Flat Lightlike Hypersurfaces in Lorentz-Minkowski 4-space

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Abstract

The lightlike hypersurfaces in Lorentz-Minkowski space are of special interest in Relativity Theory. In particular, the singularities of these hypersurfaces provide good models for the study of different horizon types. We introduce the notion of flatness for these hypersurfaces and study their singularities. The classification result asserts that a generic classification of flat lightlike hypersurfaces is quite different from that of generic lightlike hypersurfaces.

1 Introduction

The extrinsic differential geometry of submanifolds in 4-dimensional Lorentz-Minkowski space is of special interest in Relativity Theory. In particular the lightlike hypersurfaces, which can be constructed as lightlike ruled hypersurfaces over spacelike surfaces, provide good models for the study of different horizon types ([3], [23]). In this sense, the singularities of lightlike hypersurfaces are deeply related to the shapes of horizons. With the aim of studying the extrinsic geometry of lightlike hypersurfaces in 4-dimensional Lorentz-Minkowski space, M. Kossowski introduced ([17], [18]) a Gauss map on its associated spacelike surface, obtaining in this way interesting conclusions on the lightlike hypersurfaces which parallel known results for surfaces in Euclidean 3-space concerning their contacts with the model surfaces (planes and spheres). In order to generalize this method to Lorentz-Minkowski space, we considered an approach from the view point of the theory of Legendrian/Lagrangian singularities [11, 13, 14]. When working in Lorentz-Minkowski space, we observe that the properties associated to the contacts of a given submanifold with lightcones and lightlike hyperplanes have a special relevance from the geometrical viewpoint. A local classification of the generic singularities of the lightlike hypersurfaces in terms of algebraic and differential geometric invariants was obtained in [13] (cf., Theorem 2.5). In [14], we pursued with this line by describing the Lorentz invariant geometric properties of spacelike submanifolds of codimension two in Minkowski space

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that arise from their contacts with lightlike hyperplanes. For this purpose, we studied some local properties of these spacelike submanifolds. Given such a submanifold, we can arbitrarily choose a future directed timelike normal vector field $\boldsymbol{n}^T$ along it. Once $\boldsymbol{n}^T$ is fixed, there are two possibilities for the choice of a normal frame class: future directed frames $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ and orientation reversing future directed frames $(\boldsymbol{n}^T, -\boldsymbol{n}^S)$. We can associate to any one of these frames the notion of lightcone Gauss-Kronecker curvature $K_\ell(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$. This depends on the particular choice of the frame $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$, but it leads after normalization to a normalized lightcone Gauss-Kronecker curvature $\widetilde{K}_\ell^{\pm}$ which is independent of the choice of the future directed normal frame $(\boldsymbol{n}^T, \pm\boldsymbol{n}^S)$. In order to investigate its associated geometrical properties, we have chosen here the class of future directed frames, but it is clear that the results for the orientation reversing choice would run in a parallel way. We also observe that an initial choice of a past directed unit normal vector field $-\boldsymbol{n}^T$ would lead to parallel results. We analyze the geometric meaning of the normalized lightcone Gauss-Kronecker curvature from the view point of Lagrangian and Legendrian singularity theory. In the present paper we shall concentrate our attention in the properties related with flatness with respect to this curvature. Here we have, by definition, that $\widetilde{K}_\ell(p) = 0$ if and only if $K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(p) = 0$, for any future directed frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. In particular, we seek first for a characterization of flatness (i.e. $\widetilde{K}_\ell(p) = 0$) for the spacelike surfaces in Lorentz-Minkowski 4-space. Once done this, we can introduce the concept of flatness for lightlike hypersurfaces in Lorentz-Minkowski 4-space. We characterize the flat lightlike hypersurfaces as envelopes of certain families of lightlike hyperplanes and study their generic singularities.

The distribution of the paper is as follows. We include in § 2 the basic notions in Lorentz-Minkowski space that shall be used throughout the paper and give a short review on the lightcone Gauss-Kronecker curvature and the lightlike hypersurfaces, which are the main objects in this theory [11–14]. Section 3 is devoted to the study of spacelike surfaces with vanishing lightcone Gauss-Kronecker curvature. These are called lightlike flat spacelike surfaces. In particular, we shall pay especial attention to those that admit a partially parallel normal frame. By using this setting, we can introduce in §4 the notion of flat lightlike hypersurface. Such a hypersurface can be seen as one-parameter family of lightlike lines along a lightlike flat spacelike surface. We also introduce in this section the basic invariants for the flat lightlike hypersurfaces. It should be noticed that the singularities of certain classes of surfaces (i.e., kinds of “flat”surfaces) have been recently investigated by several authors ([6, 9, 15, 16, 21]) from a differential geometry viewpoint. One of the main purposes of the submanifold theory in differential geometry is to study some special classes of submanifolds in different ambient spaces such as “flat”surfaces. In general, such surfaces have singularities. Therefore the classification of the singularities of these kinds of spacial surfaces is an interesting topic in differential geometry too. In §5 we give a classification of singularities of flat lightlike hypersurfaces by using the basic invariants. As a consequence, we get that the generic singularities are the suspended cuspidal edge, the suspended swallowtail, the suspended cuspidal cross cap and the A_4-type hypersurface singularity. We emphasize that we give the exact conditions for these singularities in terms of the basic invariants. Compared with the generic singularities for the general class of lightlike hypersurfaces (cf., Theorem 2.5), the $D_4^{\pm}$-type singularities do not appear generically as singularities of flat lightlike hypersurfaces. On the other hand, we observe that the suspended cuspidal cross cap does not appear as a generic singularity in the general case of lightlike hypersurfaces. Finally, §6 contains the classification of the generic singularities of flat spacelike surfaces with partially parallel normal frame.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Local differential geometry on spacelike surface in Lorentz-Minkowski space

We introduce in this section some basic notions on Minkowski 4-space and spacelike surfaces. For basic concepts and properties, see [26].

Let $\mathbb{R}^4 = \{(x_0, x_1, x_2, x_3) \mid x_i \in \mathbb{R}\ (i = 0, 1, 2, 3)\ \}$ be an 4-dimensional cartesian space. For any $\boldsymbol{x} = (x_0, x_1, x_2, x_3),\ \boldsymbol{y} = (y_0, y_1, y_2, y_3) \in \mathbb{R}^4$, the *pseudo scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{3} x_i y_i.$$

We call $(\mathbb{R}^4, \langle , \rangle)$ the *Minkowski* 4*-space*. We shall write $\mathbb{R}^4_1$ instead of $(\mathbb{R}^4, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is defined by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have a canonical projection $\pi : \mathbb{R}^4_1 \longrightarrow \mathbb{R}^3$ defined by $\pi(x_0, x_1, x_2, x_3) = (x_1, x_2, x_3)$. Here we identify $\{\mathbf{0}\} \times \mathbb{R}^3$ with $\mathbb{R}^3$ and it is considered as Euclidean 3-space whose scalar product is induced by the pseudo scalar product $\langle , \rangle$. For a vector $\boldsymbol{v} \in \mathbb{R}^4_1$ and a real number c, we define a *hyperplane with pseudo normal* $\boldsymbol{v}$ by

$$HP(\boldsymbol{v}, c) = \{\boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{v} \rangle = c\ \}.$$

We call $HP(\boldsymbol{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if $\boldsymbol{v}$ is timelike, spacelike or lightlike respectively.

We now define the *Hyperbolic* 3*-space* by

$$H^3_+(-1) = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1, x_0 > 0\}$$

and the *de Sitter* 3*-space* by

$$S^3_1 = \{\boldsymbol{x} \in \mathbb{R}^4_1 | \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\ \}.$$

We also consider the cone

$$LC^* = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1\ | x_0 \neq 0,\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}$$

and call it *the* (*open*) *lightcone* at the origin. The *future lightcone* is the subset

$$LC^*_+ = \{\boldsymbol{x} \in LC^*\ | x_0 > 0, \}$$

If $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\widetilde{\boldsymbol{x}} = \left(1, \frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0}\right) \in S^2_+ = \{\boldsymbol{x} = (x_0, x_1, x_2, x_3) \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0,\ x_0 = 1\}.$$

We call S^2_+ the *lightcone* (or, *spacelike*) *unit sphere.*

Given vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \mathbb{R}^4_1$, we define their wedge product $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ as

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ is the canonical basis of $\mathbb{R}^4_1$ and $\boldsymbol{x}_i = (x_0^i, x_1^i, x_2^i, x_3^i)$. We can easily check that

$$\langle \boldsymbol{x}, \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3 \rangle = \det(\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3),$$

so that $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$ is pseudo orthogonal to any $\boldsymbol{x}_i$ $(i = 1, 2, 3)$.

We now recall the basic geometrical tools for the study of spacelike surfaces in Minkowski 4-space that were developed in [11, 14]). Let $\mathbb{R}^4_1$ be an oriented and timelike oriented space. We choose $\boldsymbol{e}_0 = (1, 0, 0, 0)$ as the future timelike vector field. Consider a spacelike embedding $\boldsymbol{X} : U \longrightarrow \mathbb{R}^4_1$ from an open subset $U \subset \mathbb{R}^2$ and write $M = \boldsymbol{X}(U)$ by identifying M and U through the embedding $\boldsymbol{X}$. We say that $\boldsymbol{X}$ is *spacelike* if $\boldsymbol{X}_{u_i}$ $i = 1, 2$ are always spacelike vectors. Therefore, the tangent plane T_pM of M at p is a spacelike plane (i.e., consists of spacelike vectors) for any point $p \in M$ and the pseudo-normal space N_pM is a timelike plane (i.e., Lorentz plane) (cf., [26]). We denote by $N(M)$ the pseudo-normal bundle over M. Since this is a trivial bundle, we can arbitrarily choose a future directed unit timelike normal section $\boldsymbol{n}^T(u) \in N_p(M)$, where $p = \boldsymbol{X}(u)$. Here, we say that $\boldsymbol{n}^T$ is *future directed* if $\langle \boldsymbol{n}^T, \boldsymbol{e}_0 \rangle < 0$. Therefore we can construct a spacelike unit normal section $\boldsymbol{n}^S(u) \in N_p(M)$ by

$$\boldsymbol{n}^S(u) = \frac{\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)}{\|\boldsymbol{n}^T(u) \wedge \boldsymbol{X}_{u_1}(u) \wedge \boldsymbol{X}_{u_2}(u)\|},$$

and we have $\langle \boldsymbol{n}^T, \boldsymbol{n}^T \rangle = -1$, $\langle \boldsymbol{n}^T, \boldsymbol{n}^S \rangle = 0$, $\langle \boldsymbol{n}^S, \boldsymbol{n}^S \rangle = 1$. Although we could also choose $-\boldsymbol{n}^S(u)$ as a spacelike unit normal section with the above properties, we fix the direction $\boldsymbol{n}^S(u)$ throughout this paper. We call $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ a *future directed normal frame* along $M = \boldsymbol{X}(U)$. Clearly, the vectors $\boldsymbol{n}^T(u) \pm \boldsymbol{n}^S(u)$ are lightlike. Again, we choose $\boldsymbol{n}^T + \boldsymbol{n}^S$ as a lightlike normal vector field along M. Since $\{\boldsymbol{X}_{u_1}(u), \boldsymbol{X}_{u_2}(u)\}$ is a basis of T_pM, the system $\{\boldsymbol{n}^T(u), \boldsymbol{n}^S(u), \boldsymbol{X}_{u_1}(u), \boldsymbol{X}_{u_2}(u)\}$ provides a basis for $T_p\mathbb{R}^4_1$. The following lemma has been shown in [14]:

Lemma 2.1 *Given two future directed unit timelike normal sections* $\boldsymbol{n}^T(u), \bar{\boldsymbol{n}}^T(u) \in N_p(M)$, *the corresponding lightlike normal sections* $\boldsymbol{n}^T(u) + \boldsymbol{n}^S(u), \bar{\boldsymbol{n}}^T(u) + \bar{\boldsymbol{n}}^S(u)$ *are parallel.*

Under the identification of M and U through $\boldsymbol{X}$, we have the linear mapping provided by the derivative of the lightcone normal vector field $\boldsymbol{n}^T + \boldsymbol{n}^S$ at each point $p \in M$,

$$d_p(\boldsymbol{n}^T + \boldsymbol{n}^S) : T_pM \longrightarrow T_p\mathbb{R}^4_1 = T_pM \oplus N_p(M).$$

Consider the orthogonal projections $\pi^t : T_pM \oplus N_p(M) \to T_p(M)$ and $\pi^n : T_p(M) \oplus N_p(M) \to N_p(M)$. We define

$$d_p(\boldsymbol{n}^T + \boldsymbol{n}^S)^t = \pi^t \circ d_p(\boldsymbol{n}^T + \boldsymbol{n}^S)$$

and

$$d_p(\boldsymbol{n}^T + \boldsymbol{n}^S)^n = \pi^n \circ d_p(\boldsymbol{n}^T + \boldsymbol{n}^S).$$

We respectively call the linear transformations $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S) = -d_p(\boldsymbol{n}^T+\boldsymbol{n}^S)^t$ and $d_p(\boldsymbol{n}^T+\boldsymbol{n}^S)^n$ of T_pM, the $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-*shape operator* of $M=\boldsymbol{X}(U)$ at $p=\boldsymbol{X}(u)$ and the *normal connection with respect to* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ of $M=\boldsymbol{X}(U)$ at $p=\boldsymbol{X}(u)$.

The eigenvalues of $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S)$, denoted by $\{\kappa_i(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)\}_{i=1}^2$, are called the *lightcone principal curvatures* with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at p. Then the *lightcone Gauss-Kronecker curvature* with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ at $p=\boldsymbol{X}(u)$ is defined as

$$K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)(p) = \det S_p(\boldsymbol{n}^T,\boldsymbol{n}^S).$$

We say that a point $p=\boldsymbol{X}(u)$ is a $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-*umbilic point* if all the principal curvatures coincide at p and thus $S_p(\boldsymbol{n}^T,\boldsymbol{n}^S)=\kappa(\boldsymbol{n}^T,\boldsymbol{n}^S)(p)1_{T_pM}$, for some function κ. We say that $M=\boldsymbol{X}(U)$ is *totally* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-*umbilic* if all points on M are $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-umbilic.

We deduce now the lightcone Weingarten formula. Since $\boldsymbol{X}_{u_i}$ $(i=1,2)$ are spacelike vectors, we have a Riemannian metric (the *hyperbolic first fundamental form*) on $M=\boldsymbol{X}(U)$ defined by $ds^2=\sum_{i=1}^2 g_{ij}du_idu_j$, where $g_{ij}(u)=\langle \boldsymbol{X}_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle$ for any $u\in U$. We also have a *lightcone second fundamental invariant with respect to the normal vector field* $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ defined by $h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u)=\langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i}(u),\boldsymbol{X}_{u_j}(u)\rangle$ for any $u\in U$. The following Weingarten formulae with respect to $(\boldsymbol{n}^T,\boldsymbol{n}^S)$ was shown in [14]:

(a) $(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i} = \langle \boldsymbol{n}^S,\boldsymbol{n}^T_{u_i}\rangle(\boldsymbol{n}^T+\boldsymbol{n}^S)-\sum_{j=1}^2 h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$

(b) $\pi^t\circ(\boldsymbol{n}^T+\boldsymbol{n}^S)_{u_i} = -\sum_{j=1}^2 h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\boldsymbol{X}_{u_j}$.

Here $\left(h_i^j(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)=\left(h_{ik}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)\left(g^{kj}\right)$ and $\left(g^{kj}\right)=(g_{kj})^{-1}$. It follows that we have an explicit expression for the lightcone curvature in terms of the Riemannian metric and the lightcone second fundamental invariant.

$$K_\ell(\boldsymbol{n}^T,\boldsymbol{n}^S)=\frac{\det\left(h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)\right)}{\det\left(g_{\alpha\beta}\right)}.$$

Since $\langle -(\boldsymbol{n}^T+\boldsymbol{n}^S)(u),\boldsymbol{X}_{u_j}(u)\rangle=0$, we have $h_{ij}(\boldsymbol{n}^T,\boldsymbol{n}^S)(u)=\langle \boldsymbol{n}^T(u)+\boldsymbol{n}^S(u),\boldsymbol{X}_{u_iu_j}(u)\rangle$. Therefore the lightcone second fundamental invariant at a point $p_0=\boldsymbol{X}(u_0)$ depends only on the values, $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$, respectively assumed by the vector fields $\boldsymbol{n}^T+\boldsymbol{n}^S$ and $\boldsymbol{X}_{u_iu_j}$ at the point p_0. And thus, the lightcone curvature depends only on $\boldsymbol{n}^T(u_0)+\boldsymbol{n}^S(u_0)$, $\boldsymbol{X}_{u_i}(u_0)$ and $\boldsymbol{X}_{u_iu_j}(u_0)$ too, independently of the choice of the normal vector fields $\boldsymbol{n}^T$ and $\boldsymbol{n}^S$. We write $K_\ell(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)(u_0)$ as the lightcone curvature at $p_0=\boldsymbol{X}(u_0)$ with respect to $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)=(\boldsymbol{n}^T(u_0),\boldsymbol{n}^S(u_0))$. We might also say that a point $p_0=\boldsymbol{X}(u_0)$ is $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-*umbilic* because the lightcone $(\boldsymbol{n}^T,\boldsymbol{n}^S)$-shape operator at p_0 only depends on the normal vectors $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$.

Analogously, we say that a point $p_0=\boldsymbol{X}(u_0)$ is a $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-*parabolic point* of $\boldsymbol{X}:U\longrightarrow\mathbb{R}_1^4$ if $K_\ell(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)(u_0)=0$. And we say that a point $p_0=\boldsymbol{X}(u_0)$ is a $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-*flat point* if it is an $(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)$-umbilic point and $K_\ell(\boldsymbol{n}_0^T,\boldsymbol{n}_0^S)(u_0)=0$. By Lemma 2.1, if we choose another future directed unit timelike normal section $\bar{\boldsymbol{n}}^T(u)$, then we have $\widetilde{(\boldsymbol{n}^T+\boldsymbol{n}^S)}(u)=\widetilde{(\bar{\boldsymbol{n}}^T+\bar{\boldsymbol{n}}^S)}(u)\in S_+^2$. Therefore we define the *lightcone Gauss map* of $M=\boldsymbol{X}(U)$ as

$$\begin{array}{rccc}\widetilde{\mathbb{L}}: & U & \longrightarrow & S_+^2\\ & (u) & \longmapsto & \widetilde{(\boldsymbol{n}^T+\boldsymbol{n}^S)}(u).\end{array}$$

This induces a linear mapping $d\widetilde{\mathbb{L}}_p:T_pM\longrightarrow T_p\mathbb{R}_1^4$ under the identification of U and M, where $p=\boldsymbol{X}(u)$. The following proposition is shown in [14]:

Proposition 2.2 *Under the above notation, we have the following normalized lightcone Weingarten formula:*

$$\pi^t \circ \widetilde{\mathbb{L}}_{u_i} = -\sum_{j=1}^{2} \frac{1}{\ell_0(u)} h_i^j(\boldsymbol{n}^T, \boldsymbol{n}^S)\boldsymbol{X}_{u_j},$$

where $\mathbb{L}(u) = (\ell_0(u), \ell_1(u), \ell_2(u), \ell_3(u))$.

We call the linear transformation $\widetilde{S}_p = -\pi^t \circ d\widetilde{\mathbb{L}}_p$ the *normalized lightcone shape operator* of $M = \boldsymbol{X}(U)$ at p. The eigenvalues $\{\widetilde{\kappa}_i(p)\}_{i=1}^2$ of $\widetilde{S}_p$ are called *normalized lightcone principal curvatures*. By the above proposition, we have $\widetilde{\kappa}_i(p) = (1/\ell_0)\kappa_i(\boldsymbol{n}^T, \boldsymbol{n}^S)(p)$. The *normalized lightcone Gauss-Kronecker curvature* of $M = \boldsymbol{X}(U)$ is defined to be $\widetilde{K}_\ell(u) = \det \widetilde{S}_p$. Then we have the following relation between the normalized lightcone Gauss-Kronecker curvature and the lightcone Gauss-Kronecker curvature:

$$\widetilde{K}_\ell(u) = \left(\frac{1}{\ell_0(u)}\right)^2 K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)(u).$$

It is clear from the corresponding definitions that the lightcone Gauss map, the normalized lightcone principal curvatures and the normalized lightcone Gauss-Kronecker curvatures are independent on the choice of the normal frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$.

We say that a point $u \in U$ or $p = \boldsymbol{X}(u)$ is a *lightlike umbilical point* if $\widetilde{S}_p = \widetilde{\kappa}(p)1_{T_pM}$. By the above proposition, p is a lightlike umbilic point if and only if it is a $(\boldsymbol{n}^T, \boldsymbol{n}^S)$-umbilic point for any $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. We say that $M = \boldsymbol{X}(U)$ is *totally lightlike umbilic* if all points on M are lightlike umbilic, as usual. We also say that $p = \boldsymbol{X}(u)$ is a *lightlike parabolic point* if $\widetilde{K}_\ell(u) = 0$. Moreover, p is called a *lightlike flat point* if p is both lightlike umbilic and parabolic. The spacelike surface $M = \boldsymbol{X}(U)$ is called *totally lightlike flat* provided every point of M is lightlike flat.

Given the spacelike surface $M = \boldsymbol{X}(U)$, we construct a lightlike hypersurface

$$LH_X : U \times \mathbb{R} \longrightarrow \mathbb{R}^4_1$$

given by

$$LH_X(u, r) = \boldsymbol{X}(u) + r(\boldsymbol{n}^T + \boldsymbol{n}^S)(u),$$

where $p = \boldsymbol{X}(u)$. We shall denote $LH_M = LH_X(U \times \mathbb{R})$. We call LH_M the *lightlike hypersurface* along M and the parametrization $LH_X(u, r)$ is referred to as an *adapted parametrization* of the lightlike hypersurface LH_M. We remark that we can also define $LH_X^-(p, r) = \boldsymbol{X}(u) + r(\boldsymbol{n}^T - \boldsymbol{n}^S)(u)$ as another lightlike hypersurface. Since the properties of LH_X^- are the same as those of LH_X, we shall only consider here LH_X.

In general, a hypersurface $H \subset \mathbb{R}^4_1$ is said to be a *lightlike hypersurface* if it is tangent to a lightcone at any point. It is known that any lightlike hypersurface is given, at least locally, by the construction above (cf. [18] and §6).

We define the family of *Lorentzian distance-squared functions* on a spacelike surface $M = \boldsymbol{X}(U)$ as the family $G : M \times \mathbb{R}^4_1 \longrightarrow \mathbb{R}$ given by

$$G(p, \boldsymbol{\lambda}) = G(u, \boldsymbol{\lambda}) = \langle \boldsymbol{X}(u) - \boldsymbol{\lambda}, \boldsymbol{X}(u) - \boldsymbol{\lambda} \rangle,$$

where $p = \boldsymbol{X}(u)$.

For any fixed $\boldsymbol{\lambda}_0 \in \mathbb{R}^4_1$, we write $g(p) = G_{\lambda_0}(p) = G(p, \boldsymbol{\lambda}_0)$. The following proposition has been shown in [13].

Proposition 2.3 *Let M be a spacelike surface in $\mathbb{R}^4_1$ and $G : M \times \mathbb{R}^4_1 \to \mathbb{R}$ the Lorentzian distance-squared function on M. Suppose that $p_0 \neq \boldsymbol{\lambda}_0$. Then we have the following*:

(1) $g(p_0) = \partial g/\partial u_i(p_0) = 0$ $(i = 1, 2)$ *if and only if* $p_0 - \boldsymbol{\lambda}_0 = \mu(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(p_0)$ *for some* $\mu \in \mathbb{R} \setminus \{0\}$.

(2) $g(p_0) = \partial g/\partial u_i(p_0) = \det\mathcal{H}(g)(p_0) = 0$ $(i = 1, 2)$ (*where* $\det\mathcal{H}(g)(p_0)$ *is the determinant of the Hessian matrix*) *if and only if*

$$p_0 - \boldsymbol{\lambda}_0 = \mu(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(p_0)$$

for some $\mu \in \mathbb{R} \setminus \{0\}$ *such that* $1/\mu$ *is one of the non-zero lightcone principal curvatures* $\kappa_i^{\mp}(\boldsymbol{n}^T, \boldsymbol{n}^S)(p_0), (i = 1, 2)$.

We remark that the framework used in [13] was different and thus the corresponding notation differs a little from that of the above proposition. An immediate consequence of Proposition 2.3 is that the discriminant set of the Lorentzian distance-squared function G is given by

$$\mathcal{D}_G = \left\{ \boldsymbol{\lambda} \;\middle|\; \boldsymbol{\lambda} = \boldsymbol{X}(p) + u(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(p),\ p \in M, u \in \mathbb{R} \right\},$$

which is the image of the lightlike hypersurface along M. It follows that a singular point of the lightlike hypersurface is a point $\boldsymbol{\lambda}_0 = \boldsymbol{X}(p_0)+u_0(\boldsymbol{n}^T \pm \boldsymbol{n}^S)(p_0)$ at which $u_0 = -1/\kappa_i^{\mp}(\boldsymbol{n}^T, \boldsymbol{n}^S)(p_0)$, $i = 1, 2$.

We can interpret such a correspondence from the contact geometry viewpoint. Let $\pi : PT^*\mathbb{R}^4_1 \longrightarrow \mathbb{R}^4_1$ be the projective cotangent bundle with its canonical contact structure. We next review the geometric properties of this bundle. Consider the tangent bundle $\tau : TPT^*\mathbb{R}^4_1 \to PT^*\mathbb{R}^4_1$ and the differential map $d\pi : TPT^*\mathbb{R}^4_1 \to T\mathbb{R}^4_1$ of π. For any $X \in TPT^*\mathbb{R}^4_1$, there exists an element $\alpha \in T^*\mathbb{R}^4_1$ such that $\tau(X) = [\alpha]$. Given $V \in T_x\mathbb{R}^4_1$, the property $\alpha(V) = 0$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*\mathbb{R}^4_1$ by

$$K = \{X \in TPT^*(\mathbb{R}^4_1) \mid \tau(X)(d\pi(X)) = 0\}.$$

Via the coordinates (v_0, v_1, v_2, v_3), we have the trivialization $PT^*\mathbb{R}^4_1 \cong \mathbb{R}^4_1 \times P^3(\mathbb{R})^*$, and call

$$((v_0, v_1, v_2, v_3), [\xi_0 : \xi_1 : \xi_2 : \xi_3])$$

the *homogeneous coordinates* of $PT^*\mathbb{R}^4_1$, where $[\xi_0 : \xi_1 : \xi_2 : \xi_3]$ are the homogeneous coordinates of the dual projective space $P^3(\mathbb{R})^*$.

It is easy to show that $X \in K_{(x,[\xi])}$ if and only if $\sum_{i=0}^3 \mu_i\xi_i = 0$, where $d\tilde{\pi}(X) = \sum_{i=0}^3 \mu_i\partial/\partial v_i$. An immersion $i : L \to PT^*(\mathbb{R}^4_1)$ is said to be *a Legendrian immersion* if $\dim L = 3$ and $di_q(T_qL) \subset K_{i(q)}$ for any $q \in L$. The map $\pi \circ i$ is also called *the Legendrian map* and the set $W(i) = \text{image}\, \pi \circ i$, the *wave front* of i. Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$. In Appendix A, we include a quick survey of the theory of Legendrian singularities. For additional definitions and basic results on generating families, we refer to ([1], Chapter 21). By the previous arguments we have that the lightlike hypersurface $LH^{\pm}_M$ coincides with the discriminant set of the Lorentzian distance-squared function G. We also have the following proposition (See Appendix A for the definition of a Morse family) (proven in [13]).

Proposition 2.4 *Let G be the Lorentzian distance-squared function on M. The family G is a Morse family in a neighborhood of any point $(u, \boldsymbol{\lambda}) \in G^{-1}(0)$.*

Since G is a Morse family, if we denote

$$\Sigma_*(G) = (\Delta^* G)^{-1}(0) = \{(u, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} = LH^{\pm}_M(u,t) \text{ for some } t \in \mathbb{R}\}$$

with $\Delta^* G$ a map germ as described in Appendix A, we have a Legendrian immersion

$$L^{\pm}_G : \Sigma_*(G) \longrightarrow PT^*(\mathbb{R}^3_1)$$

given by

$$L^{\pm}_G(u, \boldsymbol{\lambda}) = (\boldsymbol{\lambda}, [(X_0(u) - \lambda_0) : (\lambda_1 - X_1(u)) : (\lambda_2 - X_2(u)) : (\lambda_3 - X_3(u))]).$$

We observe that G is a generating family of the Legendrian immersion $L^{\pm}_G$ whose wave front is $LH^{\pm}_M$ (cf. Appendix A). Therefore we might say that the Lorentzian distance-squared function G on M provides a Minkowski-canonical generating family for the Legendrian lift of $LH^{\pm}_X$. In [13], we have investigated the meaning of singularities of the lightlike hypersurface $LH^{\pm}_X$ from the view point of Legendrian singularity theory and given a classification of the generic singularities of lightlike hypersurfaces as follows:

Theorem 2.5 ([13], Theorem 5.2 and Corollary 5.3) *There exists an open dense subset $\mathcal{O} \subset \mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^4_1)$ such that for any $\boldsymbol{X} \in \mathcal{O}$, the germ of the Legendrian lift of the corresponding lightlike hypersurface $LH^{\pm}_M$ at each point is Legendrian stable. Here,* $\mathrm{Emb}_{\mathrm{sp}}(U, \mathbb{R}^4_1)$ *is the space of spacelike embeddings from an open subset $U \subset \mathbb{R}^2$ equipped with the Whitney C^∞-topology.*

Moreover, by the classification results on stable Legendrian mappings (cf., [29]), *the corresponding lightlike hypersurface (wave front of $L^{\pm}_G$) $LH^{\pm}_M$ at any point $(x, y, u) \in U \times \mathbb{R}$ is diffeomorphic to the image of one of the map germs A_k $(1 \le k \le 4)$ or $D^{\pm}_4$: where A_k and $D^{\pm}_4$ represent map-germs $f : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^4, 0)$ given by*

(A_1) $f(u_1, u_2, u_3) = (u_1, u_2, u_3, 0)$,
(A_2) $f(u_1, u_2, u_3) = (3u_1^2, 2u_1^3, u_2, u_3)$,
(A_3) $f(u_1, u_2, u_3) = (4u_1^3 + 2u_1u_2, 3u_1^4 + u_2u_1^2, u_2, u_3)$,
(A_4) $f(u_1, u_2, u_3) = (5u_1^4 + 3u_2u_1^2 + 2u_1u_3, 4u_1^5 + 2u_2u_1^3 + u_3u_1^2, u_1, u_2)$,
(D_4^+) $f(u_1, u_2, u_3) = (2(u_1^2 + u_2^2) + u_1u_2u_3, 3u_1^2 + u_2u_3, 3u_2^2 + u_1u_3, u_3)$,
(D_4^-) $f(u_1, u_2, u_3) = (2(u_1^3 - u_1u_2^2) + (u_1^2 + u_2^2)u_3, u_2^2 - 3u_1^2 - 2u_1u_3, u_1u_2 - u_2u_3, u_3)$.

3 Lightlike flat spacelike surfaces with partially parallel normal frames

In this section we consider spacelike surfaces with vanishing normalized lightcone Gauss-Kronecker curvature. A surface $M = \boldsymbol{X}(U)$ is said to be *lightlike flat* if $\widetilde{K}_\ell(p) = 0$ at any point $p \in M$. By definition, $\widetilde{K}_\ell(p) = 0$ if and only if $K_\ell(\boldsymbol{n}^T, \boldsymbol{n}^S)((p) = 0$ for arbitrarily chosen future directed normal frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. Typical lightlike flat surfaces are the spacelike surfaces contained in lightlike hyperplanes. They can be characterized as the spacelike surface with constant lightcone Gauss map. In the case of surfaces located in the Euclidean space $\mathbb{R}^3_0$, we have that the lightlike flatness is equivalent to flatness in the Euclidean space, so these surfaces are the classical developable surfaces in the Euclidean space. Another interesting example is provided by the horo-flat surfaces in the hyperbolic space. These are linear Weingarten surfaces of non-Bryant (non-elliptic) type in the terminology of [7]. In this case, the surface does not admit the Weierstrass-Bryant type parametrization.

If we suppose that the spacelike surface $\boldsymbol{X} : U \longrightarrow \mathbb{R}^4_1$ has no lightlike umbilical points, then there are two lightcone principal curvature lines at each point, one of which corresponds to the vanishing lightcone principal curvature with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. We may assume that both, the u-curve and the v-curve of the coordinate system $(u,v) \in U$, coincide with the lightcone principal curvature lines. Moreover, we can assume that the u-curve corresponds to the vanishing lightcone principal curvature with respect to $(\boldsymbol{n}^T, \boldsymbol{n}^S)$. By the lightcone Weingarten formula, we have

$$\pi^t \circ (\boldsymbol{n}^T + \boldsymbol{n}^S)_u(u,v) = \boldsymbol{0} \quad \pi^t \circ (\boldsymbol{n}^T + \boldsymbol{n}^S)_v(u,v) = -\kappa(u,v)\boldsymbol{X}_v(u,v),$$

where $\kappa(u,v) \neq 0$.

In order to obtain more interesting properties we shall impose now a further condition to the spacelike surfaces with vanishing lightcone Gauss-Kronecker curvature. Recall that a normal vector field $\boldsymbol{n}$ of M is said to be *parallel* if $\pi^n \circ d\boldsymbol{n} = 0$. Moreover, given a curve $\boldsymbol{\gamma} : I \longrightarrow M$, the normal field $\boldsymbol{n}$ is said to be *parallel along* $\boldsymbol{\gamma}$ provided $\pi^n \circ (\boldsymbol{n} \circ \gamma)' = 0$. We shall say that M is a *spacelike surface with a partially parallel normal frame* if the lightlike normal field $(\boldsymbol{n}^T + \boldsymbol{n}^S)(u,v)$ is parallel along one of the lightlike curvature lines. This condition is equivalent to asking that $\pi^n \circ (\boldsymbol{n}^T, \boldsymbol{n}^S)_u(u,v) = \boldsymbol{0}$, where the u-curve is one of the lightlike curvature lines.

Of course, if $\boldsymbol{n}^T + \boldsymbol{n}^S$ is a parallel lightlike normal field, then the frame $(\boldsymbol{n}^T, \boldsymbol{n}^S)$ is a partially parallel normal frame. All the surfaces contained in the Euclidean space $\mathbb{R}^3_0$, or in the Hyperbolic space $H^3_+(-1)$ admit parallel normal frames and hence all of them have such a property.

A surface M is said to be a *lightlike flat spacelike surface with partially parallel normal frame* if it is lightlike flat and the normal frame $(\boldsymbol{n}^T + \boldsymbol{n}^S)$ is parallel along each vanishing lightcone principal curvature line in M. This is equivalent to asking that $\pi^t \circ (\boldsymbol{n}^T + \boldsymbol{n}^S)_u(u,v) = \boldsymbol{0}$ and $\pi^n \circ (\boldsymbol{n}^T + \boldsymbol{n}^S)_u(u,v) = \boldsymbol{0}$ for the above coordinate system (u,v), which is in turn equivalent to the condition $(\boldsymbol{n}^T + \boldsymbol{n}^S)_u(u,v) = \boldsymbol{0}$. It follows that

$$(\boldsymbol{n}^T + \boldsymbol{n}^S)(0,v) = (\boldsymbol{n}^T + \boldsymbol{n}^S)(u,v), \ \forall u.$$

That is, the vector field $\boldsymbol{n}^T + \boldsymbol{n}^S$ is constant along the u-curves. In order to simplify the notation we shall write $\mathbb{L} = (\boldsymbol{n}^T + \boldsymbol{n}^S)$. Then we define a function $F : \mathbb{R}^4_1 \times (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ by

$$F(\boldsymbol{x}, v) = \langle \mathbb{L}(0,v), \boldsymbol{x} - \boldsymbol{X}(0,v) \rangle,$$

for sufficiently small $\varepsilon > 0$. For any fixed $v \in (-\varepsilon, \varepsilon)$, we have a lightlike hyperplane $HP(\mathbb{L}(0,v), c)$, where $c = \langle \mathbb{L}(0,v), \boldsymbol{X}(0,v) \rangle$. Hence $F = 0$ defines a one-parameter family of lightlike hyperplanes. We have the following proposition.

Proposition 3.1 *A lightlike flat spacelike surface $M = \boldsymbol{X}(U)$ with partially parallel normal frame is a subset of the envelope of the family of lightlike hyperplanes defined by $F = 0$.*

Proof. The envelope defined by $F = 0$ is the (possibly singular) hypersurface satisfying the condition $F = F_v = 0$. Here we have

$$F_v(\boldsymbol{x}, v) = \langle \mathbb{L}_v(0,v), \boldsymbol{x} - \boldsymbol{X}(0,v) \rangle + \langle \mathbb{L}(0,v), -\boldsymbol{X}_v(0,v) \rangle.$$

Since $\mathbb{L}(0,v)$ is a normal vector and $\boldsymbol{x}_v(0,v)$ is a tangent vector of M at $\boldsymbol{X}(0,v)$, we get

$$F_v(\boldsymbol{x},v) = \langle \mathbb{L}_v(0,v), \boldsymbol{x} - \boldsymbol{X}(0,v)\rangle.$$

Consider now the function $H(u,v) = F(\boldsymbol{X}(u,v),v) = \langle \mathbb{L}(0,v), \boldsymbol{X}(u,v) - \boldsymbol{X}(0,v)\rangle$, then

$$H(0,v) = F(\boldsymbol{X}(0,v),v) = \langle \mathbb{L}(0,v), \boldsymbol{X}(0,v) - \boldsymbol{X}(0,v)\rangle = 0.$$

Since $\mathbb{L}(u,v) = \mathbb{L}(0,v)$, we have

$$H_u(u,v) = \langle \mathbb{L}(0,v), \boldsymbol{X}_u(u,v)\rangle = \langle \mathbb{L}(u,v), \boldsymbol{X}_u(u,v)\rangle = 0.$$

Therefore $H(u,v) = H(0,v) = 0$, and hence $F(\boldsymbol{X}(u,v),v) = 0$.

We now define a function G on U as follows

$$G(u,v) = F_v(\boldsymbol{X}(u,v),v) = \langle \mathbb{L}_v(0,v), \boldsymbol{X}(u,v) - \boldsymbol{X}(0,v)\rangle.$$

Obviously $G(0,v) = 0$.

On the other hand, since $\mathbb{L}_u(u,v) = 0$, we get that $\mathbb{L}_{vu}(u,v) = \mathbb{L}_{uv}(u,v) = 0$, and thus $\mathbb{L}_v(u,v) = \mathbb{L}_v(0,v)$. Therefore, we obtain

$$\pi^t \circ \mathbb{L}_v(u,v) = \pi^t \circ \mathbb{L}(0,v) \text{ and } \pi^n \circ \mathbb{L}_v(u,v) = \pi^n \circ \mathbb{L}(0,v).$$

And it follows that

$$\begin{aligned}
G_u(u,v) &= \langle \mathbb{L}_v(0,v), \boldsymbol{X}_u(u,v)\rangle \\
&= \langle \pi^t \circ \mathbb{L}_v(0,v) + \pi^n \circ \mathbb{L}_v(0,v), \boldsymbol{X}_u(u,v)\rangle \\
&= \langle \pi^t \circ \mathbb{L}_v(u,v) + \pi^n \circ \mathbb{L}_v(u,v), \boldsymbol{X}_u(u,v)\rangle \\
&= \langle \pi^t \circ \mathbb{L}_v(u,v), \boldsymbol{X}_u(u,v)\rangle \\
&= \langle -\kappa(u,v)\boldsymbol{X}_v(u,v), \boldsymbol{X}_u(u,v)\rangle.
\end{aligned}$$

Since $\boldsymbol{X}_u(u,v), \boldsymbol{X}_v(u,v)$ are the eigenvectors of S_p at $p = \boldsymbol{X}(u,v)$, they are orthogonal, and thus $G_u(u,v) = 0$. Therefore $G(u,v) = G(0,v) = 0$, and we get

$$F_v(\boldsymbol{X}(u,v),v) = 0.$$

Consequently, we have that $\boldsymbol{X}(u,v)$ satisfies both the conditions,

$$F(\boldsymbol{X}(u,v),v) = F_v(\boldsymbol{X}(u,v),v) = 0,$$

which means that $M = \boldsymbol{X}(U)$ is a subset of the envelope of the family of lightlike hyperplanes defined by $F = 0$. $\square$

We now associate to the spacelike surface $M = \boldsymbol{X}(U)$ a lightlike hypersurface. This is given by the embedding $\widetilde{\boldsymbol{X}} : \mathbb{R} \times \mathbb{R} \times I \longrightarrow \mathbb{R}^4_1$, defined by

$$\widetilde{\boldsymbol{X}}(r,s,v) = \boldsymbol{X}(0,v) + s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r\mathbb{L}(0,v),$$

where $I \subset \mathbb{R}$ is an open interval. We have the following proposition.

Proposition 3.2 *Under the conditions of Proposition 3.1, the hypersurface $\widetilde{M} = \widetilde{\boldsymbol{X}}(\mathbb{R}\times\mathbb{R}\times J)$ is the envelope of the family of lightlike hyperplanes defined by $F = 0$.*

Proof. We remind that $\mathbb{L}(u,v) = \boldsymbol{n}^T(u,v) + \boldsymbol{n}^S(u,v)$ is a lightlike normal vector field along $M = \boldsymbol{X}(U)$. It follows that

$$F(\widetilde{\boldsymbol{X}}(r,s,v),v)) = \left\langle \mathbb{L}(0,v), s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r\mathbb{L}(0,v)\right\rangle = 0.$$

By differentiating the relation $\langle \mathbb{L}(u,v), \mathbb{L}(u,v)\rangle = 0$ with respect to the v-variable, we get that $\langle \mathbb{L}_v(u,v), \mathbb{L}(u,v)\rangle = 0$. Since $\pi^t \circ \mathbb{L}_v(0,v) = -\kappa(0,v)\boldsymbol{X}_v(0,v)$ and $\langle \pi^n \circ \mathbb{L}_v(0,v), \boldsymbol{X}_u(0,v)\rangle = 0$, we obtain

$$F_v(\widetilde{\boldsymbol{X}}(r,s,v),v) = \left\langle \mathbb{L}_v(0,v), s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r\mathbb{L}(0,v)\right\rangle = -\frac{s\kappa(0,v)}{\|\boldsymbol{X}_u(0,v)\|}\langle \boldsymbol{X}_v(0,v), \boldsymbol{X}_u(0,v)\rangle.$$

Both the u-curve and the v-curve are the lines of curvature, so we have that $\langle \boldsymbol{X}_v(0,v), \boldsymbol{X}_u(0,v)\rangle = 0$. This means that $F_v(\widetilde{\boldsymbol{X}}(r,s,v),v) = 0$, which completes the proof. $\square$

By Propositions 3.1 and 3.2, we can view any lightlike flat spacelike surface with partially parallel normal frame as a subset of the lightlike hypersurface parameterized (at least locally) by

$$\widetilde{\boldsymbol{X}}(r,s,v) = \boldsymbol{X}(0,v) + s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r\mathbb{L}(0,v).$$

The image of $\boldsymbol{X}(u,v)$ is a subset of the image of $\widetilde{\boldsymbol{X}}(r,s,v)$, where v is the common parameter. Therefore we have a reparametrization

$$\boldsymbol{X}(u,v) = \boldsymbol{X}(0,v) + s(u,v)\boldsymbol{a}_1(v) + r(u,v)\mathbb{L}(0,v),$$

where $\boldsymbol{a}_1(v) = \boldsymbol{X}_u(0,v)/\|\boldsymbol{X}_u(0,v)\| \in S^3_1$. By assumption,

$$\begin{aligned}
\boldsymbol{X}_u(u,v) &= s_u(u,v)\boldsymbol{a}_1(v) + r_u(u,v)\mathbb{L}(0,v),\\
\boldsymbol{X}_v(u,v) &= \boldsymbol{X}_v(0,v) + s_v(u,v)\boldsymbol{a}_1(v) + s(u,v)\boldsymbol{a}_{1v}(v) + r_v(u,v)\mathbb{L}(0.v) + r(u,v)\mathbb{L}_v(0,v)
\end{aligned}$$

are spacelike vectors. If $s_u(u_0,v) = 0$, then $\boldsymbol{X}_u(u_0,v) = t_u(u_0,v)\mathbb{L}(0,v)$ is lightlike, so $s_u(u,v) \neq 0$. By an adequate parameter transformation, we may (at least locally) put

$$\boldsymbol{X}(s,v) = \boldsymbol{X}(0,v) + s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r(s,v)\mathbb{L}(0,v).$$

We now analyze the meaning of the above parametrization. Fix $v = v_0$ and denote

$$\boldsymbol{a}_0 = \boldsymbol{n}^T(0,v_0),\ \ \boldsymbol{a}_1 = \frac{\boldsymbol{X}_u(0,v_0)}{\|\boldsymbol{X}_u(0,v_0)\|},\ \ \boldsymbol{a}_2 = \boldsymbol{n}^S(0,v_0).$$

Then we have a curve

$$\boldsymbol{\sigma}(s) = \boldsymbol{X}(s,v_0) = \boldsymbol{X}(0,v_0) + s\boldsymbol{a}_1 + r(s,v_0)(\boldsymbol{a}_0 + \boldsymbol{a}_2).$$

Since $\boldsymbol{\sigma}'(s) = \boldsymbol{a}_1 + r_s(s,v_0)(\boldsymbol{a}_0 + \boldsymbol{a}_2)$, we have that $\langle \boldsymbol{\sigma}'(s), \boldsymbol{\sigma}'(s)\rangle = \langle \boldsymbol{a}_1, \boldsymbol{a}_1\rangle = 1$. Therefore $\boldsymbol{\sigma}(s)$ is a unit speed spacelike curve. This means that $\boldsymbol{t}(s) = \boldsymbol{a}_1 + r_s(s,v_0)(\boldsymbol{a}_0 + \boldsymbol{a}_2)$ is a unit spacelike vector. Moreover, the vector $\boldsymbol{\sigma}(s) - \boldsymbol{X}(0,v_0)$ belongs to the lightlike plane $\langle \boldsymbol{a}_1, \boldsymbol{a}_0 + \boldsymbol{a}_2\rangle_{\mathbb{R}}$, so the surface can be seen as a one-parameter family of lightlike plane curves, where by a *lightlike plane curve* we understand a spacelike curve contained in a lightlike plane.

Theorem 3.3 *Suppose that $M \subset \mathbb{R}^4_1$ is a lightlike flat spacelike surface with partially parallel normal frame and no lightlike umbilic points, then M is a one-parameter family of lightlike plane curves. Moreover, each one of these lightlike plane curve is a vanishing lightcone principal curvature line in M.*

Proof. The first assertion of the theorem is a consequence of the above arguments. As for the second, we assume that $M = \boldsymbol{X}(U)$ and that both the u-curve and the v-curve are the curvature lines as above, so they satisfy that $\mathbb{L}_u(u,v) = 0$ and $\pi^t \circ \mathbb{L}_v(u,v) = -\bar{\kappa}(u,v)\boldsymbol{X}_v(u,v)$. We consider now the parametrization

$$\widetilde{\boldsymbol{X}}(s,v) = \boldsymbol{X}(0,v) + s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r(s,v)\mathbb{L}(0,v)$$

of $M = \boldsymbol{X}(U)$. By a straightforward calculation, we get

$$\begin{aligned}
\widetilde{\boldsymbol{X}}_s(s,v) &= \frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r_s(s,v)\mathbb{L}(0,v),\\
\widetilde{\boldsymbol{X}}_v(s,v) &= \boldsymbol{X}_v(0,v) + s\left(\frac{\boldsymbol{X}_{uv}(0,v)}{\|\boldsymbol{X}_u(0,v)\|} - \frac{2\langle \boldsymbol{X}_u(0,v), \boldsymbol{X}_{uv}(0,v)\rangle}{\|\boldsymbol{X}_u(0,v)\|^2}\boldsymbol{X}_u(0,v)\right)\\
&\quad + r_v(s,v)\mathbb{L}(0,v) + r(s,v)\mathbb{L}_v(0,v).
\end{aligned}$$

Since $\langle \mathbb{L}(0,v), \boldsymbol{X}_u(0,v)\rangle = 0$, we have that $\langle \mathbb{L}_v(0,v), \boldsymbol{X}_u(0,v)\rangle + \langle \mathbb{L}(0,v), \boldsymbol{X}_{uv}(0,v)\rangle = 0$. Since the v-curve is a curvature line satisfying that $\pi^t \circ \mathbb{L}_v(0,v) = -\bar{\kappa}(0,v)\boldsymbol{X}_v(0,v)$, we have $\langle \mathbb{L}_v(0,v), \boldsymbol{X}_u(0,v)\rangle = \langle \pi^n \circ \mathbb{L}_v(0,v) + \pi^t \circ \mathbb{L}_v(0,v), \boldsymbol{X}_(0,v)\rangle = \langle \pi^t \circ \mathbb{L}_v(0,v), \boldsymbol{X}_u(0,v)\rangle = -\bar{\kappa}(0,v)\langle \boldsymbol{X}_v(0,v), \boldsymbol{X}_u(0,v)\rangle = 0$. Therefore we get that $\langle \mathbb{L}(0,v), \boldsymbol{X}_{uv}(0,v)\rangle = 0$. Since $\mathbb{L}(0,v)$ is the lightlike normal vector of $M = \boldsymbol{X}(U)$ at $\boldsymbol{X}(0,v)$, we obtain

$$\langle \mathbb{L}(0,v), \widetilde{\boldsymbol{X}}_s(s,v)\rangle = \langle \mathbb{L}(0,v), \widetilde{\boldsymbol{X}}_v(s,v)\rangle = 0.$$

This means that $\mathbb{L}(0,v)$ is the lightlike normal of $M = \boldsymbol{X}(U)$ at $\widetilde{\boldsymbol{X}}(s,v)$. Therefore we have that the lightlike normal $\mathbb{L}$ is constant along the s-curve. Since $(u,v) = (u(s,v),v)$, we get that $\mathbb{L}_s(u(s,v),v) = \mathbb{L}_u(u(s,v),v)u_s(s,v) = \mathbf{0}$, so the s-curve must be the vanishing lightcone principal curvature line. □

4 Flat lightlike hypersurfaces

We introduce in this section a notion of flatness for the lightlike hypersurfaces in $\mathbb{R}^4_1$. Let $LH_X : U \times \mathbb{R} \longrightarrow \mathbb{R}^4_1$ be the lightlike hypersurface along the spacelike surface $M = \boldsymbol{X}(U)$. We say that LH_M is *flat* if $M = \boldsymbol{X}(U)$ is a lightlike flat spacelike surface with partially parallel normal frame. We must show now that this notion of flatness on lightlike hypersurfaces is independent of the choice of the parametrization. Let $\overline{U}$ an open subset of $\mathbb{R}^2$ and $\overline{\boldsymbol{X}} : \overline{U} \longrightarrow \mathbb{R}^4_1$ a spacelike embedding such that $\overline{M} = \overline{\boldsymbol{X}}(\overline{U}) \subset LH_M$. Then we can write

$$\overline{\boldsymbol{X}}(\overline{u},\overline{v}) = \boldsymbol{X}(u(\overline{u},\overline{v}), v(\overline{u},\overline{v})) + r(\overline{u},\overline{v})\mathbb{L}(u(\overline{u},\overline{v}), v(\overline{u},\overline{v})).$$

Moreover, we assume that there is a diffeomorphism (parameter transformation) $\phi : \overline{U} \longrightarrow U$ defined by $\phi(\overline{u},\overline{v}) = (u(\overline{u},\overline{v}), v(\overline{u},\overline{v}))$.

Under these notations, we have the following proposition.

Proposition 4.1 *If $\boldsymbol{X}$ is a lightlike flat spacelike surface with partially parallel normal frame, then $\overline{\boldsymbol{X}}$ is lightlike flat.*

Proof. We may assume that the u-curves and the v-curves are the curvatures lines on M and that the u-curves correspond to the vanishing lightcone curvature. This means that $\mathbb{L}_u \equiv 0$. By a straight forward calculation, we have

$$\begin{aligned} \overline{\boldsymbol{X}}_u &= \boldsymbol{X}_u + (r_{\overline{u}}\overline{u}_u + r_{\overline{v}}\overline{v}_u)\mathbb{L} + r(\overline{u},\overline{v})\mathbb{L}_u = \boldsymbol{X}_u + (r_{\overline{u}}\overline{u}_u + r_{\overline{v}}\overline{v}_u)\mathbb{L} \\ \overline{\boldsymbol{X}}_v &= \boldsymbol{X}_v + (r_{\overline{u}}\overline{u}_v + r_{\overline{v}}\overline{v}_v)\mathbb{L} + r(\overline{u},\overline{v})\mathbb{L}_v = (1-r\kappa)\boldsymbol{X}_v + (r_{\overline{u}}\overline{u}_v + r_{\overline{v}}\overline{v}_v)\mathbb{L}, \end{aligned}$$

and thus $\mathbb{L}(u(\overline{u},\overline{v}), v(\overline{u},\overline{v}))$ is normal to $\overline{\boldsymbol{X}}_u, \overline{\boldsymbol{X}}_v$. However, we have that $\overline{\boldsymbol{X}}_{\overline{u}} = \overline{\boldsymbol{X}}_u u_{\overline{u}} + \overline{\boldsymbol{X}}_v v_{\overline{u}}, \overline{\boldsymbol{X}}_{\overline{v}} = \overline{\boldsymbol{X}}_u u_{\overline{v}} + \overline{\boldsymbol{X}}_v v_{\overline{v}}$, so $\mathbb{L}(u(\overline{u},\overline{v}), v(\overline{u},\overline{v}))$ is a lightlike normal to $\overline{\boldsymbol{X}}(\overline{u},\overline{v})$. Therefore we can choose $(\overline{\boldsymbol{n}}^T, \overline{\boldsymbol{n}}^S)$ as a future directed normal frame along $\overline{\boldsymbol{X}}$, such that $(\overline{\boldsymbol{n}}^T + \overline{\boldsymbol{n}}^S)(\overline{u},\overline{v})$ is parallel to $\mathbb{L}(u(\overline{u},\overline{v}), v(\overline{u},\overline{v}))$. It follows that there exists a function $\lambda(u,v)$, such that $\mathbb{L}(u((\overline{u},\overline{v})), v((\overline{u},\overline{v}))) = \lambda(u(\overline{u},\overline{v}), v(\overline{u},\overline{v}))(\overline{\boldsymbol{n}}^T + \overline{\boldsymbol{n}}^S)(\overline{u},\overline{v})$. Denote $\overline{\mathbb{L}} = (\overline{\boldsymbol{n}}^T + \overline{\boldsymbol{n}}^S)$. It can be seen that $\mathbb{L}_u = \lambda_u\overline{\mathbb{L}} + \lambda\overline{\mathbb{L}}_u$. Since $\mathbb{L}_u \equiv 0$ and $\overline{\mathbb{L}}$ is a lightlike normal, we get that the tangent component of $\overline{\mathbb{L}}_u$ is identically zero. This implies that the $\phi(u, v_0)$-curve on $\overline{M}$ is a vanishing lightcone curvature line. $\square$

By the proposition above we have that the definition of flat lightlike hypersurface is independent of the choice of the lightlike flat spacelike surface with partially parallel normal frame along which it is defined. Moreover, in the proof of the above proposition, we have that the diffeomorphism $\phi : \overline{U} \longrightarrow U$ satisfies $\overline{\mathbb{L}}(\overline{u},\overline{v}) = \lambda(\overline{u},\overline{v})\mathbb{L}(u(\overline{u},\overline{v}), v(\overline{u},\overline{v}))$. This extends to a diffeomorphism $\Phi : \overline{U} \times \mathbb{R} \longrightarrow U \times \mathbb{R}$ given by

$$\Phi(\overline{u},\overline{v},\overline{r}) = (u(\overline{u},\overline{v}), v(\overline{u},\overline{v}), r(\overline{u},\overline{v}) + \overline{r}\lambda(\overline{u},\overline{v})).$$

By definition, we have $LH_X(\Phi(\overline{U} \times \mathbb{R})) = LH_{\overline{X}}(\overline{U} \times \mathbb{R})$. We call Φ an *adapted parameter transformation* of LH_M.

Proposition 4.2 *Given a lightlike hypersurface $LH_M = LH_X(U \times \mathbb{R})$, we can find (at least locally) a spacelike surface $\overline{\boldsymbol{X}} : \overline{U} \longrightarrow \mathbb{R}^4_1$ with partially parallel normal frame and an adapted parameter transformation $\Phi : \overline{U} \times \mathbb{R} \longrightarrow U \times \mathbb{R}$ of LH_M.*

Proof. For a given spacelike embedding $\boldsymbol{X} : U \longrightarrow \mathbb{R}^4_1$, we consider the lightlike normal $\mathbb{L}(u,v) = (\boldsymbol{n}^T + \boldsymbol{n}^S)(u,v)$. We denote $\mathbb{R}^3_c = \{(c, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid c \in \mathbb{R}\ \}$, which is a spacelike hyperplane in $\mathbb{R}^4_1$. For any $(u_0, v_0) \in U$, let c be the first component of $\boldsymbol{X}(u_0, v_0)$. Then $LH_X(U \times \mathbb{R}) \cap \mathbb{R}^3_c$ is a non-singular surface near the point $\boldsymbol{X}(u_0, v_0)$. It follows that there is a neighborhood $\overline{U} \subset \mathbb{R}^2$ of $(u_0 v_0)$, $W \subset \mathbb{R}^4_1$ of $\boldsymbol{X}(u_0, v_0)$ and a embedding $\overline{\boldsymbol{X}} : V \longrightarrow W$ such that $\boldsymbol{X}(\overline{U}) = W \cap LH_X(U \times \mathbb{R}) \cap R^3_c$. Then we choose $\boldsymbol{n}^T(u,v) = \boldsymbol{e}_0 = (1,0,0,0)$ as a future directed timelike unit normal of $\overline{\boldsymbol{X}}$. We also choose the spacelike normal field $\boldsymbol{n}^S$ as the (unique) unit normal vector field in $\mathbb{R}^3_c$ such that $\boldsymbol{n}^T + \boldsymbol{n}^S$ is parallel to $\mathbb{L}$. It follows that $(\boldsymbol{n}^T, \boldsymbol{n}^s)$ is a parallel normal frame along $\overline{\boldsymbol{X}}$. The remaining assertions are trivial by construction. $\square$

Consider now a flat lightlike hypersurface $LH_X : U \times \mathbb{R} \longrightarrow \mathbb{R}^4_1$, where the u-curves of $M = \boldsymbol{X}(U)$ are the family of vanishing lightcone principal curvature lines and the v-curves are the orthogonal lightcone principal curvature lines. This implies that $\mathbb{L}_u(u,v) \equiv 0$. By the results in §3, the lightlike hypersurface defined by

$$\widetilde{\boldsymbol{X}}(r,s,v) = \boldsymbol{X}(0,v) + s\frac{\boldsymbol{X}_u(0,v)}{\|\boldsymbol{X}_u(0,v)\|} + r\mathbb{L}(0,v)$$

is the envelope of lightlike hyperplanes given by

$$F(\boldsymbol{x}, v) = \langle \mathbb{L}(0, v), \boldsymbol{x} - \boldsymbol{X}(0, v) \rangle.$$

Since $\mathbb{L}(u, v) = \mathbb{L}(u, v)$, we can show the following proposition by using exactly the same arguments as in the proof of Proposition 3.2.

Proposition 4.3 *Suppose that $LH_X : U \times \mathbb{R} \longrightarrow \mathbb{R}^4_1$ is a flat lightlike hypersurface along a lightlike flat spacelike surface $M = \boldsymbol{X}(U)$ with partially parallel normal frame. If the u-curves are the family vanishing lightcone curvature lines, then LH_M is the envelope of the lightlike hyperplanes defined by $F(\boldsymbol{x}, v) = \langle \mathbb{L}(0, v), \boldsymbol{x} - \boldsymbol{X}(0, v) \rangle$.*

As a consequence of this we have that the flat lightlike hypersurface LH_M can be reparametrized (at least locally) by the mapping $\widetilde{\boldsymbol{X}}(r, s, v)$. If we denote

$$t = v, \boldsymbol{\gamma}(t) = \boldsymbol{X}(0, t),\ \boldsymbol{a}_0(t) = \boldsymbol{n}^T(0, t),\ \boldsymbol{a}_1(t) = \frac{\boldsymbol{X}_u(0, t)}{\|\boldsymbol{X}_u(0, t)\|},\ \boldsymbol{a}_2(0, t) = \boldsymbol{n}^S(0, t),$$

then we have

$$\widetilde{\boldsymbol{X}}(r, s, t) = \boldsymbol{\gamma}(t) + s\boldsymbol{a}_1(t) + r(\boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)).$$

With this we arrive to the central point of the paper: Let $\boldsymbol{\gamma} : I \longrightarrow \mathbb{R}^4_1$ be a regular spacelike curve (ie., $\langle \boldsymbol{\gamma}'(t), \boldsymbol{\gamma}'(t) \rangle > 0$), and $\boldsymbol{a}_0 : I \longrightarrow H^3_+(-1)$ and $\boldsymbol{a}_i : I \longrightarrow S^3_1$ $(i = 1, 2)$, smooth curves such that $\langle \boldsymbol{a}_i(t), \boldsymbol{a}_j(t) \rangle = 0$, where I denotes an open interval. We now write $\boldsymbol{\ell}(t) = \boldsymbol{a}_0(t) + \boldsymbol{a}_2(t)$ and define a mapping

$$F_{(\gamma, a_0, a_1, a_2)} : \mathbb{R} \times \mathbb{R} \times I \longrightarrow \mathbb{R}^4_1$$

by

$$F_{(\gamma, a_0, a_1, a_2)}(s, r, t) = \boldsymbol{\gamma}(t) + s\boldsymbol{a}_1(t) + r\boldsymbol{\ell}(t).$$

We observe that for any fixed $t = t_0$, we have a lightlike plane $F_{(\gamma, a_0, a_1, a_2)}(s, r, t_0)$.

We call $F_{(\gamma, a_0, a_1, a_2)}$ (or the image of it) a *lightlike planar hypersurface.* Each lightlike plane $F_{(\gamma, a_0, a_1, a_2)}(s, r, t_0)$ is called a *generating lightlike plane.*

We can summarize the consequences of the previous arguments together with Proposition 4.3 as follows:

Proposition 4.4 *A flat lightlike hypersurface is a lightlike planar hypersurface.*

For each $t \in I$, we can take the unit spacelike vector $\boldsymbol{a}_3(t) = \boldsymbol{a}_0(t) \wedge \boldsymbol{a}_1(t) \wedge \boldsymbol{a}_2(t)$, so that we have a pseudo-orthonormal frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ of $\mathbb{R}^4_1$. By using the above curve $\boldsymbol{\gamma}(t)$ and the pseudo-orthonormal frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$, we define the following fundamental invariants for lightlike planar hypersurfaces:

$$\begin{aligned}
c_1(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_1(t) \rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_1'(t) \rangle, & c_4(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_2(t) \rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_2'(t) \rangle, \\
c_2(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_2(t) \rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_2'(t) \rangle, & c_5(t) &= \langle \boldsymbol{a}_1'(t), \boldsymbol{a}_3(t) \rangle = -\langle \boldsymbol{a}_1(t), \boldsymbol{a}_3'(t) \rangle, \\
c_3(t) &= \langle \boldsymbol{a}_0'(t), \boldsymbol{a}_3(t) \rangle = -\langle \boldsymbol{a}_0(t), \boldsymbol{a}_3'(t) \rangle, & c_6(t) &= \langle \boldsymbol{a}_2'(t), \boldsymbol{a}_3(t) \rangle = -\langle \boldsymbol{a}_2(t), \boldsymbol{a}_3'(t) \rangle
\end{aligned}$$

and

$$b_0(t) = \langle \boldsymbol{\gamma}'(t), \boldsymbol{a}_0(t) \rangle,\ b_1(t) = \langle \boldsymbol{\gamma}'(t), \boldsymbol{a}_1(t) \rangle,\ b_2(t) = \langle \boldsymbol{\gamma}'(t), \boldsymbol{a}_2(t) \rangle,\ b_3(t) = \langle \boldsymbol{\gamma}'(t), \boldsymbol{a}_3(t) \rangle.$$

It can be shown that the following fundamental differential equations hold:

$$\begin{cases} \boldsymbol{a}_0'(t) & = c_1(t)\boldsymbol{a}_1(t) + c_2(t)\boldsymbol{a}_2(t) + c_3(t)\boldsymbol{a}_3(t) \\ \boldsymbol{a}_1'(t) & = c_1(t)\boldsymbol{a}_0(t) + c_4(t)\boldsymbol{a}_2(t) + c_5(t)\boldsymbol{a}_3(t) \\ \boldsymbol{a}_2'(t) & = c_2(t)\boldsymbol{a}_0(t) - c_4(t)\boldsymbol{a}_1(t) + c_6(t)\boldsymbol{a}_3(t) \\ \boldsymbol{a}_3'(t) & = c_3(t)\boldsymbol{a}_0(t) - c_5(t)\boldsymbol{a}_1(t) - c_6(t)\boldsymbol{a}_2(t), \end{cases} \tag{4.1}$$

and

$$\boldsymbol{\gamma}'(t) = b_0(t)\boldsymbol{a}_0(t) + b_1(t)\boldsymbol{a}_1(t) + b_2(t)\boldsymbol{a}_2(t) + b_3(t)\boldsymbol{a}_3(t). \tag{4.2}$$

This can be written in the following form:

$$\begin{pmatrix} \boldsymbol{a}_0'(t) \\ \boldsymbol{a}_1'(t) \\ \boldsymbol{a}_2'(t) \\ \boldsymbol{a}_3'(t) \end{pmatrix} = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ c_1(t) & 0 & c_4(t) & c_5(t) \\ c_2(t) & -c_4(t) & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{a}_0(t) \\ \boldsymbol{a}_1(t) \\ \boldsymbol{a}_2(t) \\ \boldsymbol{a}_3(t) \end{pmatrix}.$$

Where we remark that

$$C(t) = \begin{pmatrix} 0 & c_1(t) & c_2(t) & c_3(t) \\ c_1(t) & 0 & c_4(t) & c_5(t) \\ c_2(t) & -c_4(t) & 0 & c_6(t) \\ c_3(t) & -c_5(t) & -c_6(t) & 0 \end{pmatrix} \in \mathfrak{so}(3,1).$$

Here $\mathfrak{so}(3,1)$ denotes the Lie algebra of the Lorentzian group $SO_0(3,1)$. If $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$ is the above pseudo-orthonormal frame field, we have that the above 4×4-matrix defines a smooth curve $A : I \longrightarrow SO_0(3,1)$. Therefore we get the relation that $A'(t) = C(t)A(t)$. Conversely, let $A : I \longrightarrow SO_0(3,1)$ be a smooth curve, then we can show that $A'(t)A(t)^{-1} \in \mathfrak{so}(3,1)$. Moreover, given any smooth curve $C : I \longrightarrow \mathfrak{so}(3,1)$, we can apply the fundamental theorem for linear systems of ordinary differential equations, and conclude that there exists a unique curve $A : I \longrightarrow SO_0(3,1)$ such that $C(t) = A'(t)A(t)^{-1}$ with initial data $A(t_0) \in SO(3,1)$. Moreover, for any spacelike curve $\boldsymbol{b}(t) = (b_0(t), b_1(t), b_2(t), b_3(t)) \in \mathbb{R}^4$ with $\langle \boldsymbol{b}(t), \boldsymbol{b}(t) \rangle > 0$, we have $\boldsymbol{\gamma}(t)$. Therefore, a smooth curve $(\boldsymbol{b}, C) : I \longrightarrow \mathbb{R}^4 \times \mathfrak{so}(3,1)$ might be identified with a lightlike planar hypersurface in $\mathbb{R}^4_1$. Let $C : I \longrightarrow \mathfrak{so}(3,1)$ be a smooth curve with $C(t) = A'(t)A(t)^{-1}$ and $B \in SO(3,1)$, then we have $C(t) = (A(t)B)'(A(t)B)^{-1}$. This means that the curve $C : I \longrightarrow \mathfrak{so}(3,1)$ is a Lorentzian invariant of the pseudo-orthonormal frame $\{\boldsymbol{a}_0(t), \boldsymbol{a}_1(t), \boldsymbol{a}_2(t), \boldsymbol{a}_3(t)\}$, and thus it is a Lorentzian invariant of the corresponding lightlike planar hypersurface. For simplicity, we shall write in what follows $F_{(\gamma,A)}$ instead of $F_{(\gamma,a_0,a_1,a_2)}$.

Let $C^\infty(I, \mathbb{R}^4_1 \times \mathfrak{so}(3,1))$ be the space of smooth curves into $\mathfrak{so}(3,1)$ equipped with Whitney C^∞-topology. We consider an open subset

$$C^\infty_{sp}(I, \mathbb{R}^4_1 \times \mathfrak{so}(3,1)) = \{(\boldsymbol{b}, C) \in C^\infty(I, \mathbb{R}^4_1 \times \mathfrak{so}(3,1)) \mid \langle \boldsymbol{b}(t), \boldsymbol{b}(t) \rangle > 0 \ \}.$$

By the above arguments, we may regard $C^\infty_{sp}(I, \mathbb{R}^4_1 \times \mathfrak{so}(3,1))$ as the space of lightlike planar surfaces, where I is an open interval, or the unit circle.

Motivated by the result of Proposition 4.3, we have the following definition: For any lightlike planar hypersurface $F_{(\gamma,A)}$, consider the surface defined by

$$F_{(\gamma,A;r(s,t))}(s,t) = \boldsymbol{\gamma}(t) + s\boldsymbol{a}_1(t) + r(s,t)\boldsymbol{\ell}(t),$$

where $r = r(s,t)$ is a smooth function. By a straight forward calculation, we can show that the surface $F_{(\gamma,A;r(s,t))}$ is spacelike.

On the other hand, let $M = \boldsymbol{X}(u,v)$ be a lightlike flat spacelike surface without umbilics and with partially parallel normal frame $\boldsymbol{n}^T(u,v), \boldsymbol{n}^S(u,v)$, whose u-curves are the vanishing lightcone principal curvature lines. Suppose that $F_{(\gamma,A;r(s,t))}(s,t)$ provides a (local) parametrization for $M = \boldsymbol{X}(u,v)$. By using the same arguments than in §3, we can write $(u,v) = (u(s,t),t)$ with $u_s(s,t) \neq 0$. Moreover, we may assume that

$$\boldsymbol{\gamma}(t) = \boldsymbol{X}(u(0,t),t), \boldsymbol{a}_0(t) = \boldsymbol{n}^T(u(0,t),t), \boldsymbol{a}_1(t) = \frac{\boldsymbol{X}_u(u(0,t),t)}{\|\boldsymbol{X}_u(u(0,t),t)\|}, \boldsymbol{a}_2(t) = \boldsymbol{n}^S(u(0,t),t).$$

Since $\mathbb{L}_u \equiv 0$, $\mathbb{L}_s = \mathbb{L}_u u_s \equiv 0$, and hence $\mathbb{L}(u,v) = \mathbb{L}(u(0,t),t) = \boldsymbol{\ell}(t)$. Therefore, $\boldsymbol{\ell}(t)$ is a lightlike normal vector at (s,t). Conversely, if $\boldsymbol{\ell}(t)$ is a lightlike normal vector at (s,t), then $\boldsymbol{\ell}_s(t) \equiv 0$ means that $F_{(\gamma,A;r(s,t))}$ is lightlike flat spacelike surface with partially parallel normal frame. It follows that the s-curves coincide with the vanishing lightcone curvature lines. We can now show the following.

Proposition 4.5 *The vector $\boldsymbol{\ell}(t)$ is a lightlike normal of the surface $F_{(\gamma,A;r(s,t))}$ at any regular point (s,t) if and only if*

$$(b_2(t) - b_0(t)) + s(c_4(t) - c_1(t)) = 0.$$

Proof. By using the basic invariants, we obtain

$$\begin{aligned}
\frac{\partial F_{(\gamma,A;r(s,t))}}{\partial t}(s,t) &= (b_0 + sc_1)\boldsymbol{a}_0 + b_1\boldsymbol{a}_1 + (b_2 + sc_4)\boldsymbol{a}_2 + (r_t + rc_2)\boldsymbol{\ell} + (b_3 + sc_5 + r(c_3 + c_6))\boldsymbol{a}_3,\\
\frac{\partial F_{(\gamma,A;r(s,t))}}{\partial s}(s,t) &= \boldsymbol{a}_1 + r_s\boldsymbol{\ell}.
\end{aligned}$$

On the other hand, we have that $\langle(\partial F_{(\gamma,A;r(s,t))}/\partial s)(s,t), \boldsymbol{\ell}(t)\rangle \equiv 0$, $\boldsymbol{\ell}(t)$ is a normal vector at (s,t) if and only if

$$0 = \langle \frac{\partial F_{(\gamma,A;r(s,t))}}{\partial t}(s,t), \boldsymbol{\ell}(t)\rangle = -(b_0(t) + sc_1(t)) + (b_2(t) + sc_4(t)),$$

which completes the proof.

□

We remark that $b_2(t) - b_0(t) + s(c_4(t) - c_1(t)) = 0$ for any (s,t) if and only if $b_2(t) - b_0(t) = c_4(t) - c_1(t) = 0$. Therefore, we say that a lightlike planar hypersurface $F_{(b,)A}(r,s,t)$ is *flat* if $b_2(t) - b_0(t) = c_4(t) - c_1(t) = 0$. In such case, we have that a spacelike surface of the form $F_{(\gamma,A;r(s,t))}$ must be lightlike flat. In particular, $F_{(\gamma,A;0)}$, which is said to be a *lightlike flat spacelike ruled surface.*

By the above arguments, we can consider the linear subspace of $\mathfrak{so}(3,1)$ defined by

$$\mathfrak{fl}(3,1) = \left\{ C = \begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ c_1 & 0 & c_4 & c_5 \\ c_2 & -c_4 & 0 & c_6 \\ c_3 & -c_5 & -c_6 & 0 \end{pmatrix} \in \mathfrak{so}(3,1) \;\middle|\; c_1 - c_4 = 0 \right\}$$

and the lightlike hyperplane $HP(\boldsymbol{e}_0 + \boldsymbol{e}_2, 0) = \{\boldsymbol{b} = (b_0, b_1, b_2, b_3) \in \mathbb{R}^4_1 \mid b_0 = b_2 \}$. Then the *space of flat lightlike planar hypersurfaces* can be regarded as the space

$$C^\infty_{sp}(I, HP(\boldsymbol{e}_0 + \boldsymbol{e}_2, 0) \times \mathfrak{fl}(3,1)) = \{(\boldsymbol{b}, C) : I \longrightarrow HP(\boldsymbol{e}_0 + \boldsymbol{e}_2, 0) \times \mathfrak{fl}(3,1) \mid \langle \boldsymbol{b}(t), \boldsymbol{b}(t)\rangle > 0 \}$$

with Whitney C^∞-topology.

5 Singularities of flat lightlike hypersurfaces

In this section, we shall work with flat lightlike planar hypersurfaces. According to Proposition 4.3, such a hypersurface has a parametrization $F_{(\gamma,A)}(s,t,r) = \boldsymbol{\gamma}(t) + s\boldsymbol{a}_1(t) + r\boldsymbol{\ell}(t)$, satisfying

$$\langle \boldsymbol{\ell}, \boldsymbol{\gamma}' \rangle \equiv 0 \ (b_0 \equiv b_2) \text{ and } \langle \boldsymbol{\ell}', \boldsymbol{a}_1 \rangle \equiv 0 \ (c_1 \equiv c_4). \tag{5.1}$$

Fix a point $p_0 = (s_0, t_0, r_0)$ and denote $F_{(\gamma,A)}(p_0) = \boldsymbol{x}_0$. We define a function $H(t, \boldsymbol{x}) = \langle \boldsymbol{\ell}(t), \boldsymbol{x} - \boldsymbol{\gamma}(t) \rangle$, $(\boldsymbol{x} = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1)$. Then we have the following lemma.

Lemma 5.1 *If $c_3 + c_6 \neq 0$ then the discriminant set*

$$D_H = \left\{ \boldsymbol{x} \in \mathbb{R}^4_1 \;\middle|\; \text{there exists } t \in \mathbb{R} \text{ such that } H(t, \boldsymbol{x}) = \frac{\partial H}{\partial t}(t, \boldsymbol{x}) = 0 \right\}$$

of H is equal to the image $\mathrm{Im}(F_{(\gamma,A)})$.

The proof of this Lemma is essentially included in the proof of Proposition 4.3. However, we give a proof using the parametrization above.

Proof. We shall use the abbreviation $c_{36} = c_3 + c_6$. It is easy to check $\mathrm{Im}(F_{(\gamma,A)}) \subset D_H$. Assume that $\boldsymbol{x} \in D_H$. By (5.1), we have that $\boldsymbol{x} \in D_H$ if and only if $\langle \boldsymbol{\ell}(t), \boldsymbol{x} - \boldsymbol{\gamma}(t) \rangle = \langle \boldsymbol{\ell}'(t), \boldsymbol{x} - \boldsymbol{\gamma}(t) \rangle = 0$, for some t. We can take real numbers $\alpha_i(t)$ $(i = 0, \ldots, 3)$ such that $\boldsymbol{x} - \boldsymbol{\gamma}(t) = \sum_{i=0}^{3} \alpha_i(t) \boldsymbol{a}_i(t)$. Then, since $c_{36} \neq 0$, we have $\alpha_2(t) = \alpha_0(t)$ and $\alpha_3(t) = 0$. Hence $\boldsymbol{x} - \boldsymbol{\gamma}(t) = \alpha_0(t)\boldsymbol{\ell}(t) + \alpha_1(t)\boldsymbol{a}_1(t)$ holds, which means that $\boldsymbol{x} \in \mathrm{Im}(F_{(\gamma,A)})$. □

By (4.1), we have that $c_{36} \neq 0$ if and only if $\boldsymbol{\ell}$ and $\boldsymbol{\ell}'$ are linearly independent. Provided $c_{36} \neq 0$, we have that the map

$$\Delta^* H = \left(H, \frac{\partial H}{\partial t} \right) : (\mathbb{R} \times \mathbb{R}^4, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}, \mathbf{0})$$

is submersive, namely H is a Morse family (see Appendix A.). In fact, it is submersive if and only if the matrix

$$\left(\frac{\partial H}{\partial \boldsymbol{x}}, \frac{\partial^2 H}{\partial \boldsymbol{x} \partial t} \right) = (\boldsymbol{\ell}, \boldsymbol{\ell}')$$

has the maximal rank. Moreover, we use the following (see [2, Section 6.10]):

Lemma 5.2 *Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $F : (\mathbb{R} \times \mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ be functions such that $F(t, \mathbf{0}) = f(t)$. Assume f has type A_k at $t = 0$ ($f'(0) = \cdots f^{(k)}(0) = 0$ and $f^{(k+1)}(0) \neq 0$). Write $(k-1)$-jet of F as*

$$\sum_{i=0}^{n} \alpha_{0,i}\, x_i + \sum_{i=0}^{n} \alpha_{1,i}\, x_i t + \sum_{i=0}^{n} \alpha_{2,i}\, x_i t^2 + \cdots + \sum_{i=0}^{n} \alpha_{k-1,i}\, x_i t^{k-1}, \quad \boldsymbol{x} = (x_1, \ldots, x_n).$$

Then F is a $\mathcal{K}$-versal unfolding (See Appendix A.*) if and only if the $(k \times n)$ matrix of coefficients*

$$\mathcal{V}_F = \begin{pmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{k-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{k-1,2} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{0,n} & \alpha_{1,n} & \cdots & \alpha_{k-1,n} \end{pmatrix}$$

has rank k.

Since H is defined by $H(t,\boldsymbol{x})=\langle \boldsymbol{\ell}, \boldsymbol{x}-\boldsymbol{\gamma}\rangle$, the matrix $\mathcal{V}_H$ is $(\boldsymbol{\ell},\boldsymbol{\ell}',\dots,\boldsymbol{\ell}^{(k)})$. Therefore, if $H(t,\boldsymbol{x}_0)$ is of type A_k $(k\le 4)$ at 0 and $\{\boldsymbol{\ell},\boldsymbol{\ell}',\dots,\boldsymbol{\ell}^{(k)}\}$ is linearly independent, then H is a Morse family and a $\mathcal{K}$-versal deformation of h. Moreover, by uniqueness of the $\mathcal{K}$-versal deformations (see [2, Section 6.5].), H must be P-$\mathcal{K}$-equivalent to $F_k(t,\boldsymbol{x})=t^{(k+1)}+x_0+x_1t+\cdots+x_{k-1}t^{k-1}$ $(k=2,3,4)$. Since the discriminant set of F_k is the *cusp* $\times\mathbb{R}^2$ (if $k=2$), the *swallowtail* $\times\mathbb{R}$ (if $k=3$) or an A_4-singularity (if $k=4$), the map germ of $F_{(\gamma,A)}$ at p is $\mathcal{A}$-equivalent to the *cusp* $\times\mathbb{R}^2$, the *swallowtail* $\times\mathbb{R}$ or an A_4-singularity, for each $k=2,3,4$.

By summarizing these discussions, we have that under the condition that $c_{36}(t_0)\neq 0$, the image germ of $F_{(\gamma,A)}$ at p is diffeomorphic to the *cusp* $\times\mathbb{R}^2$ if and only if $H''(t_0,\boldsymbol{x}_0)=0$, $H'''(t_0,\boldsymbol{x}_0)\neq 0$ and $\{\boldsymbol{\ell},\boldsymbol{\ell}',\boldsymbol{\ell}''\}$ is linearly independent at $(t_0,\boldsymbol{x}_0)$. Similarly, the image germ of $F_{(\gamma,A)}$ at p is diffeomorphic to the *swallowtail* $\times\mathbb{R}$ if and only if $H''(t_0,\boldsymbol{x}_0)=H'''(t_0,\boldsymbol{x}_0)=0$, $H^{(4)}(t_0,\boldsymbol{x}_0)\neq 0$ and $\{\boldsymbol{\ell},\boldsymbol{\ell}',\boldsymbol{\ell}'',\boldsymbol{\ell}'''\}$ is linearly independent at $(t_0,\boldsymbol{x}_0)$. Finally, the image germ of $F_{(\gamma,A)}$ at p is diffeomorphic to an A_4-singularity if and only if $H''(t_0,\boldsymbol{x}_0)=H'''(t_0,\boldsymbol{x}_0)=H^{(4)}(t_0,\boldsymbol{x}_0)=0$, $H^{(5)}(t_0,\boldsymbol{x}_0)\neq 0$ and $\{\boldsymbol{\ell},\boldsymbol{\ell}',\boldsymbol{\ell}'',\boldsymbol{\ell}''',\boldsymbol{\ell}^{(4)}\}$ is linearly independent at $(t_0,\boldsymbol{x}_0)$.

By taking the derivative of $H(t,\boldsymbol{x})=\langle\boldsymbol{\ell}(t),\boldsymbol{x}-\boldsymbol{\gamma}(t)\rangle$ with respect to t and using the derivations of the relations $\langle\boldsymbol{\ell},\boldsymbol{\gamma}'\rangle\equiv 0$ and $\langle\boldsymbol{\ell}',\boldsymbol{a}_1\rangle\equiv 0$, we have

$$
\begin{aligned}
H''(t_0,\boldsymbol{x}_0) &= -\langle\boldsymbol{\gamma}'+s\boldsymbol{a}_1'+r\boldsymbol{\ell}',\boldsymbol{\ell}'\rangle\\
H'''(t_0,\boldsymbol{x}_0) &= -\langle 2\boldsymbol{\gamma}'+2s\boldsymbol{a}_1'+3r\boldsymbol{\ell}',\boldsymbol{\ell}''\rangle-\langle\boldsymbol{\gamma}''+s\boldsymbol{a}_1'',\boldsymbol{\ell}'\rangle\,.
\end{aligned}
$$

We can calculate $H^{(4)}(0,\mathbf{0})$ and $H^{(5)}(0,\mathbf{0})$ analogously. The derivatives of $\boldsymbol{\ell}$ is calculated as follows:

$$
\begin{aligned}
\boldsymbol{\ell}' &= c_2a_0+c_2a_2+c_{36}a_3\\
\boldsymbol{\ell}'' &= \Big(c_2^2+c_3c_{36}+c_2'\Big)a_0-c_5c_{36}a_1+\Big(c_2^2-c_6c_{36}+c_2'\Big)a_2+\Big(c_2c_{36}+c_{36}'\Big)a_3\\
\boldsymbol{\ell}''' &= \Big(-c_1c_5c_{36}+3c_3c_{36}-c_3c_6'+c_3'c_6\Big)a_0+\Big(c_1c_{36}^2-(c_2c_5+c_5')c_{36}-2c_5c_{36}'\Big)a_1\\
&\quad+\Big((-c_1c_5-c_6')c_{36}-2c_6c_{36}'\Big)a_2+\Big((c_2^2-c_5^2+2c_2'+c_3^2-c_6^2)c_{36}+c_2c_{36}'+c_{36}''\Big)a_3\\
\boldsymbol{\ell}'''' &= \alpha_0a_0+\alpha_1a_1+\alpha_2a_2+\alpha_3a_3
\end{aligned}
$$

where,

$$
\begin{aligned}
\alpha_0 &= c_2^4+c_3^4-c_3^2c_5^2+c_3^3c_6-c_3c_5^2c_6-c_3^2c_6^2-c_3c_6^3+c_1^2c_{36}^2-c_3c_5c_1'-c_5c_6c_1'\\
&\quad+3c_3^2c_2'+2c_3c_6c_2'-c_6^2c_2'+3(c_2')^2+c_2^2((2c_3-c_6)c_{36}+6c_2')\\
&\quad+3c_3'c_{36}'-c_1(2c_{36}c_5'+3c_5c_{36}')\\
&\quad-c_2(2c_1c_5c_{36}-3c_3c_3'+2c_6c_3'+5c_6c_6'-4c_2'')+4c_3c_3''+c_6c_3''+3c_3c_6''+c_2'''\\
\alpha_1 &= -(c_3^3c_5)+c_5^3c_6+c_5c_6^3-c_2^2c_5c_{36}+c_6^2c_1'+c_3^2(-(c_5c_6)+c_1')\\
&\quad-3c_5c_6c_2'+5c_1c_6c_3'-3c_3'c_5'+5c_1c_6c_6'-3c_5'c_6'-c_2(c_{36}c_5'+2c_5c_{36}')\\
&\quad-3c_5c_3''+c_3(c_5^3+2c_6c_1'+c_5(c_6^2-3c_2')+5c_1c_{36}'-c_5'')-c_6c_5''-3c_5c_6''\\
\alpha_2 &= c_2^4-c_3^3c_6+c_3c_5^2c_6-c_3^2c_6^2+c_5^2c_6^2+c_3c_6^3+c_6^4+c_1^2(c_3+c_6)^2-c_3c_5c_1'-c_5c_6c_1'\\
&\quad+c_3^2c_2'-2c_3c_6c_2'-3c_6^2c_2'+3(c_2')^2+c_2^2((c_3-2c_6)c_{36}+6c_2')-3c_6'c_{36}'\\
&\quad-c_1(2c_{36}c_5'+3c_5c_{36}')\\
&\quad+c_2(-2c_1c_5c_{36}+5c_3c_3'+(2c_3-3c_6)c_6'+4c_2'')-3c_6c_3''-(c_3+4c_6)c_6''+c_2'''\\
\alpha_3 &= c_2^3c_{36}+c_2^2c_{36}'+3c_3^2(2c_3'+c_6')+3c_3(-(c_5c_5')+c_6(c_3'-c_6')+c_2'')\\
&\quad-3(c_5c_6c_5'+c_5^2c_{36}'-c_2'c_{36}'+c_6^2(c_3'+2c_6')-c_6c_2'')\\
&\quad+c_2(c_{36}(c_3^2-c_5^2-c_6^2+5c_2')+c_{36}'')+c_{36}'''
\end{aligned}
\;.
$$

Now, by using (4.1) and (4.2), we can calculate similarly the fourth derivatives of $\boldsymbol{\gamma}$ and $\boldsymbol{a}_1$. From these calculations, we obtain the following.

Proposition 5.3 *Suppose that $c_{36}(t_0) \neq 0$.*

- *The image germ of $F_{(\gamma,A)}$ at p is diffeomorphic to the $3/2$-cusp $\times\mathbb{R}^2$ if and only if the following holds at p,*

$$\begin{aligned}&\Xi_1 := sc_5 + rc_{36} + b_3 = 0 \qquad \textit{and}\\ &\Xi_2 := (b_2 + sc_1 + rc_2)c_{36} - sc_5' - c_{36}' + b_1c_5 - b_3' \neq 0.\end{aligned}$$

- *The image germ of $F_{(\gamma,A)}$ at p is diffeomorphic to the swallowtail $\times\mathbb{R}$ if and only if the following holds at p,*

$$\begin{aligned}&\Xi_1 = \Xi_2 = 0 \qquad \textit{and}\\ &\Xi_3 := (sc_1 - b_1c_1)c_{36} + 2(b_2 + sc_1)c_{36}' - rc_{36}'' + \big((-b_2 - sc_1)c_{36} + 2rc_{36}'\big)c_2\\ &\qquad\qquad -rc_{36}c_2^2 + rc_2'c_{36} + 2b_1c_5' - sc_5'' - b_3'' \neq 0.\end{aligned}$$

- *The image germ of $F_{(\gamma,A)}$ at p is diffeomorphic to the A_4 singularity if and only if the following holds at p,*

$$\begin{aligned}&\Xi_1 = \Xi_2 = \Xi_3 = 0 \qquad \textit{and}\\ &\Xi_4 := (-c_1b_1' - 2b_1c_1' + b_2'' + sc_1'')c_{36} + (-3b_1c_1 + 3b_2' + 3sc_1')c_{36}' + 3(b_2 + sc_1)c_{36}'' - rc_{36}'''\\ &\quad +\Big((b_1c_1 - sc_1' - b_2')c_{36} - 3(sc_1 + b_2)c_{36} + 3rc_{36}''\Big)c_2 + \Big(sc_1 + b_2)c_{36} - 3rc_{36}'\Big)c_2^2 + rc_{36}c_2^3\\ &\quad +\Big((-2sc_1 - 2b_2 - 3rc_2)c_{36} + 3rc_{36}'\Big)c_2' + rc_{36}c_2'' + b_1''c_5 + 3b_1'c_5' + 3b_1c_5'' - sc_5''' - b_3''' \neq 0.\end{aligned}$$

On the other hand, we can consider a singular point p with $c_{36}(t_0) = 0$ as an application of the criterion in Appendix B. The area density function for $F_{(\gamma,A)}$ is $b_3 + sc_5 + rc_{36}$. A point p is non-degenerate if and only if $b_3' + sc_5' + rc_{36}' \neq 0$ or $c_5 \neq 0$. Therefore, the null vector field is given by $(-1, b_1, b_0 + sc_1 + rc_2)$. Since the singular set is $S = S(F_{(\gamma,A)}) = \{b_3 + sc_5 + rc_{36} = 0\}$, the tangent plane T_pS is spanned by $v_1 = (-c_5, b_3' + sc_5' + rc_{36}', 0)$ and $v_2 = (0, 0, 1)$. Hence $\eta \notin T_pS(f)$ if and only if $b_3' + sc_5' + rc_{36}' - c_5b_1 \neq 0$. Therefore the function $\tilde{\psi}$ defined in Appendix B is $\psi = \langle \boldsymbol{\ell}, a_3' \rangle = c_{36}$. As a summary of these arguments, we can state the following.

Proposition 5.4 *$F_{(\gamma,A)}$ at p is diffeomorphic to cuspidal cross cap $\times\mathbb{R}$ if and only if the following conditions hold:*

- $b_3 + sc_5 + rc_{36} = 0$
- $c_{36} = 0$
- $b_3' + sc_5' + rc_{36}' \neq 0$ *or* $c_5 \neq 0$
- $c_{36}' \neq 0$.

Observe that since flat lightlike planar hypersurfaces are determined up to Lorentzian motion by functions $(c_1, c_2, c_3, c_5, c_6, b_0, b_1, b_3)$, the space of flat lightlike planar hypersurfaces FLP can be defined by the set of these functions $FLP = C^\infty(I, \mathbb{R}^8)$ with the Whitney C^∞-topology. Then, as a consequence of the above results, we obtain the following generic classification.

Theorem 5.5 *There exists an open and dense subset $\mathcal{O}_1 \subset FLP$ such that for any $a \in \mathcal{O}_1$, the map germ $F_{(\gamma,A)}$ defined by a at any point is $\mathcal{A}$-equivalent to either a regular point, a cusp $\times\mathbb{R}^2$, a swallowtail $\times\mathbb{R}$, an A_4-singularity, or a cuspidal cross cap $\times\mathbb{R}$.*

Proof. We first define following sets:

$$
\begin{aligned}
&\Xi^i := \{(s,t,r) \in \mathbb{R} \times J^3(I,\mathbb{R}^8) \times \mathbb{R} \,|\, \Xi_i = 0\}, \qquad (i = 1,\ldots,4)\\
&\Xi^5 = \{j^3a(t) \in J^3(I,\mathbb{R}^8) \,|\, c_{36}(t) = 0\},\\
&\Xi^6 = \{j^3a(t) \in J^3(I,\mathbb{R}^8) \,|\, c_5(t) = 0\} \text{ and}\\
&\Xi^7 = \{j^3a(t) \in J^3(I,\mathbb{R}^8) \,|\, c_{36}'(t) = 0\}.
\end{aligned}
$$

It is easy to check $\Xi^5 \cap \Xi^6$ and $\Xi^5 \cap \Xi^7$ are both closed submanifolds of $J^3(I,\mathbb{R}^8)$ codimension 2. Next we show that $\cap_{i=1}^4 \Xi_i$ is a codimension 4 submanifold of $\mathbb{R} \times J^3(I,\mathbb{R}^8) \times \mathbb{R}$. Denote $\Xi = (\Xi_1,\ldots,\Xi_4)$, then $\cap_{i=1}^4 \Xi_i = \Xi^{-1}(0,0,0,0)$. Hence it is sufficient to prove that $(0,0,0,0)$ is a regular value of Ξ. We calculate the derivative of Ξ with respect to the coordinates of $\mathbb{R} \times J^3(I,\mathbb{R}^8) \times \mathbb{R}$ corresponding to the zero, first, second and third derivatives of b_3. It coincides with

$$
\begin{pmatrix}
1 & 0 & 0 & 0\\
* & -1 & 0 & 0\\
* & * & -1 & 0\\
* & * & * & -1.
\end{pmatrix}
$$

This has the maximal rank at any point. So, the image $\pi(\cap_{i=1}^4 \Xi_i)$ by the projection $\pi : \mathbb{R} \times J^3(I,\mathbb{R}^8) \times \mathbb{R} \to J^3(I,\mathbb{R}^8)$ is a closed semi-algebraic set of codimension 2. Hence it admits a canonical stratification. By applying the Thom-Mather jet transversality theorem to the strata $\Xi^5 \cap \Xi^6$ and $\Xi^5 \cap \Xi^7$, we have that there exists an open and dense set $\mathcal{O}_1$ such that for any $j^3a(t) \in \mathcal{O}_1$ is transverse to $\Xi^5 \cap \Xi^6$ and $\Xi^5 \cap \Xi^7$. Since they have codimensions lesser than 2, the transversality conditions means that there are no intersection points. Therefore $\mathcal{O}_1$ has the desired properties. $\square$

6 Singularities of lightlike flat spacelike surfaces with partially parallel normal frame

In this last section, we shall study the generic behavior of the lightlike flat spacelike surfaces with partially parallel normal frame. By Theorem 3.3, such a surface can be parameterized as

$$F(s,t) = \gamma(t) + s\boldsymbol{a}_1(t) + r(s,t)\boldsymbol{\ell}(t), \qquad b_0 \equiv b_2,\ c_1 \equiv c_4$$

where we are using the frame given in (4.1) and (4.2). Here, we consider that a lightlike flat spacelike surface with partially parallel normal frame is determined by functions $\alpha = (c_1, c_2, c_3, c_5, c_6, b_0, b_1, b_3, r) \in C^\infty(I \times \mathbb{R}, \mathbb{R}^9)$, where c_i and b_j are seen as maps $c_i(s,t) = c_i(t)$, $b_j(s,t) = b_j(t)$.

We define the space of lightlike flat spacelike surfaces with partially parallel normal frame as $LFSP := C^\infty(I \times \mathbb{R}, \mathbb{R}^9)$ with the Whitney C^∞-topology. In order to study these surfaces, we try to find a special curve on the surface along which the singularities are located. Such curves are usually called the *striction curves* of the surface. A point (s,t) is said to be *noncylindrical* if $c_5 + r_s c_{36} \neq 0$ holds. Let us assume that any (s,t) is noncylindrical. A curve $(s(t),t)$ is the striction curve of F if $b_3(t)+s(t)c_5(t)+r(s(t),t)c_{36}(t)=0$ holds. Now we study the singularities of F. A point (s,t) is a singular point of F if and only if

$$\hat{\Xi}_1 := b_3 + sc_5 + rc_{36} = 0,\ \hat{\Xi}_2 := b_0 + sc_1 + r' + rc_2 - b_1 r_s = 0. \tag{6.1}$$

So we have that the singular points are located on the striction curve. It follows from the same arguments as those in Whitney [28] that the map germ F at (s,t) is $\mathcal{A}$-equivalent to the cross cap in $\mathbb{R}^4$ if and only if (6.1) and

$$F_s,\ b_1F_{ss} - F_s',\ -b_1^2F_{ss} + F''$$

are linearly independent, where, $' = \partial/\partial t$. Here, the cross cap in $\mathbb{R}^4$ is the map germ defined by $(x,y,z) \mapsto (x^2, xy, y, z)$ at the origin. By a straightforward calculation, the above condition is equivalent to

$$\hat{\Xi}_3 := \det\begin{pmatrix} r_s & b_1r_{ss} - r_{st} - c_1 - r_sc_2 & w_1 \\ 1 & 0 & b_1' \\ 0 & -c_5 - r_sc_{36} & w_2 \end{pmatrix} \neq 0, \tag{6.2}$$

where,

$$\begin{array}{rcl} w_1(t,s) & = & b_0' + b_1c_1 + b_0c_2 + s(c_1' + c_1c_2) + r'' + 2r'c_2 + r(c_2' + c_2^2) - b_1^2r_{ss} \\ w_2(t,s) & = & b_1(r_sc_{36} + c_5) + b_3' + sc_5' + r'c_{36} + rc_{36}' \end{array}.$$

We have the following lemma.

Lemma 6.1 *There is an open and dense subset $\mathcal{O}_2 \subset LFSP$ such that for any $\alpha \in \mathcal{O}_2$, the lightlike flat spacelike surface with partially parallel normal frame determined by α is non-singular at each cylindrical point.*

Proof. We define a map $NC : J^3(I\times\mathbb{R},\mathbb{R}^9) \to \mathbb{R}^3$ by

$$NC(j^3(\alpha(s,t))) = (c_5 + r_sc_{36}, \hat{\Xi}_1, \hat{\Xi}_2).$$

Since $(0,0,0)$ is a regular value, $NC^{-1}(0,0,0)$ is a submanifold of $J^3(I\times\mathbb{R},\mathbb{R}^9)$ with codimension 3. By using the same method as in the proof of Theorem 5.5, we reach the desired conclusion. □

Theorem 6.2 *There exists an open and dense subset $\mathcal{O}_3$ such that for any $\alpha_1 \in \mathcal{O}_3$, the map germ of the lightlike flat spacelike surfaces with partially parallel normal frame determined by α_1 at any one of its points is either $\mathcal{A}$-equivalent to a regular point, or to the cross cap in $\mathbb{R}^4$.*

Proof. We define a map $\hat{\Xi} : J^3(I\times\mathbb{R},\mathbb{R}^9)\setminus NC^{-1}(0,0,0) \to \mathbb{R}^4$ by

$$\hat{\Xi}(j^3(\alpha(s,t))) = (\hat{\Xi}_1, \hat{\Xi}_2, \hat{\Xi}_3).$$

We calculate the derivative of $\hat{\Xi}$ with respect to the coordinates of $J^3(I\times\mathbb{R},\mathbb{R}^9)$ corresponding to b_3, b_0 and b_0', where b_0' means the first derivative of b_0 with respect to t. The Jacobi matrix is given by

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & -(c_5 + r_sc_{36}) \end{pmatrix}. \tag{6.3}$$

In fact, since $\hat{\Xi}_1$ is equal to $b_3 + sc_5 + rc_{36}$, the derivative of $\hat{\Xi}_1$ with respect to the coordinates corresponding to b_3 is equal to 1. Furthermore, since $\hat{\Xi}_3$ is given in (6.2), its derivative with respect to the coordinates corresponding to b'_0 is equal to $-(c_5 + r_s c_{36})$. Moreover $\hat{\Xi}_1$ and $\hat{\Xi}_2$ have no terms in b'_0, and thus we have the third row of (6.3).

This matrix has maximal rank at any point. Hence $NC^{-1}(0,0,0) \cup \hat{\Xi}^{-1}(0,0,0)$ is a closed algebraic set in $J^3(I \times \mathbb{R}, \mathbb{R}^9)$. By using the same method as in the proof of Theorem 5.5, we arrive to the desired conclusion. $\square$

We say that a lightlike flat spacelike surface with partially parallel normal frame is a *cylinder* if $c_5 + r_s c_{36} \equiv 0$. Such a surface seems to have the same type geometrical properties as cylinders in $\mathbb{R}^3$. The investigation of this surface is left for a future work.

A Generating families

Here we give a quick survey on the theory of Legendrian singularities mainly developed by Arnol'd-Zakalyukin [1,29]. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is *a Morse family* if the map germ

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \ldots, \frac{\partial F}{\partial q_k}\right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is submersive, where $(q, x) = (q_1, \ldots, q_k, x_1, \ldots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)$-dimensional submanifold

$$\Sigma_*(F) = \left\{ (q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \;\middle|\; F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \cdots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}$$

and the map germ $\mathcal{L}_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\mathcal{L}_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \cdots : \frac{\partial F}{\partial x_n}(q, x)\right]\right)$$

is a Legendrian immersion. Then we have the following fundamental theorem in the theory of Legendrian singularities ([1] §20.7 [29], Page 27).

Proposition A.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F *a generating family* of $\mathcal{L}_F$, and the corresponding wave front is $W(\mathcal{L}_F) = \pi_n(\Sigma_*(F))$, where $\pi_n : \mathbb{R}^k \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the canonical projection.

We now introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves the fibers of π and $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighborhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighborhood of the original point such that each Legendrian immersion belonging to the first neighborhood has in the second neighborhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined by the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition A.2 *Let $i : (L,p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i$ and $\pi \circ i'$ are dense. Then i, i' are Legendrian equivalent if and only if their wave front sets, $W(i)$ and $W(i')$, are diffeomorphic as set germs. Here $\pi : PT^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the canonical projection of the projective cotangent bundle.*

This result was firstly pointed out by Zakalyukin ([30], Assertion 1.1). In his original assertion, he assumed that the representatives of $\pi \circ i$ and $\pi \circ i'$ are proper. However, we remark that we can get rid of such an assumption. The assumption in the above proposition is a generic condition for i, i'. In particular, if i and i' are Legendrian stable, then they satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote by $\mathcal{E}_n$ the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal $\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are *P-$\mathcal{K}$-equivalent* if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back $\mathbb{R}$-algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *$\mathcal{K}$-versal deformation of $f = F|\mathbb{R}^k \times \{\mathbf{0}\}$* if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1}|\mathbb{R}^k \times \{\mathbf{0}\}, \ldots, \frac{\partial F}{\partial x_n}|\mathbb{R}^k \times \{\mathbf{0}\} \right\rangle_{\mathbb{R}}, \tag{A.1}$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \ldots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [20].) The main result in the theory ([1], §20.8 and [29], Theorem 2) is the following:

Theorem A.3 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Then*

(1) *Φ_F and Φ_G are Legendrian equivalent if and only if F, G are P-$\mathcal{K}$-equivalent, and*

(2) *Φ_F is Legendrian stable if and only if F is a $\mathcal{K}$-versal deformation of $F \mid \mathbb{R}^k \times \{\mathbf{0}\}$.*

Since F and G are function germs on the common space germ $(\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$, we do not need the notion of stably P-$\mathcal{K}$-equivalences under this situation (cf. [29], Page 27). As a consequence of the uniqueness of the $\mathcal{K}$-versal deformation of a function germ, we have the following classification result of Legendrian stable germs (cf. [10]). Given a map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, *the local ring of f* is defined by $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p)\mathcal{E}_n$.

Proposition A.4 *Let F and $G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families. Suppose that $\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian stable. The the following conditions are equivalent.*

(1) *$(W(\mathcal{L}_F), \mathbf{0})$ and $(W(\mathcal{L}_G), \mathbf{0})$ are diffeomorphic as germs.*

(2) *$\mathcal{L}_F$ and $\mathcal{L}_G$ are Legendrian equivalent.*

(3) *$Q(f)$ and $Q(g)$ are isomorphic as $\mathbb{R}$-algebras,*

where $f = F|\mathbb{R}^k \times \{\mathbf{0}\}, g = G|\mathbb{R}^k \times \{\mathbf{0}\}$.

B Criteria for the cuspidal cross cap $\times\mathbb{R}^k$

In this section, we describe a criteria to characterize the *cuspidal cross cap* $\times\mathbb{R}^k$. A useful criterion for the cuspidal cross cap has been given in [6]. The *cuspidal cross cap* $\times\mathbb{R}^k$ is the k-dimensional suspension of cuspidal cross cap. This is a map germ at origin defined by

$$f_{\mathrm{ccr}} : (u, v, \boldsymbol{w}) \mapsto (u, v^2, uv^3, \boldsymbol{w}), \quad \boldsymbol{w} = (w_1, \dots, w_k).$$

Let $U \subset \mathbb{R}^{k+2}$ be a domain. We can identify the projective cotangent bundle $PT^*\mathbb{R}^{k+3}$ with its canonical contact structure with the projective tangent bundle $PT\mathbb{R}^{k+3} = \mathbb{R}^{k+3} \times P^{k+2}$. A smooth map $f : U \to \mathbb{R}^{k+3}$ is said to be a *frontal* if there exists a non-zero vector field ν of $\mathbb{R}^{k+3}$ along f such that $L := (f, [\nu]) : U \to \mathbb{R}^{k+3} \times P^{k+2}$ is an *isotropic map*. That is, the pull-back of the canonical contact form α of $PT\mathbb{R}^{k+3}$ vanishes on U. Since this condition is equivalent to asking that ν be perpendicular to df, we call ν a *normal vector* of f. An *area density function* of a frontal f is

$$\lambda(u, v, \boldsymbol{w}) = \det(f_u, f_v, f_{w_1}, \dots, f_{w_k}, \nu).$$

A singular point $p \in U$ of f is *non-degenerate* if $d\lambda(p) \neq 0$. Suppose that p is a non-degenerate singular point of f. Then, in a small enough neighborhood of p, the set of singular points $S(f)$ is a submanifold of codimension 1, and there exists a non-zero vector field η satisfying that for any $q \in S(f)$, $df_q(\eta_q) = 0$. We call this vector field a *null vector field.*

Theorem B.1 *The germ of a frontal $f : U \to \mathbb{R}^{k+3}$ at a singular point p is $\mathcal{A}$-equivalent to the cuspidal cross cap $\times\mathbb{R}^k$ if and only if p is non-degenerate, η is transverse to $S(f)$ at p and it satisfies the following condition: For any linearly independent tangent vector field $(u_1, \dots, u_{k+1})$ of $S(f)$, the function ψ on $S(f)$ given by*

$$\psi = \det(f_{u_1}, \dots, f_{u_{k+1}}, D^f_\eta \nu, \nu) \tag{B.1}$$

satisfies that $\psi(p) = 0$ and $d\psi(p) \neq 0$. Here, D^f is the canonical covariant derivative along f induced by the canonical connection on $\mathbb{R}^{k+3}$, and f_{u_i} means the directional derivative $u_i f$, $(1 \leq i \leq k+1)$.

In order to prove this theorem we shall follow a method introduced in [6]. We start with the following key lemma.

Lemma B.2 *The conditions of Theorem B.1 do not depend on the choice of coordinates in both U and $\mathbb{R}^{k+3}$, neither on the representative ν, or on the choice of η.*

Proof. The non-degeneracy and transversality conditions do not depend on the coordinates. One can easily check that these conditions do not depend on scalar multiplications by non-zero functions to ψ. Hence they do not depend on the choice of representative of ν, neither on the choice of η. Next, we prove that it does not depend on the choice of the vector fields $(u_1, \dots, u_{k+1})$ too. Assume that $(v_1, \dots, v_{k+1})$ is another vector field satisfying these conditions. Then by the non-degeneracy, for all $q \in S(f)$, we have $\langle f_{u_1}(q), \dots, f_{u_{k+1}}(q) \rangle = T_q f(S(f))$, $f_{v_i}(q) = \sum_{j=1}^{k+1} a_{ij}(q) f_{u_j}(q)$ $(1 \leq i \leq k+1)$. And hence we get,

$$\det(f_{v_1}(q), \dots, f_{v_{k+1}}(q), D^f_\eta \nu, \nu) = \det\big((a_{ij}(q))_{1 \leq i,j \leq k+1}\big)\psi(q).$$

So, the conditions do not depend on the choice of the vector field. Since the conditions of Theorem B.1 do not use the coordinates in the source, we have that they do not depend on these coordinates. Now, we only need to prove that these conditions do not depend on the choice of coordinates in the target. Since they do not depend on the choice of coordinates on U, neither on the choice of η, we may assume that the coordinates $(u, v, \boldsymbol{w})$ on U satisfy that $S(f) = \{v = 0\}$, $\eta = \partial v$, and the vector field $(u_1, \ldots, u_{k+1})$ is given by $\partial u, \partial w_1, \ldots, \partial w_k$. Moreover, we can assume that the representative ν satisfies $|\nu| = 1$. Under this assumption, we have

$$\psi = \det(f_u, f_{w_1}, \ldots, f_{w_k}, \nu_v, \nu).$$

Consider a diffeomorphism $\tilde{\Omega}$ of $\mathbb{R}^{k+3}$ and denote its differential by $\tilde{\Omega}_*$. If we denote $\tilde{f} = \tilde{\Omega} \circ f$, then

$$\tilde{\nu} = {}^t\tilde{\Omega}_*(f)^{-1}\nu$$

gives the normal vector of $\tilde{f}$, where we consider $\tilde{\Omega}_*$ as a $GL(3, \mathbb{R})$-valued function. For notation simplification, Ω_* denotes the matrix valued function $\tilde{\Omega}_*(f)$. Since

$$({}^t\Omega_*^{-1}\nu)_v = ({}^t\Omega_*^{-1})_v\nu + {}^t\Omega_*^{-1}\nu_v$$

and ∂v is the null vector field on $S(f)$, $({}^t\Omega_*^{-1}\nu)_v = {}^t\Omega_*^{-1}\nu_v$ on $S(f)$. Therefore the function $\tilde{\psi}$ of $\tilde{f}$ defined by (B.1) is

$$\tilde{\psi} = \det(\Omega_* f_u, \Omega_* f_{w_1}, \ldots, \Omega_* f_{w_k}, {}^t\Omega_*^{-1}\nu_v, {}^t\Omega_*^{-1}\nu).$$

It is sufficient to prove that $\psi(p) = 0$ (respectively, $\psi(p) = 0$ and $d\psi(p) \neq 0$) implies $\widetilde{\psi}(p) = 0$ (respectively, $\widetilde{\psi}(p) = 0$ and $d\widetilde{\psi}i(p) \neq 0$) . We have that $\langle f_u, \nu_v \rangle = \langle f_v, \nu_u \rangle = 0$ and $\langle f_{w_i}, \nu_v \rangle = \langle f_v, \nu_{w_i} \rangle = 0$ for any $i = 1, \ldots, k$ on $S(f)$. Moreover, since we assume $|\nu| = 1$, we have $\langle \nu, \nu_v \rangle = 0$. Thus, $\psi(p) = 0$ implies that $\nu_v(p) = 0$, in particular $\tilde{\psi}(p) = 0$ holds.

Next, we assume that $\psi(p) = 0$ and $d\psi(p) \neq 0$. Since $\nu_v(p) = 0$, we can assume

$$\psi_u(p) = \det(f_u, f_{w_1}, \ldots, f_{w_k}, \nu_{uv}, \nu)(p) \neq 0$$

by permutating the coordinates if necessary. This implies that $\nu_{uv} \notin \langle f_u, f_{w_1}, \ldots, f_{w_k}, \nu \rangle_{\mathbb{R}}$ at p. In particular, $\nu_{uv} \notin \langle \nu \rangle_{\mathbb{R}}$, and we have ${}^t\Omega_*^{-1}\nu_{uv} \notin \langle {}^t\Omega_*^{-1}\nu \rangle_{\mathbb{R}}$ at p.

On the other hand, $\langle f_u, \nu_v \rangle = \langle f_{w_i}, \nu_v \rangle = 0$ holds on $S(f)$ for $i = 1, \ldots, k$. By taking the derivative of these formulae and applying $\nu_v(p) = 0$, we get

$$\langle f_u, \nu_{uv} \rangle (p) = \langle f_{w_i}, \nu_{uv} \rangle (p) = 0 \tag{B.2}$$

Hence we have

$$\left\langle {}^t\Omega_*^{-1}\nu_{uv}, \Omega_* f_u \right\rangle (p) = \left\langle \nu_{uv}, \Omega_*^{-1}\Omega_* f_u \right\rangle (p) = \langle \nu_{uv}, f_u \rangle (p) = 0, \quad \left\langle {}^t\Omega_*^{-1}\nu_{uv}, \Omega_* f_{w_i} \right\rangle (p) = 0.$$

Clearly,

$$\left\langle {}^t\Omega_*^{-1}\nu, \Omega_* f_u \right\rangle = 0, \quad \left\langle {}^t\Omega_*^{-1}\nu, \Omega_* f_{w_i} \right\rangle = 0 \text{ at } p$$

hold, and we have

$${}^t\Omega_*^{-1}\nu, \quad {}^t\Omega_*^{-1}\nu_{uv} \in \left(\langle \Omega_* f_u, \Omega_* f_{w_1}, \ldots, \Omega_* f_{w_k} \rangle_{\mathbb{R}} \right)^{\perp} \text{ at } p.$$

Therefore, ${}^t\Omega_*^{-1}\nu_{uv} \notin \langle {}^t\Omega_*^{-1}\nu \rangle_{\mathbb{R}}$ implies that

$$\frac{\partial \tilde{\psi}}{\partial u}(p) = \det(\Omega_* f_u, \Omega_* f_{w_1}, \ldots, \Omega_* f_{w_k}, {}^t\Omega_*^{-1}\nu_{uv}, {}^t\Omega_*^{-1}\nu)(p) \neq 0.$$

This completes the proof. $\square$

We finally give the proof of Theorem B.1: Let us assume that a singular point p of a frontal f satisfies the condition of the Theorem. Since p is of rank $k+1$, by Lemma B.2 we may assume that

$$f = (g_1(u, v, \boldsymbol{w}), g_2(u, v, \boldsymbol{w}), u, \boldsymbol{w}).$$

By an appropriate coordinate change in the target, we can write

$$f = (v\overline{g_1}(u, v, \boldsymbol{w}), v\overline{g_2}(u, v, \boldsymbol{w}), u, \boldsymbol{w}).$$

We may also assume that the singular set is $\{v = 0\}$ and that the null vector field is ∂v. Then we can put

$$f = (v^2\tilde{g}_1(u, v, \boldsymbol{w}), v^2\tilde{g}_2(u, v, \boldsymbol{w}), u, \boldsymbol{w}).$$

Since $d\lambda(p) \neq 0$, it follows that $\tilde{g}_1(p) \neq 0$. By applying now the coordinate change $\overline{u} = u\sqrt{|\tilde{g}_1|}, \overline{v} = v, \overline{\boldsymbol{w}} = \boldsymbol{w}$, we may assume that

$$f = (\overline{v}^2, \overline{v}^2 h(\overline{u}, \overline{v}, \overline{\boldsymbol{w}}), \overline{u}, \overline{\boldsymbol{w}}).$$

By decomposing h into an odd function and an even function with respect to $\overline{v}$, and using a coordinate change in the target, we may assume that

$$f = (\overline{v}^2, \overline{v}^3\overline{h}(\overline{u}, \overline{v}^2, \overline{\boldsymbol{w}}), \overline{u}, \overline{\boldsymbol{w}}).$$

By the assumption of the theorem, and transposing the coordinates $\overline{u}, \overline{\boldsymbol{w}}$ if necessary, we have that $\overline{h}_{\overline{u}}(p) \neq 0$. By applying now the coordinate change $U = \overline{h}(\overline{u}, \overline{v}^2, \overline{\boldsymbol{w}}), V = \overline{v}, \boldsymbol{W} = \overline{\boldsymbol{w}}$, we get that f is $\mathcal{A}$-equivalent to

$$(V^2, UV^3, \hat{h}(U, V, \boldsymbol{W}), \boldsymbol{W}).$$

Since $u = \hat{h}(\overline{h}(u, v^2, \boldsymbol{w}), v, \boldsymbol{w})$, we have that $\hat{h}$ is an even function with respect to V, namely $\hat{f}(U, V, \boldsymbol{W}) = \tilde{h}(U, V^2, \boldsymbol{W})$. Moreover, it is clear that $\tilde{h}_U(p) \neq 0$. By considering the inverse (diffeomorphism) of

$$(x, y, z, \boldsymbol{w}) \mapsto (x, y, \tilde{h}(z, x, \boldsymbol{w}), \boldsymbol{w}),$$

we obtain that f is $\mathcal{A}$-equivalent to

$$(v^2, uv^3, u, \boldsymbol{w}).$$

This completes the proof. $\square$

Remark B.3 *Since η is the null vector field, the derivative D can be chosen to be an arbitrary linear connection on $\mathbb{R}^{k+3}$. Moreover, for any vector field ξ along $f|_{S(f)}$ satisfying that $\xi \in \nu^{\perp}$ and that ξ is transverse to $f_*(TS(f))$, we define a function $\tilde{\psi} := \left\langle D^f_\eta \xi, \nu \right\rangle$. Then, there is a non-zero function α such that $\psi = \alpha\tilde{\psi}$. This also holds if $\langle\ ,\ \rangle$ is a pseudo inner product.*

Proof. For any ξ, we can find tangential vector fields $(u_1, \dots, u_{k+1})$ and functions $a(\neq 0), b_i$ $(1 \leq i \leq k+1)$ such that

$$\xi = a(f_{u_1} \wedge \cdots \wedge f_{u_{k+1}} \wedge \nu) + \sum_{i=1}^{k+1} b_i f_{u_i}.$$

Since $\langle \xi, \nu \rangle = 0$ and η is null direction, we have $\left\langle D^f_\eta \xi, \nu \right\rangle + \left\langle \xi, D^f_\eta \nu \right\rangle = 0$. Hence

$$\tilde{\psi} = -\left\langle \xi, D^f_\eta \nu \right\rangle = -a \left\langle f_{u_1} \wedge \cdots \wedge f_{u_{k+1}} \wedge \nu, D^f_\eta \nu \right\rangle = -a\psi.$$

This completes the proof. The last part of the remark is obvious, for pseudo inner products are non-degenerate symmetric bilinear forms, so we can follow the discussion given in this section. □

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