcations of the $S_2$ torus, $S_1$ torus, and chaotic attractors originating from the $S_1$ torus.

A. Bifurcations from $S_2$ torus to $S_1$ torus

First, we follows a branch of $S_2$ torus which is the start point of the route to chaos. Figure 5 shows a two-parameter bifurcation diagram for the $S_2$ torus. The $S_2$ torus is stable in region IV, and was obtained by following the $S_2$ torus solution from $\alpha=1$ to $\alpha=1$. Unlike the result for $\alpha=-1$, there is no secondary Hopf bifurcation point which generates an $S_2$ torus for $\alpha=1$. Furthermore, trivial fixed point and partial antiphase solutions are unstable at any $(\kappa,\beta)$. Before we discuss the detail of the bifurcation structure, we make some comments on the solutions which are not related to this route. We note that the other types of stable solutions, rotating solution described in Eq. (3) and quasiperiodic solution bifurcated from the rotating solution, also exist in a part of the parameter space shown in Fig. 5. A trajectory of the quasiperiodic solution is on a torus which is invariant under a cyclic permutation (i.e., $Z_3$ symmetric). We confirm that those periodic and quasiperiodic solutions, except for the $S_1$ torus, are unstable above the upper boundary of region IV, which is labeled PF in Fig. 5, by following the branches numerically.

We consider the bifurcation structure of an $S_2$ torus which is invariant under a permutation $(z_2,z_3)$. If $\beta$ is increased from within region IV then the $S_2$ torus becomes unstable when $\beta$ passes through the pitchfork bifurcation point of the $S_2$ torus. The pitchfork bifurcation points correspond to the upper solid boundary line of region IV in Fig. 5. The lower solid boundary line to the left of point $P$ is also a pitchfork bifurcation line of the $S_2$ torus. The pitchfork bifurcation line on the upper boundary is supercritical for $\kappa>-1.19$ and sub-critical for $\kappa<-1.19$. A region where stable $S_1$ tori exist is illustrated by the gray region in Fig. 5. For $\kappa>-1.19$ stable $S_1$ tori bifurcate from the supercritical pitchfork bifurcation points, whereas for $\kappa<-1.19$ the branches are unstable. For $-3.76<\kappa<-1.19$ [Fig. 6(b)], the branches of unstable $S_1$ tori fold and become stable from saddle-node bifurcations. For $\kappa<-3.76$ [Fig. 6(a)], branches of $S_1$ tori, which bifurcated from the upper boundary of region IV, connect to the branch which bifurcated from the lower solid boundary.

A boundary labeled HC exists above the point $P$. Here we mention about the $S_2$ torus near the boundary labeled HC in Fig. 5 only briefly. As $\kappa$ approaches to HC from the left, the $S_2$ torus approaches two unstable partial in-phase solutions, and the trajectory spends increasingly longer times passing through the neighborhoods of partial in-phase solutions. For example, if the $S_2$ torus is invariant under a permutation $(z_2,z_3)$, the orbit approaches two unstable solutions; one is characterized by $z_1=z_2$ and the other one is characterized by $z_1=z_3$. On reaching HC the $S_1$ torus becomes a heteroclinic cycle connecting the pair of partial in-phase solutions.

B. Period doubling route to $S_1$ chaotic attractor

We investigate the bifurcation diagram of a stable $S_1$ torus at $\kappa=-3.7$ (Fig. 7). As $\beta$ is increased period-doubling bifurcations occur on a stable $S_1$ torus branch. The first period-doubling bifurcation line (PD) is illustrated in Fig. 5. An inset of Fig. 7 shows that the stable $S_1$ torus develops into a chaotic attractor after period-doubling bifurcations. We call this attractor, which is not invariant for any permutation of $(z_1,z_2,z_3)$ except the identity permutation, an $S_1$ chaotic attractor for convenience.

C. Attractor-merging crisis I: From $S_1$ chaotic attractors to $S_2$ chaotic attractor

Although each $S_1$ chaotic attractor is not symmetric itself, a pair of $S_1$ chaotic attractors which originally bifurcated
under a permutation of $S_3$. Furcated from the same $S_2$ torus, the two attractors merge into one attractor which has $S_3$ symmetry: invariant under a permutation of $(z_2, z_3)$. This type of global bifurcation is called an attractor-merging crisis [19].

The attractor-merging crisis induces intermittent switching between two chaotic attractors which exist before the crisis. The trajectory spends a long time $\tau$ in a region where the attractor existed before the crisis. After this time the orbit moves to the region where the other attractor previously existed. Figure 9 shows intermittent switching among two chaotic states. It can be seen that each chaotic state consists of short laminar phases, which stay near one of the two original $S_2$ tori branches, and burst phases. This property persists even when $\beta$ is not near $\beta_1$. Figure 10 shows that the average transient lifetime $\langle \tau \rangle$ scales as follows:

$$\langle \tau \rangle \propto (\beta - \beta_1)^{-\nu}$$

for $\beta$ close to $\beta_1$.

D. Attractor-merging crisis II: From $S_2$ chaotic attractors to $S_3$ chaos

Since this system has $S_3$ symmetry, there are three isolated $S_2$ chaotic attractors for $\beta_1 < \beta < \beta_2$. As $\beta$ exceeds $\beta_2$, the three mutually symmetric attractors are replaced by one large $S_3$ symmetric attractor, which is invariant to permutation $(z_1, z_2, z_3)$. We refer to the large $S_3$ symmetric attractor as $S_3$ chaos as in the previous section.

For $\beta$ slightly larger than $\beta_2$, the trajectory of $S_3$ chaos shows intermittent switching among the three $S_3$ chaotic regions which correspond to $S_1$ chaotic attractors before the crisis, as shown in Fig. 11. $\langle \tau_2 \rangle$ which represents the average lifetime of an $S_2$ chaotic region, becomes smaller with in-

FIG. 7. Bifurcation diagram of an $S_1$ torus on a Poincaré section $r_2=r_3$ for $\alpha=1.0, \kappa=-3.7, \beta_1=12.04774$, and $\beta_2=12.611$. Points on the section are plotted only if $r_3-r_2>0$. For $\beta<\beta_1$, black dots and gray dots represent the different attractors which originally bifurcated from the same $S_2$ torus. The two attractors merge into an $S_2$ chaotic attractor at $\beta_1$. Three $S_2$ chaotic attractors merge into one $S_3$ chaos at $\beta_2$ which is accompanied by a sudden widening. Note that the other two $S_2$ chaotic attractors are not plotted for $\beta<\beta_2$.

from the same $S_2$ torus is mutually symmetric, because of the symmetry of the system. The total number of $S_1$ chaotic attractors is 6 since three pairs exist. Increasing $\beta$, the distance between a pair of $S_1$ chaotic attractors decreases until the distance goes to zero and a crisis occurs at $\beta_1$. Figure 8 shows two $S_1$ chaotic attractors simultaneously colliding with an $S_2$ torus which is on the basin boundary between them, as $\beta$ passes through $\beta_1$. As $\beta$ exceeds the critical value $\beta_1$, the two mutually symmetric $S_1$ chaotic attractors will merge into one attractor which has $S_2$ symmetry: invariant under a permutation of $(z_2, z_3)$. This type of global bifurcation is called an attractor-merging crisis [19].

The attractor-merging crisis induces intermittent switching between two chaotic attractors which exist before the crisis. The trajectory spends a long time $\tau$ in a region where the attractor existed before the crisis. After this time the orbit moves to the region where the other attractor previously existed. Figure 9 shows intermittent switching among two chaotic states. It can be seen that each chaotic state consists of short laminar phases, which stay near one of the two original $S_2$ tori branches, and burst phases. This property persists even when $\beta$ is not near $\beta_1$. Figure 10 shows that the average transient lifetime $\langle \tau \rangle$ scales as follows:

$\langle \tau \rangle \propto (\beta - \beta_1)^{-\nu}$

for $\beta$ close to $\beta_1$.

D. Attractor-merging crisis II: From $S_2$ chaotic attractors to $S_3$ chaos

Since this system has $S_3$ symmetry, there are three isolated $S_2$ chaotic attractors for $\beta_1 < \beta < \beta_2$. As $\beta$ exceeds $\beta_2$, the three mutually symmetric attractors are replaced by one large $S_3$ symmetric attractor, which is invariant to permutation $(z_1, z_2, z_3)$. We refer to the large $S_3$ symmetric attractor as $S_3$ chaos as in the previous section.

For $\beta$ slightly larger than $\beta_2$, the trajectory of $S_3$ chaos shows intermittent switching among the three $S_3$ chaotic regions which correspond to $S_1$ chaotic attractors before the crisis, as shown in Fig. 11. $\langle \tau_2 \rangle$ which represents the average lifetime of an $S_2$ chaotic region, becomes smaller with in-

FIG. 8. Phase portrait of (a) $S_1$ chaotic attractors and (b) $S_2$ chaotic attractor on the Poincaré section $z_1=z_2$ for $\alpha=1, \kappa=-3.7$. Points on the section are plotted only if $r_3-r_2>0$. A white disk represents an unstable $S_3$ torus. Two $S_1$ chaotic attractors touch the $S_2$ torus at $\beta=\beta_2$ and an attractor-merging crisis occurs.

$\langle \tau \rangle \propto (\beta - \beta_1)^{-\nu}$

for $\beta$ close to $\beta_1$.

D. Attractor-merging crisis II: From $S_2$ chaotic attractors to $S_3$ chaos

Since this system has $S_3$ symmetry, there are three isolated $S_2$ chaotic attractors for $\beta_1 < \beta < \beta_2$. As $\beta$ exceeds $\beta_2$, the three mutually symmetric attractors are replaced by one large $S_3$ symmetric attractor, which is invariant to permutation $(z_1, z_2, z_3)$. We refer to the large $S_3$ symmetric attractor as $S_3$ chaos as in the previous section.

For $\beta$ slightly larger than $\beta_2$, the trajectory of $S_3$ chaos shows intermittent switching among the three $S_3$ chaotic regions which correspond to $S_1$ chaotic attractors before the crisis, as shown in Fig. 11. $\langle \tau_2 \rangle$ which represents the average lifetime of an $S_2$ chaotic region, becomes smaller with in-

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D. Attractor-merging crisis II: From $S_2$ chaotic attractors to $S_3$ chaos

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For $\beta$ slightly larger than $\beta_2$, the trajectory of $S_3$ chaos shows intermittent switching among the three $S_3$ chaotic regions which correspond to $S_1$ chaotic attractors before the crisis, as shown in Fig. 11. $\langle \tau_2 \rangle$ which represents the average lifetime of an $S_2$ chaotic region, becomes smaller with in-

FIG. 9. Time series of an $S_2$ chaotic attractor on the Poincaré section $z_1=z_2$ at $\alpha=1, \kappa=-3.7$, and $\beta=12.06$. Points on the section are plotted only if $r_3-r_2>0$. Intermittent switching among two states is clearly shown.

FIG. 10. Dependence of the average lifetime $\langle \tau \rangle$ with $\beta_1$ =12.01774. The dotted line indicates the linear fitting which yields a critical exponent $\nu=-0.59$. 
increasing $\beta - \beta_2$. When the trajectory is in an $S_2$ chaotic region, it shows switching among two $S_1$ chaotic regions, and frequently passes near the $S_2$ torus which exists between them, just as the trajectory on an $S_1$ chaotic attractor did. Note that these properties of the switching behavior are also observed if $\beta$ is substantially larger than $\beta_2$.

E. Symmetry transitions

To investigate how the symmetry of the attractor changes through the bifurcation diagram, we focus on the following properties of the symmetry: For $S_3$ chaos, $\langle r_j \rangle = \langle r_k \rangle = \langle r_l \rangle$ holds. On the other hand, for $S_2$ chaos and torus, there exists $j, k$, and $l$ with $j \neq k$ and $k \neq l$ such that $\langle r_j \rangle = \langle r_k \rangle \neq \langle r_l \rangle$. We introduce the order parameters $P_1 = |(r_1 + r_2 e^{i \pi/3} + r_3 e^{i 2\pi/3})|$ and $P_2 = |(r_1 - r_2)(r_2 - r_3)(r_3 - r_1)|$. $P_1$ becomes 0 if the attractor is invariant under any permutation of elements. Therefore $P_1$ is assumed to be the index of $S_3$ symmetry for the attractor. $P_2$ becomes 0 if the attractor is invariant under a permutation of one pair of elements $(r_j, r_k)$, therefore $P_2$ is assumed to be the index of $S_2$ symmetry. For $S_3$ chaotic attractors, $P_1$ and $P_2$ obey the following relations:

\[
S_3 \text{ chaos: } P_1 = 0, \quad P_2 = 0,
\]
\[
S_2 \text{ chaotic attractor: } P_1 \geq 0, \quad P_2 = 0,
\]

\[
S_1 \text{ chaotic attractor: } P_1 \geq 0, \quad P_2 \geq 0,
\]

Figure 12 shows that $P_1$ jumps down to 0 as $\beta$ exceeds $\beta_2$, and $P_2$ jumps down to 0 as $\beta$ exceeds $\beta_1$. This results imply that the symmetry of attractor changes from $S_1$ to $S_2$ as $\beta$ exceeds $\beta_1$, and from $S_2$ to $S_1$ and as $\beta$ exceeds $\beta_2$, respectively. Sharp peaks of $P_1$ and $P_2$ correspond to windows of the bifurcation diagram as shown in Fig. 12. For example, $S_1$ and $S_2$ symmetric tori appear at $\beta = 12.650, \ 13.102$, respectively.

V. DISCUSSION

We have presented several oscillating patterns and switching behaviors in three repulsively coupled Stuart-Landau equations. The number of the dimensions for our model is essentially 5. This model could be one of the simplest models which shows intermittent switching among three and more states. For sufficiently small $\beta$, the rotating solution is the only stable solution. For large $\beta$, $S_2$ torus and chaotic solutions appear. Our result implies that chaotic element and external noise are not necessary for the switching behaviors. For three globally coupled identical oscillators, it can be said that spontaneous switching behavior may occur when the amplitude dependency and repulsive coupling are sufficiently strong.

We reported two different routes to $S_1$ chaos for large $\beta$. One route is $S_2$ torus $\rightarrow$ $S_3$ chaos for $\alpha = -1$ [Fig. 13(b)]. In this route, intermittent switching among three $S_2$ tori are observed, and the average time between bursts scales as a power law in the difference of a parameter from its critical value. The other route consists of two parts. The first part is the creation of chaotic attractors through the period-doubling cascade: $S_2$ torus $\rightarrow$ $S_1$ torus $\rightarrow$ $S_1$ chaotic attractor. The second part is two successive attractor-merging crises: $S_1$ cha-
otic attractor $\rightarrow$ $S_2$ chaotic attractor $\rightarrow$ $S_3$ chaos, for $\alpha=1$ [Fig. 13(a)].

In the first route to $S_3$ chaos we found that the time series of an $S_1$ chaotic attractor showed switching among three laminar states near the pitchfork bifurcation point of the $S_2$ torus. Each laminar state corresponds to the neighborhood of an $S_2$ torus. In each laminar state, one element oscillates with a small amplitude and the other two elements oscillate with a large amplitude. We can explain the intermittent switching among $S_2$ tori (Fig. 3) as follows. We are far from the parameter regime where $S_2$ chaotic attractors merge into $S_3$ chaos. The trajectories therefore easily switch among $S_2$ chaotic regions. On the other hand, from the weak linear instability of the $S_2$ torus, the average lifetime of a laminar state which corresponds to an $S_2$ torus becomes large near the pitchfork bifurcation point. Due to the structure of the $S_2$ chaotic region that contains the $S_2$ torus, the laminar state therefore becomes dominant in each $S_2$ chaotic region and it is also dominant in the time series (Fig. 3) if the set of parameters is sufficiently near the bifurcation point.

To show the continuity of $S_1$ chaos from $\alpha=-1$ to $\alpha=1$, we perform the following transformation. If $\alpha-3\kappa>0$, Eq. (1) can be transformed to the following form:

$$\frac{dZ_j}{dt} = \left(1 + \frac{i\kappa}{\eta}\right)Z_j - (1 - i\beta)|Z_j|^2Z_j + \eta\sum_{n=1}^{N} Z_n,$$

where $Z_j = (\alpha-3\kappa)\kappa_j$, $t' = (\alpha-3\kappa)t$, $\eta = \kappa/(\alpha-3\kappa)$. $\eta < -1/3$ corresponds to $\kappa < -1/3$ at $\alpha=1$, and $\eta > -1/3$ corresponds to $\kappa < 0$ at $\alpha=1$. $\eta = -1/3$ corresponds to the limit $\kappa \to -\infty$ at $\alpha = \pm 1$. Figure 14 shows a one-parameter bifurcation diagram of an $S_2$ torus at $\beta=12.0$. It shows two routes to the onset of $S_3$ chaos which then seems to be continuous within the intermediate region.

As mentioned in Sec. II, for the $S_2$ torus, one element oscillates with smaller amplitude, and the other two elements oscillate with larger amplitude. This type of separation of elements is an important role in our system. The study of whether separation of elements has an important role for coupled $N$ oscillator systems, and detailed global bifurcation analyses of these systems, will be interesting future work.

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