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エネルギー最小化の視点からの、空間1次元の Activator-Inhibitor system の最安定定常解 (On the most stable steady states of Activator-Inhibitor system in view of minimizing an free energy)

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1 Introduction

In the present paper, we consider the stationary problem about the following activator-inhibitor system in one space dimension:

$$\begin{aligned} u_t &= \varepsilon^2 u_{xx} + f(u) - \sigma v \quad \text{in} \quad (0,1), \\ \delta v_t &= v_{xx} + u - m - \gamma v \quad \text{in} \quad (0,1), \end{aligned} \tag{1.1}$$

with the homogeneous Neumann boundary condition and an adequate initial condition. Here $f(u)$ is a so-called balanced nonlinearity of Fitz-Hugh Nagumo type. Actually, in the main results (Theorem 1.1 and 1.2) in this paper, $f(u)$ may as well be more general nonlinearity of Fitz-Hugh Nagumo type like non-balanced one without the constraint. In

fact, by use of a kind of affine transformation, we may assume that $f(u)$ is balanced without loss of generality. But we do not assume that this is symmetric. Precisely, we assume the following (A1), (A2), (A3), and (A4) about the nonlinearity $f(u)$: We define the double-well potential $F(u)$ as the primitive function of $-f(u)$ (i.e., $F'(u) = -f(u)$) and there exist constants $a < 0$ and $b > 0$ such that

(A1) $F(z) = 0$, if $z = a$ and $z = b$, and $F(z) > 0$, otherwise,

(A2) $F \in C^3$, $F'(a) = F'(0) = F'(b) = 0$, and $F''(a)$, $F''(b) > 0$, $F''(0) < 0$.

(A3) $F'(z) < 0$, if $-\infty < z < a$ or $0 < z < b$, and $F'(z) > 0$, if $a < z < 0$ or $b < z < \infty$.

(A4) $\underline{\lim}_{z \to \pm\infty} |F'(z)| > 0$.

The other parameters are that $0 < \varepsilon \ll 1$, $\sigma > 0$, $\delta \geq 0$, $m \in (a, b)$, and $\gamma > 0$.

The stationary problem of (1.1) is

$$
\begin{aligned}
0 &= \varepsilon^2 u_{xx} + f(u) - \sigma v \quad \text{in} \quad (0,1), \\
0 &= v_{xx} + u - m - \gamma v \quad \text{in} \quad (0,1),
\end{aligned} \tag{1.2}
$$

with the homogeneous Neumann boundary condition.

There are long history and fruitful results of research about exsitence and stability of stationary solutions of (1.2). See, for instance, [1], [3], [4], and references therein. Recently, we are primarily concerned with structure of stable stationary solutions in an adequate space of functions. This is because it is known that there are "infinitely" many stable steady states of (1.2), as ε tends to zero. More precisely speaking, we have proved that, for given $n \in \mathbf{N}$, $\sigma > 0$, $m \in (a, b)$, and $\gamma > 0$, there exists a constant $\varepsilon_0 > 0$ such that, if $0 < \varepsilon < \varepsilon_0$, then n-layered stationary solution of (1.2) is exponentially stable by use of the SLEP method which has been developed by the second author. For details, refer to [3] again. Therefore, if ε is very small, there are enormously many stationary solutions of (1.2), all of which are exponentially stable in sense of (1.1). We would like to seek a criterion by that we judge which one is the "most" stable. For the purpose we come up with the following functional:

$$
J_{\varepsilon,\sigma,\gamma}(u) := \int_0^1 \left(\frac{\varepsilon^2}{2} |u_x|^2 + F(u) + \frac{\sigma}{2} |(-\Delta_N + \gamma I)^{-\frac{1}{2}} (u - m)|^2 \right) dx, \tag{1.3}
$$

because (1.2) is the Euler-Lagrange equation of $J_{\varepsilon,\sigma,\gamma}$ in the admissible space $H^1(0,1)$. Here $(-\Delta_N)^{-1}$ means the inverse of the minus Laplacian with the homogeneous Neumann boundary condition, and the fractional power is defined by the spectral decomposition.

Moreover, if δ is equal to 0, then (1.1) is the L^2-gradient flow of $J_{\varepsilon,\sigma,\gamma}$. That is easily seen by solving the second equation of (1.1) in v and putting it into the first equation.

Therefore, as calculating the value of the energy of each stable steady state, we judge it conveniently. We can use it as a criterion, only when δ is equal to 0 in (1.1). Then, surely, (1.1) is a system of equations, but there also is a property similar to a single equation. This is because the variable v in (1.1) changes quite immediately according to u's changing. The field of inhibitor has no time lag to reach a stationary state, when the activator changes. In point of view of phenomena, that is a meaning of the fact that the system (1.1) with $\delta = 0$ is the gradient flow of (1.3). Hence, there is no possibility for Hopf bifurcation to occur, and it is seen that there is no time periodic solution in this system (1.1) with $\delta = 0$ by use of general theory of gradient systems. On the other hand, in mathematically technical point of view, we can use some variational methods to solve the stationary system (1.2). We therefore have a possibility to get more information about the structure of stationary solutions of (1.2) than about the one of a system of equations without variational structure. In fact, we show the following theorem:

Theorem 1.1 *For given $\sigma > 0$, $\gamma > 0$, $m \in (a,b)$, and the nonlinearity f, there exists $\varepsilon_1 > 0$ such that, if $0 < \varepsilon < \varepsilon_1$, then the following properties hold:*

(1) If u^ε is a global minimizer of $J_{\varepsilon,\sigma,\gamma}$ in $H^1(0,1)$, then u^ε is spatially periodic and its minimum period P^ε satisfies

$$P^\varepsilon = 2\left(3A\left(\frac{b-a}{(b-m)(m-a)}\right)^2\frac{\varepsilon}{\sigma}\right)^{\frac{1}{3}} + O(\varepsilon^{\frac{2}{3}}).$$

(2) There are at most two distinct global minimizers up to translation.

Here we have defined by $A := \sqrt{2}\int_a^b \sqrt{F(\tau)}d\tau$.

Remarks:

(1) At the global minimizers, all of three components of energy, $J_{\varepsilon,\sigma,\gamma}$, have the same value. Namely, when a steady states has the global minimum energy, the energy is distributed equaly to the three terms of $J_{\varepsilon,\sigma,\gamma}$. This is also proved rigorously at the same time as we get Theorem 1.1. Moreover, if a stable steady states has less frequency than the global minimizers, the third term has more energy than the first and the second terms do. Inversely, in more frequent steady states, the first and the second terms are bigger than the third term. Furthermore, some numerical computations suggest that there is a rugged landscape of a graph of energy, $J_{\varepsilon,\sigma,\gamma}$, in which the bottom of energy is the global minimizer that we have characterized in the above theorem, and the more or less frequency a steady state have, the more energy it has. On the way to prove Theorem 1.1 (and Theorem 1.2 below), we have also proved those rigorously, as long as we consider about normal n-layered solutions for any $n \in \mathbf{N}$.

In the present paper, we basically concentrate a non-conserved case of (1.1). But we have gotten the similar characterization of global minimizers in a conserved case. Here the conserved case means that the integration $\int_0^1 u dx$ is conserved, if u evolves in time. Therefore, the time evolution equation and the stationary problem must be changed. In fact, we have considered the following Cahn-Hilliard type parabolic system as the time evolution equation

$$
\begin{aligned}
(1.4) \qquad u_t &= -(\varepsilon^2 u_{xx} + f(u) - \sigma v)_{xx} \quad \text{in} \quad (0,1), \\
\delta v_t &= v_{xx} + u - m - \gamma v \quad \text{in} \quad (0,1), \\
u(x,0) &= u_0(x), v(x,0) = v_0(x),
\end{aligned}
$$

with the homogeneous Neumann boundary condition. If $\int_0^1 u_0(x)dx = m$, then $\int_0^1 u(x,t)dx = m$ for any $t > 0$, as long as the solution of (1.4) exists. If $\delta = 0$, then this is the H^{-1}-gradient flow of $J_{\varepsilon,\sigma,\gamma}(u)$. The reason why we consider the Cahn-Hilliard type time evolution equation is that the activator's action must be local. If in the conserved case we dare to write the equation similar to the socond order one (1.1), we need to put the integral term $-\int_0^1 f(u)dx$ and $\sigma \int_0^1 v dx$ in it, namely,

$$
\begin{aligned}
(1.5) \qquad u_t &= \varepsilon^2 u_{xx} + f(u) - \int_0^1 f(u)dx - \sigma v + \sigma \int_0^1 v dx \quad \text{in} \quad (0,1), \\
\delta v_t &= v_{xx} + u - m - \gamma v \quad \text{in} \quad (0,1), \\
u(x,0) &= u_0(x),
\end{aligned}
$$

with the homogeneous Neumann boundary condition. This is non-local and are not prefered by the principle of local action. But in both cases, the stationary problem can be written as the same and is the following:

$$
\begin{aligned}
(1.6) \qquad 0 &= \varepsilon^2 u_{xx} + f(u) - \int_0^1 f(u)dx - \sigma v + \sigma \int_0^1 v dx \quad \text{in} \quad (0,1), \\
0 &= v_{xx} + u - m - \gamma v \quad \text{in} \quad (0,1), \\
\int_0^1 u dx &= m,
\end{aligned}
$$

with the homogeneous Neumann boundary condition. This can be regarded as the Euler-Lagrange equation of $J_{\varepsilon,\sigma,\gamma}(u)$ under the constraint $\int_0^1 u dx = m$, so that the term of integration of the first equaiton of (1.6) can be considered of as the Lagrange multiplier. We also refer to [10] about a background of the equations (1.6). In the conserved case, we also consider the case when $\gamma = 0$. This is a model equation of micro-phase separation phenomena of diblock copolymer melts. We also refer to

[5], [8], [9], and rederences therein about its background. We note that it is also known that, if $\delta = 0$, then the stability of steady states in the sense of the non-local second order parabolic system (1.5) completely agrees with the one in the sense of the Cahn-Hilliard type system (1.4). See [6].

(3) Because of some technical reasons, we need to assume "good" periodicity of global minimizers to get the theorem of the conserved case. Here we define the "good" periodicity of a function in C^1-class by a periodic function which acrosses x-axis only once in its half period. For example, a normal n-layered solution has this "good" periodicity. We refer to [3] about a normal n-layered solution. We exactly state the theorem of the conserved case:

Theorem 1.2 *For given $\sigma > 0$, $\gamma > 0$, $m \in (a, b)$, and the nonlinearity f, there exists $\varepsilon_2 > 0$ such that, if $0 < \varepsilon < \varepsilon_2$ and if any global minimizer of $J_{\varepsilon,\sigma,\gamma}$ in $H^1(0,1)$ with the integral constraint has the "good" periodicity, then the following properties hold:*

(1) If u^ε is a global minimizer, then u^ε's minimum period P^ε satisfies

$$P^\varepsilon = 2\left(3A\left(\frac{b-a}{(b-m)(m-a)}\right)^2 \frac{\varepsilon}{\sigma}\right)^{\frac{1}{3}} + O(\varepsilon^{\frac{2}{3}}).$$

(2) There are at most two distinct global minimizers up to translation.

The theorem is weaker than Theorem 1.1 of the non-conserved case, although we can judge which one is the most stable steady state among all the stable steady states gotten by n-times flipping a 1-layered solution for any $n \in \mathbf{N}$. In this paper we mainly prove it about the non-conserved case, and we remark differences and difficulties of the conserved case, occasionally. We also note that the results of both cases are independent of γ.

(4) Professor Stefan Müller has first proved this kind of characterization of global minimizers in the conserved case with $\gamma = 0$, with $m = 0$, and with the symmetric nonlinearity, which is derived from the problem in a different context of physics. See [2]. In the conserved case, we cannot simply extend his result to the conserved case with non-symmetric nonlinearity (even if it is balanced), unless we assume the "good" periodicity of global minimizers, as we have remarked in (3).

(5) Professor Xiaofen Ren and Professor Juncheng Wei have studied the similar problem of the conserved case in [10]. Their case is in $\sigma = O(\varepsilon)$, because they basically adopt Γ-convergence technique to characterize global minimizers and to seek other local minimizers. Although the parameter value is special a little, their results have similar meaning to ours, but they are represented in a quite different way.

(6) According to Theorem 1.1 and Theorem 1.2, the global minimizers have the spatial period which is proportional to $\varepsilon^{1/3}$. On the other hand, recently, we have made numerical experiments in which we take many kinds of randomly frequent functions whose modes are from one to a thousand as initial data and we solve the time evolution equation (1.1) numerically by use of discretization. These numerical computations suggest us that the spatial period of the most frequently appearing final steady states have another dependency on ε. Namely, it seems that it is propotional to $\varepsilon^{1/2}$. We will report it in the forthcoming paper [7] after we have done enough numbers of numerical experiments to conclude it.

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