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# A Generalization of the Crossed Extensor

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## Abstract

Extensor analysis, a generalization of tensor analysis, offers many mathematical models designed to meet requirement of application to physical science and/or engineering. Since extensor analysis is concerned with the calculation of higher order differentials, it also plays an important role in the study of higher order space.

The purpose of this paper is to show the existence of a new extensor, referred to as a hybrid extensor, and to investigate its properties. It is a possible generalization of the crossed extensor which has been studied elsewhere.


**Contents**



## 1. Introduction

The concept of extensor with several important properties has been established by H. V. Craig[1] and A. Kawaguchi[2),3),4]. Craig has also shown the existence of a certain special extensor in his work[5], which he refers to as the crossed extensor.

Since then, works on crossed extensor have been done by some investigators[6),7),9]. Among them, Y. Katsurada extended the concept of crossed extensor in the manifold of line-elements to that in the manifold of surface-elements of a higher order, and she introduced the concept of generalized crossed extensor[6]. She, however, treats either covariant or contravariant components of it. Therefore, in this paper the authors will supplement and explain in detail the generalized crossed extensor from the point of view of a new extensor which will be called a hybrid extensor. Although we discuss only the case of one parameter, it can easily be extended to that of multiparameter. Also it will be shown that this new extensor contains an ordinary extensor, crossed extensor and generalized crossed extensor as a special case.

## 2. Notations and Preliminaries

The notations used in this paper are mostly the same as those defined in the

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previous paper[9]. The line-elements of order $M$ with respect to the curve $x^i=x^i(t)$ in an $N$-dimensional differentiable manifold are represented by

$$(x^i, x^{(1)i}, x^{(2)i}, \cdots, x^{(M)i}),$$

and Craig's expoint transformation by

$$\begin{cases} x^a = x^a(x^i) \\ x^{(\alpha)a} = X^{(\alpha-1)a}_{(\rho-1)i}\, x^{(\rho)i}\,. \qquad (\alpha, \rho = 1, 2, \cdots, M) \end{cases}$$

The coefficients $X^{(\alpha)a}_{(\rho)i}$ of expoint transformation have the following available relation:

$$X^{(\alpha)a}_{(\rho)i} = \binom{\alpha}{\rho} X^{a(\alpha-\rho)}_{i} = \binom{\alpha}{\rho}\left(\frac{\partial x^a}{\partial x^i}\right)^{(\alpha-\rho)} \qquad (\text{for } \alpha \geqq \rho)$$

$$= 0\,, \qquad (\text{for } \alpha < \rho)$$

where $\binom{\alpha}{\rho}$ is a binomial coefficient. In connection with these coefficients, we get the following identities

$$\sum_{\beta=\mu}^{\alpha} X^{(\beta)a}_{(\mu)i} X^{(\alpha)b}_{(\beta)j} = \binom{\alpha}{\mu} (X^a_i X^b_j)^{(\alpha-\mu)}\,,$$

$$\sum_{\beta=\mu}^{\nu} X^{(\beta)a}_{(\mu)i} X^{(\nu)j}_{(\beta)b} = \binom{\nu}{\mu} (X^a_i X^j_b)^{(\nu-\mu)}\,,$$

from which we can deduce the following general form:

$$\sum_{\alpha_1=\mu}^{\nu} \cdots \sum_{\alpha_{r+s-1}=\mu}^{\nu} X^{(\alpha_1)a_1}_{(\mu)i_1} X^{(\alpha_2)a_2}_{(\alpha_1)i_2} X^{(\alpha_3)a_3}_{(\alpha_2)i_3} \cdots X^{(\alpha_r)a_r}_{(\alpha_{r-1})i_r} X^{(\alpha_{r+1})j_1}_{(\alpha_r)b_1} X^{(\alpha_{r+2})j_2}_{(\alpha_{r+1})b_2} \cdots X^{(\nu)j_s}_{(\alpha_{r+s-1})b_s}$$
$$= \binom{\nu}{\mu} (X^{a_1}_{i_1} X^{a_2}_{i_2} X^{a_3}_{i_3} \cdots X^{a_r}_{i_r} X^{j_1}_{b_1} X^{j_2}_{b_2} \cdots X^{j_s}_{b_s})^{(\nu-\mu)}\,.$$
$$(\mu, \nu = 0, 1, 2, \cdots, M;\ \nu \geqq \mu)$$

Here, we shall put, for the sake of simplicity, as follows:*

$$X^{a_1a_2\cdots a_r\,j_1j_2\cdots j_s}_{i_1i_2\cdots i_r\,b_1b_2\cdots b_s} \equiv X^{a_1}_{i_1} X^{a_2}_{i_2} \cdots X^{a_r}_{i_r} X^{j_1}_{b_1} X^{j_2}_{b_2} \cdots X^{j_s}_{b_s}\,,$$

$$X_{(\mu)i_1i_2\cdots i_r\,b_1b_2\cdots b_s}^{\;a_1a_2\cdots a_r\,j_1j_2\cdots j_s(\nu)} \equiv \sum_{\alpha_1=\mu}^{\nu} \cdots \sum_{\alpha_{r+s-1}=\mu}^{\nu} X^{(\alpha_1)a_1}_{(\mu)i_1} X^{(\alpha_2)a_2}_{(\alpha_1)i_2} X^{(\alpha_3)a_3}_{(\alpha_2)i_3} \cdots X^{(\alpha_r)a_r}_{(\alpha_{r-1})i_r}$$
$$\times X^{(\alpha_{r+1})j_1}_{(\alpha_r)b_1} X^{(\alpha_{r+2})j_2}_{(\alpha_{r+1})b_2} \cdots X^{(\nu)j_s}_{(\alpha_{r+s-1})b_s}\,.$$

Then we find the following propositions:

*Proposition 1.* We assume that $(x^a, x^b, x^c, \cdots)$, $(x^i, x^j, x^k, \cdots)$ and $(x^r, x^s, x^t, \cdots)$ are three different coordinate systems. Then the equalities

$$X_{(\delta)ibj}^{\;akc(\rho)} X_{(\rho)rkt}^{\;isj(\lambda)} = X_{(\delta)rbt}^{\;asc(\lambda)}$$

hold, and they are generally represented by

$$X_{(\mu)i_1\cdots i_r\,b_1\cdots b_s}^{\;a_1\cdots a_r\,j_1\cdots j_s(\nu)} X_{(\nu)t_1\cdots t_r\,j_1\cdots j_s}^{\;i_1\cdots i_r\,r_1\cdots r_s(\omega)} = X_{(\mu)t_1\cdots t_r\,b_1\cdots b_s}^{\;a_1\cdots a_r\,r_1\cdots r_s(\omega)}\,.$$

*Proposition 2.* In regard to two different coordinate systems $(x^a, x^b, x^c, \cdots)$ and $(x^i, x^j, x^k, \cdots)$, it follows that

* We find these notations in 5).

$$X_{(\delta)ibk}^{\;\;ajc(\rho)} X_{(\rho)djf}^{\;\;iek(\lambda)} = \delta_\delta^\lambda \delta_d^a \delta_b^e \delta_f^c$$

and generally

$$X_{(\mu)i_1\cdots i_r\, b_1\cdots b_s}^{\;\;a_1\cdots a_r\, j_1\cdots j_s(\nu)} X_{(\nu)c_1\cdots c_r\, j_1\cdots j_s}^{\;\;i_1\cdots i_r\, d_1\cdots d_s(\omega)} = \delta_\mu^\omega \delta_{c_1\cdots c_r\, b_1\cdots b_s}^{a_1\cdots a_r\, d_1\cdots d_s},$$

where

$$\delta_{c_1\cdots c_r\, b_1\cdots b_s}^{a_1\cdots a_r\, d_1\cdots d_s} \equiv \delta_{c_1}^{a_1}\cdots\delta_{c_r}^{a_r}\delta_{b_1}^{d_1}\cdots\delta_{b_s}^{d_s}.$$

These two propositions suggest that we can define a certain extensor which has $X_{(\mu)i_1\cdots i_r\, b_1\cdots b_s}^{\;\;a_1\cdots a_r\, j_1\cdots j_s(\nu)}$ for the coefficients of transformation.

## 3. Hybrid Extensor

*Definition 3. 1.* The quantities $T^{i_1\cdots i_r(\mu)}$ and $T_{j_1\cdots j_r(\nu)}$ are called the components of an excontravariant and the components of an excovariant blocked extensors of range $M$ respectively, when they are transformed as follows:

$$T^{i_1\cdots i_r(\mu)} = X_{(\alpha)a_1\cdots a_r}^{\;\;i_1\cdots i_r(\mu)} T^{a_1\cdots a_r(\alpha)},$$

$$T_{j_1\cdots j_r(\nu)} = X_{(\nu)j_1\cdots j_r}^{\;\;b_1\cdots b_r(\beta)} T_{b_1\cdots b_r(\beta)}$$

$$(r\geqq 1\,;\ \alpha, \beta, \mu, \nu = 0, 1, 2, \cdots, M)$$

under the expoint transformation.

*Definition 3. 2.* If the quantities $T_{i_1\cdots i_r}{}^{(\mu)}$ and $T^{j_1\cdots j_r}{}_{(\nu)}$ are transformed

$$T_{i_1\cdots i_r}{}^{(\mu)} = X_{(\alpha)i_1\cdots i_r}^{\;\;a_1\cdots a_r(\mu)} T_{a_1\cdots a_r}{}^{(\alpha)},$$

$$T^{j_1\cdots j_r}{}_{(\nu)} = X_{(\nu)b_1\cdots b_r}^{\;\;j_1\cdots j_r(\beta)} T^{b_1\cdots b_r}{}_{(\beta)}$$

$$(r\geqq 1\,;\ \alpha, \beta, \mu, \nu = 0, 1, 2, \cdots, M)$$

under the expoint transformation, then we shall call these quantities the components of an excontravariant generalized crossed extensor* of range $M$ and the components of an excovariant one, respectively.

*Definition 3. 3.* Under the expoint transformation, the components $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)}$ of an excontravariant hybrid extensor of range $M$ and the components $T^{k_1\cdots k_r}{}_{l_1\cdots l_s(\nu)}$ of an excovariant one are varied in the rules

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)} = X_{(\alpha)a_1\cdots a_r\, j_1\cdots j_s}^{\;\;i_1\cdots i_r\, b_1\cdots b_s(\mu)} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\alpha)},$$

$$T^{k_1\cdots k_r}{}_{l_1\cdots l_s(\nu)} = X_{(\nu)c_1\cdots c_r\, l_1\cdots l_s}^{\;\;k_1\cdots k_r\, d_1\cdots d_s(\beta)} T^{c_1\cdots c_r}{}_{d_1\cdots d_s(\beta)}.$$

$$(r, s\geqq 0,\ r+s\geqq 1\,;\ \alpha, \beta, \mu, \nu = 0, 1, 2, \cdots, M)$$

*Remark*: The concepts of blocked and generalized crossed extensors are involved in one of the hybrid extensors as a special case, since we have the components of excontravariant blocked or generalized crossed extensors $T^{i_1\cdots i_r(\mu)}$ or $T_{j_1\cdots j_s}{}^{(\mu)}$, putting $s=0$ or $r=0$ for the components of excontravariant hybrid extensor $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)}$. Also the blocked extensors involve an ordinary extensor, i.e. if we put $r=1$ or $s=1$ for the components of blocked extensors $T^{i_1\cdots i_r(\mu)}$ or $T_{j_1\cdots j_s(\nu)}$, then

* Although there is a slight difference between the generalized crossed extensor in the sense of Y. Katsurada and our extensor defined here, the latter being more limited than the former, we shall call our extensor a generalized crossed extensor.

we can regard $T^{i_1(\mu)}$, $T_{j_1(\nu)}$ as the components of ordinary extensor from the definition of the coefficients of transformation. Similarly, we see that the crossed extensor defined in 9) is involved in the generalized crossed extensor.

*Theorem 3. 1.* If the quantities $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)}$ are the components of any excontravariant hybrid extensor of range $M$, then the quantities

$$\binom{M}{\nu} T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(M-\nu)} \qquad (\nu=0, 1, 2, \cdots, M)$$

are the components of excovariant hybrid extensor of range $M$.

*Proof.* Under the expoint transformation, we have

$$\begin{aligned}
&\binom{M}{\nu} T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(M-\nu)}\\
&=\binom{M}{\nu} X_{(\alpha)a_1\cdots a_r\, j_1\cdots j_s}^{\ \ i_1\cdots i_r\, b_1\cdots b_s(M-\nu)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\alpha)}\\
&=\binom{M}{\nu}\binom{M-\nu}{\alpha}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(M-\nu-\alpha)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\alpha)}\\
&=\binom{M}{\nu}\binom{M-\nu}{M-\beta}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\beta-\nu)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(M-\beta)}
\end{aligned}$$

(putting $\alpha=M-\beta$)

$$\begin{aligned}
&=\binom{\beta}{\nu}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\beta-\nu)}\binom{M}{\beta} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(M-\beta)}\\
&= X_{(\nu)a_1\cdots a_r\, j_1\cdots j_s}^{\ \ i_1\cdots i_r\, b_1\cdots b_s(\beta)}\binom{M}{\beta} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(M-\beta)}
\end{aligned}$$

and the theorem is proved. Q. E. D.

*Corollary 3. 1.* If two quantities $T^{i_1\cdots i_r(\mu)}$ and $T_{k_1\cdots k_s}{}^{(\nu)}$ are the components of any excontravariant blocked and excontravariant generalized crossed extensor of range $M$ respectively, then the quantities

$$\binom{M}{\rho} T^{i_1\cdots i_r(M-\rho)}, \qquad \binom{M}{\sigma} T_{k_1\cdots k_s}{}^{(M-\sigma)}$$

are the components of the excovariant generalized crossed and of the excovariant blocked extensors of range $M$ respectively.

As an example of the hybrid extensor, we shall consider the following quantities. If the quantities $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}$ are the differentiable components of any tensor field defined along an arc of class $C^M$, $x^i=x^i(t)$, then assuming the required differentiability, the quantities which are the $\mu$-th derivative of $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}$ with respect to $t$

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)} \qquad (\mu=0, 1, 2, \cdots, M)$$

are the components of the excontravariant hybrid extensor, and if we represent the quantities which are the $(M-\nu)$-th derivative of $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}$ with respect to $t$, multiplying by the binomial coefficient $\binom{M}{\nu}$, as

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\nu)} \equiv \binom{M}{\nu} T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(M-\nu)}, \qquad (\nu=0, 1, 2, \cdots, M)$$

then they are the components of excovariant hybrid extensor.

*Theorem 3. 2.** If $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)}$ and $T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\nu)}$ are the components of two arbitrary excontravarinat hybrid extensors of range $M$, then the quantities

$$T^{i_1\cdots i_r k_1\cdots k_p}{}_{j_1\cdots j_s l_1\cdots l_q}{}^{(\omega)} \equiv \sum_{\nu=0}^{\omega}\binom{\omega}{\nu} T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\omega-\nu)} T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\nu)} \qquad (\omega=0,1,2,\cdots,M)$$

are also the components of excontravariant hybrid extensor of range $M$.

*Proof.* By making use of the transformation of the hybrid extensor, we can calculate as follows:

$$\sum_{\nu=0}^{\omega}\binom{\omega}{\nu} T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\omega-\nu)} T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\nu)}$$

$$=\sum_{\nu=0}^{\omega}\binom{\omega}{\nu} X_{(\alpha)a_1\cdots a_r\, j_1\cdots j_s}^{\ i_1\cdots i_r\, b_1\cdots b_s(\omega-\nu)} X_{(\beta)c_1\cdots c_p\, l_1\cdots l_q}^{\ k_1\cdots k_p\, d_1\cdots d_q(\nu)} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\alpha)} T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\beta)}$$

$$=\sum_{\nu=0}^{\omega}\sum_{\beta=0}^{\nu}\sum_{\alpha=0}^{\omega-\nu}\binom{\omega}{\nu}\binom{\omega-\nu}{\alpha}\binom{\nu}{\beta}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r j_1\cdots j_s})^{(\omega-\nu-\alpha)}$$
$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\nu-\beta)} T^{a_1\cdots a_r}{}_{b_1\cdots b_k}{}^{(\alpha)} T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\beta)}$$

$$=\sum_{\beta=0}^{\omega}\sum_{\alpha=0}^{\omega-\beta}\sum_{\nu=0}^{\omega-\alpha}\binom{\omega}{\alpha+\beta}\binom{\alpha+\beta}{\beta}\binom{\omega-(\alpha+\beta)}{\nu-\beta}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\omega-\nu-\alpha)}$$
$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\nu-\beta)} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\alpha)} T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\beta)}$$

$$=\sum_{\beta=0}^{\omega}\sum_{\gamma=\beta}^{\omega}\sum_{\nu=\beta}^{\omega-(\gamma-\beta)}\binom{\omega}{\gamma}\binom{\gamma}{\beta}\binom{\omega-\gamma}{\nu-\beta}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\omega-\nu-(\gamma-\beta))}$$
$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots b_p\, l_1\cdots l_q})^{(\nu-\beta)} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\gamma-\beta)} T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\beta)}$$

$$(\text{putting } \alpha=\gamma-\beta)$$

$$=\sum_{\beta=0}^{\omega}\sum_{\gamma=\beta}^{\omega}\binom{\omega}{\gamma}\binom{\gamma}{\beta}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s} X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\omega-\gamma)}$$
$$\times T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\gamma-\beta)} T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\beta)}$$

$$=X_{(\gamma)a_1\cdots a_r\, c_1\cdots c_p\, j_1\cdots j_s\, l_1\cdots l_q}^{\ i_1\cdots i_r\, k_1\cdots k_p\, b_1\cdots b_s\, d_1\cdots d_q(\omega)}\sum_{\beta=0}^{\gamma}\binom{\gamma}{\beta} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\gamma-\beta)} T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\beta)}.$$

Q. E. D.

*Theorem 3. 3.* If $T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\sigma)}$ and $T^{k_1\cdots k_p}{}_{l_1\cdots l_q(\rho)}$ are the components of two arbitrary excovariant hybrid extensors of range $M$, then the quantities

$$T^{i_1\cdots i_r\, k_1\cdots k_p}{}_{j_1\cdots j_s l_1\cdots l_q(\gamma)} \equiv \sum_{\beta=0}^{M-\gamma}\binom{M}{\beta}^{-1}\binom{\gamma+\beta}{\beta} T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\gamma+\beta)} T^{k_1\cdots k_p}{}_{l_1\cdots l_q(M-\beta)} \qquad (\gamma=0,1,2,\cdots,M)$$

are also the components of excovariant hybrid extensor of range $M$.

*Proof.* Under successive applications of the transformation we have the following equalities:

$$\sum_{\beta=0}^{M-\gamma}\binom{M}{\beta}^{-1}\binom{\gamma+\beta}{\beta} T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\gamma+\beta)} T^{k_1\cdots k_p}{}_{l_1\cdots l_q(M-\beta)}$$

* This theorem corresponds to the Theorem 1.5 in 6).

$$=\sum_{\beta=0}^{M-\gamma}\binom{M}{\beta}^{-1}\binom{\gamma+\beta}{\beta}X_{(\gamma+\beta)a_1\cdots a_r\, j_1\cdots j_s}^{\;\;i_1\cdots i_r\, b_1\cdots b_s(\alpha)}X_{(M-\beta)c_1\cdots c_p\, l_1\cdots l_q}^{\;\;k_1\cdots k_p\, d_1\cdots d_q(\delta)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\alpha)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(\delta)}$$

$$=\sum_{\beta=0}^{M-\gamma}\sum_{\alpha=\gamma+\beta}^{M}\sum_{\delta=M-\beta}^{M}\binom{M}{\beta}^{-1}\binom{\gamma+\beta}{\beta}\binom{\alpha}{\gamma+\beta}\binom{\delta}{M-\beta}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\alpha-\gamma-\beta)}$$
$$\times(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\delta-M+\beta)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\alpha)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(\delta)}$$

$$=\sum_{\beta=0}^{M-\gamma}\sum_{\alpha=\gamma+\beta}^{M}\sum_{\rho=0}^{\beta}\binom{M}{\beta}^{-1}\binom{\gamma+\beta}{\beta}\binom{\alpha}{\gamma+\beta}\binom{M-\rho}{M-\beta}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\alpha-\gamma-\beta)}$$
$$\times(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\beta-\rho)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\alpha)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}\qquad(\text{putting }\delta=M-\rho)$$

$$=\sum_{\alpha=\gamma}^{M}\sum_{\rho=0}^{\alpha-\gamma}\sum_{\mu=0}^{\alpha-\gamma-\rho}\binom{M}{\mu+\rho}^{-1}\binom{\gamma+\mu+\rho}{\mu+\rho}\binom{\alpha}{\gamma+\mu-\rho}\binom{M-\rho}{M-\mu-\rho}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\alpha-\gamma-\mu-\rho)}$$
$$\times(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\mu)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\alpha)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}\qquad(\text{putting }\beta=\mu+\rho)$$

$$=\sum_{\rho=0}^{M-\gamma}\sum_{\mu=0}^{M-\gamma-\rho}\sum_{\omega=\gamma+\mu}^{M-\rho}\binom{M}{\mu+\rho}^{-1}\binom{\gamma+\mu+\rho}{\mu+\rho}\binom{\omega+\rho}{\gamma+\mu+\rho}\binom{M-\rho}{M-\mu-\rho}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\omega-\gamma-\mu)}$$
$$\times(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\mu)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\omega+\rho)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}\qquad(\text{putting }\alpha=\omega+\rho)$$

$$=\sum_{\omega=\gamma}^{M}\sum_{\mu=0}^{\omega-\gamma}\binom{\omega}{\gamma}\binom{\omega-\gamma}{\mu}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\omega-\gamma-\mu)}(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\mu)}$$
$$\times\sum_{\rho=0}^{M-\omega}\binom{M}{\rho}^{-1}\binom{\omega+\rho}{\rho}T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\omega+\rho)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}$$

$$=X_{(\gamma)a_1\cdots a_r\, c_1\cdots c_p\, j_1\cdots j_s\, l_1\cdots l_q}^{\;\;i_1\cdots i_r\, k_1\cdots k_p\, b_1\cdots b_s\, d_1\cdots d_q(\omega)}\sum_{\rho=0}^{M-\omega}\binom{M}{\rho}^{-1}\binom{\omega+\rho}{\rho}T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\omega+\rho)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}.$$

Q. E. D.

*Theorem 3. 4.* If $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)}$ and $T^{k_1\cdots k_p}{}_{l_1\cdots l_q(\nu)}$ are the components of any excontravariant and excovariant hybrid extensors of range $M$ respectively, then the quantities

$$T^{i_1\cdots i_r\, k_1\cdots k_p}{}_{j_1\cdots j_s\, l_1\cdots l_q}{}^{(\beta)}\equiv\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\binom{M}{\alpha}^{-1}T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\beta-\alpha)}T^{k_1\cdots k_p}{}_{l_1\cdots l_q(M-\alpha)}$$
$$(\beta=0,1,2,\cdots,M)$$

are the components of excontravariant hybrid extensor of range $M$.

*Proof.* Applying successively the transformation for hybrid extensors, we get

$$\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\binom{M}{\alpha}^{-1}T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\beta-\alpha)}T^{k_1\cdots k_p}{}_{l_1\cdots l_q(M-\alpha)}$$

$$=\sum_{\alpha=0}^{\beta}\binom{\beta}{\alpha}\binom{M}{\alpha}^{-1}X_{(\mu)a_1\cdots a_r\, j_1\cdots j_s}^{\;\;i_1\cdots i_r\, b_1\cdots b_s(\beta-\alpha)}X_{(M-\alpha)c_1\cdots c_p\, l_1\cdots l_q}^{\;\;k_1\cdots k_p\, d_1\cdots d_q(\nu)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\mu)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(\nu)}$$

$$=\sum_{\alpha=0}^{\beta}\sum_{\mu=0}^{\beta-\alpha}\sum_{\nu=M-\alpha}^{M}\binom{\beta}{\alpha}\binom{M}{\alpha}^{-1}\binom{\beta-\alpha}{\mu}\binom{\nu}{M-\alpha}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\beta-\alpha-\mu)}$$
$$\times(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\nu-M+\alpha)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\mu)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(\nu)}$$

$$=\sum_{\alpha=0}^{\beta}\sum_{\mu=0}^{\beta-\alpha}\sum_{\rho=0}^{\alpha}\binom{\beta}{\alpha}\binom{M}{\alpha}^{-1}\binom{\beta-\alpha}{\mu}\binom{M-\rho}{M-\alpha}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\beta-\alpha-\mu)}$$
$$\times(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\alpha-\rho)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\mu)}T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}\qquad(\text{putting }\nu=M-\rho)$$

$$= \sum_{\mu=0}^{\beta}\sum_{\rho=0}^{\beta-\mu}\sum_{\tau=0}^{\beta-\mu-\rho}\binom{\beta}{\tau+\rho}\binom{M}{\tau+\rho}^{-1}\binom{\beta-\tau-\rho}{\mu}\binom{M-\rho}{M-\tau-\rho}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\beta-\tau-\rho-\mu)}$$

$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\tau)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\mu)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)} \qquad (\text{putting } \alpha=\tau+\rho)$$

$$= \sum_{\rho=0}^{\beta}\sum_{\tau=0}^{\beta-\rho}\sum_{\omega=\rho}^{\beta-\tau}\binom{\beta}{\tau+\rho}\binom{M}{\tau+\rho}^{-1}\binom{\beta-\tau-\rho}{\omega-\rho}\binom{M-\rho}{M-\tau-\rho}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\beta-\tau-\omega)}$$

$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\tau)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\omega-\rho)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)} \qquad (\text{putting } \mu=\omega-\rho)$$

$$= \sum_{\omega=0}^{\beta}\sum_{\tau=0}^{\beta-\omega}\binom{\beta}{\omega}\binom{\beta-\omega}{\tau}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\beta-\omega-\tau)}(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\tau)}$$

$$\times \sum_{\rho=0}^{\omega}\binom{\omega}{\rho}\binom{M}{\rho}^{-1} T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\omega-\rho)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}$$

$$= X_{(\omega)a_1\cdots a_r\, c_1\cdots c_p\, j_1\cdots j_s\, l_1\cdots l_q}^{i_1\cdots i_r\, k_1\cdots k_p\, b_1\cdots b_s\, d_1\cdots d_q(\beta)} \sum_{\rho=0}^{\omega}\binom{\omega}{\rho}\binom{M}{\rho}^{-1} T^{a_1\cdots a_r}{}_{j_1\cdots b_s}{}^{(\omega-\rho)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q(M-\rho)}\,.$$

Q. E. D.

*Theorem 3. 5.* If $T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\sigma)}$ and $T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\rho)}$ are the components of any excovariant and excontravariant hybrid extensors of range $M$ respectively, then the quanttities

$$T^{i_1\cdots i_r\, k_1\cdots k_p}{}_{j_1\cdots j_s\, l_1\cdots l_q(\beta)} \equiv \sum_{\alpha=0}^{M-\beta}\binom{\beta+\alpha}{\alpha} T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\beta+\alpha)}\, T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\alpha)}$$

$$(\beta = 0, 1, 2, \cdots, M)$$

are the components of excovariant hybrid extensor of range $M$.

*Proof.* With the same process as the proof of the preceding theorem, we have the following equalities:

$$\sum_{\alpha=0}^{M-\beta}\binom{\beta+\alpha}{\alpha} T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\beta+\alpha)}\, T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\alpha)}$$

$$= \sum_{\alpha=0}^{M-\beta}\binom{\beta+\alpha}{\alpha} X_{(\beta+\alpha)a_1\cdots a_r\, j_1\cdots j_s}^{i_1\cdots i_r\, b_1\cdots b_s(\mu)}\, X_{(\nu)c_1\cdots c_p\, l_1\cdots l_q}^{k_1\cdots k_p\, d_1\cdots d_q(\alpha)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\mu)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\nu)}$$

$$= \sum_{\alpha=0}^{M-\beta}\sum_{\mu=\beta+\alpha}^{M}\sum_{\nu=0}^{\alpha}\binom{\beta+\alpha}{\alpha}\binom{\mu}{\beta+\alpha}\binom{\alpha}{\nu}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\mu-\alpha-\beta)}$$

$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\alpha-\nu)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\mu)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\nu)}$$

$$= \sum_{\mu=\beta}^{M}\sum_{\nu=0}^{\mu-\beta}\sum_{\rho=0}^{\mu-\beta-\nu}\binom{\beta+\rho+\nu}{\rho+\nu}\binom{\mu}{\beta+\rho+\nu}\binom{\rho+\nu}{\nu}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\mu-\rho-\nu-\beta)}$$

$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\rho)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\mu)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\nu)} \qquad (\text{putting } \alpha=\rho+\nu)$$

$$= \sum_{\nu=0}^{M-\beta}\sum_{\tau=\beta}^{M-\nu}\sum_{\rho=0}^{\tau-\beta}\binom{\beta+\rho+\nu}{\rho+\nu}\binom{\tau+\nu}{\beta+\rho+\nu}\binom{\rho+\nu}{\nu}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\tau-\beta-\rho)}$$

$$\times (X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\rho)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\tau+\nu)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\nu)} \qquad (\text{putting } \mu=\tau+\nu)$$

$$= \sum_{\tau=\beta}^{M}\sum_{\rho=0}^{\tau-\beta}\binom{\tau}{\beta}\binom{\tau-\beta}{\rho}(X^{i_1\cdots i_r\, b_1\cdots b_s}_{a_1\cdots a_r\, j_1\cdots j_s})^{(\tau-\beta-\rho)}(X^{k_1\cdots k_p\, d_1\cdots d_q}_{c_1\cdots c_p\, l_1\cdots l_q})^{(\rho)}$$

$$\times \sum_{\nu=0}^{M-\nu}\binom{\tau+\nu}{\nu} T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\tau+\nu)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\nu)}$$

$$= X_{(\beta)a_1\cdots a_r\, c_1\cdots c_p\, j_1\cdots j_s\, l_1\cdots l_q}^{i_1\cdots i_r\, k_1\cdots k_p\, b_1\cdots b_s\, d_1\cdots d_q(\tau)} \sum_{\nu=0}^{M-\tau}\binom{\tau+\nu}{\nu} T^{a_1\cdots a_r}{}_{b_1\cdots b_s(\tau+\nu)}\, T^{c_1\cdots c_p}{}_{d_1\cdots d_q}{}^{(\nu)}\,. \qquad \text{Q. E. D.}$$

According to the results of these theorems, we can establish the relations among blocked extensors, generalized crossed extensors and hybrid extensors. Assuming that $T^{i_1\cdots i_r(\mu)}$ and $T_{j_1\cdots j_s(\nu)}$ are the components of any excontravariant and excovariant blocked extensors respectively, we construct the components of excontravariant and excovariant hybrid extensors as follows:

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\omega)} \equiv \sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\binom{M}{\nu}^{-1} T^{i_1\cdots i_r(\omega-\nu)}\, T_{j_1\cdots j_s(M-\nu)}\,,$$

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\nu)} \equiv \sum_{\mu=0}^{M-\nu}\binom{\nu+\mu}{\mu} T^{i_1\cdots i_r(\mu)}\, T_{j_1\cdots j_s(\nu+\mu)}\,.$$

Similarly, from the components of any excontravariant generalized crossed extensor and the components of any excovariant one, we get the following hybrid extensors,

$$T^{k_1\cdots k_s}{}_{i_1\cdots i_r}{}^{(\omega)} \equiv \sum_{\nu=0}^{\omega}\binom{\omega}{\nu}\binom{M}{\nu}^{-1} T_{i_1\cdots i_r}{}^{(\omega-\nu)}\, T^{k_1\cdots k_s}{}_{(M-\nu)}\,,$$

$$T^{i_1\cdots i_r}{}_{k_1\cdots k_s(\nu)} \equiv \sum_{\mu=0}^{M-\nu}\binom{\nu+\mu}{\mu} T_{k_1\cdots k_s}{}^{(\mu)}\, T^{i_1\cdots i_r}{}_{(\nu+\mu)}\,.$$

Also from the components of any blocked extensor and of any generalized crossed extensor, we construct hybrid extensors as follows:

$$T^{i_1\cdots i_r}{}_{k_1\cdots k_s}{}^{(\omega)} \equiv \sum_{\nu=0}^{\omega}\binom{\omega}{\nu} T^{i_1\cdots i_r(\omega-\nu)}\, T_{k_1\cdots k_s}{}^{(\nu)}\,,$$

$$T^{i_1\cdots i_r}{}_{k_1\cdots k_s(\nu)} \equiv \sum_{\mu=0}^{M-\nu}\binom{M}{\nu}^{-1}\binom{\nu+\mu}{\mu} T_{k_1\cdots k_s(\mu+\nu)}\, T^{i_1\cdots i_r}{}_{(M-\mu)}\,.$$

## 4. Contractions and Reduced Range

In the hybrid extensor, the operation of contraction is the same as that of an ordinary extensor. Then, we obtain the following contraction of hybrid extensors.

*Theorem 4. 1.* For an arbitrary given hybrid extensor of range $M$, the following $M+1$ kinds of quantities

$$\sum_{\rho=A}^{M}\binom{\rho}{A} T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\rho-A)}\, T^{j_1\cdots j_s}{}_{i_1\cdots i_r(\rho)} \qquad (A=0, 1, 2, \cdots, M)$$

are invariant.

*Proof.* Applying the transformation for hybrid extensor, we know that these quantities are invariant:

$$\begin{aligned}
&\sum_{\rho=A}^{M}\binom{\rho}{A} T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\rho-A)}\, T^{j_1\cdots j_s}{}_{i_1\cdots i_r(\rho)}\\
&=\sum_{\rho=A}^{M}\binom{\rho}{A} X_{(\nu)a_1\cdots a_r\, j_1\cdots j_s}^{\ \ i_1\cdots i_r\, b_1\cdots b_s(\rho-A)}\, X_{(\rho)c_1\cdots c_s\, i_1\cdots i_r}^{\ \ j_1\cdots j_s\, d_1\cdots d_r(\omega)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\nu)}\, T^{c_1\cdots c_s}{}_{d_1\cdots d_r(\omega)}\\
&=\sum_{\rho=A}^{M}\sum_{\nu=0}^{\rho-A}\sum_{\omega=\rho}^{M}\binom{\rho}{A} X_{(\nu)a_1\cdots a_r\, j_1\cdots j_s}^{\ \ i\cdots i_r\, b_1\cdots b_s(\rho-A)}\, X_{(\rho)c_1\cdots c_s\, i_1\cdots i_r}^{\ \ j_1\cdots j_s\, d_1\cdots d_r(\omega)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\nu)}\, T^{c_1\cdots c_s}{}_{d_1\cdots d_r(\omega)}\\
&=\sum_{\beta=0}^{M-A}\sum_{\nu=0}^{\beta}\sum_{\omega=\beta+A}^{M}\binom{\beta+A}{A} X_{(\nu)a_1\cdots a_r\, j_1\cdots j_s}^{\ \ i_1\cdots i_r\, b_1\cdots b_s(\beta)}\, X_{(\beta+A)c_1\cdots c_s\, i_1\cdots i_r}^{\ \ b_1\cdots j_s\, d_1\cdots d_r(\omega)}\, T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\nu)}\, T^{c_1\cdots c_s}{}_{d_1\cdots d_r(\omega)}
\end{aligned}$$

(putting $\rho=\beta+A$)

$$=\sum_{\beta=0}^{M-A}\sum_{\nu=0}^{\beta}\sum_{\rho=\beta}^{M-A}\binom{\beta+A}{A}X_{(\nu)a_1\cdots a_r\,j_1\cdots j_s}^{\;\;i_1\cdots i_r\,b_1\cdots b_s(\beta)}X_{(\beta+A)c_1\cdots c_s\,i_1\cdots i_r}^{\;\;j_1\cdots j_s\,d_1\cdots d_r(\rho+A)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\nu)}T^{c_1\cdots c_s}{}_{d_1\cdots d_r(\rho+A)}$$

(putting $\omega=A+\rho$)

$$=\sum_{\beta=0}^{M-A}\sum_{\nu=0}^{\beta}\sum_{\rho=\beta}^{M-A}\binom{\beta+A}{A}\binom{\rho+A}{\beta+A}\binom{\rho}{\beta}^{-1}X_{(\nu)a_1\cdots a_r\,j_1\cdots j_s}^{\;\;i_1\cdots i_r\,b_1\cdots b_s(\beta)}$$

$$\times X_{(\beta)c_1\cdots c_s\,i_1\cdots i_r}^{\;\;j_1\cdots j_s\,d_1\cdots d_r(\rho)}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\nu)}T^{c_1\cdots c_s}{}_{d_1\cdots d_r(\rho+A)}$$

$$=\sum_{\nu=0}^{M-A}\sum_{\rho=\nu}^{M-A}\binom{\rho+A}{A}\delta^{\rho}_{\nu}\delta^{d_1\cdots d_r\,b_1\cdots b_s}_{a_1\cdots a_r\,c_1\cdots c_s}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\nu)}T^{c_1\cdots c_s}{}_{d_1\cdots d_r(\rho+A)}$$

$$=\sum_{\rho=0}^{M-A}\binom{\rho+A}{A}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\rho)}T^{b_1\cdots b_s}{}_{a_1\cdots a_r(\rho+A)}$$

$$=\sum_{\delta=A}^{M}\binom{\delta}{A}T^{a_1\cdots a_r}{}_{b_1\cdots b_s}{}^{(\delta-A)}T^{b_1\cdots q_s}{}_{a_1\cdots a_r(\delta)}\,.$$

(putting $\rho=\delta-A$) Q. E. D.

From the results of Theorem 3.2, 3.3, 3.4 and 3.5, we obtain the following other kind of contraction.

*Theorem 4.2.* If $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\nu)}$ and $T^{k_1\cdots k_s}{}_{l_1\cdots l_q(\mu)}$ are the components of arbitrary excontravariant and excovariant hybrid extensor of range $M$ respectively, then the quantities

$$T^{i_1\cdots i_r}{}_{l_1\cdots l_q(\beta)}\equiv\sum_{\alpha=0}^{M-\beta}\binom{\beta+\alpha}{\alpha}T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\alpha)}T^{j_1\cdots j_s}{}_{l_1\cdots l_q(\beta+\alpha)}$$

are also the components of excovariant hybrid extensor of range $M$.

The existence of the hybrid extensor of reduced range is shown by the following theorem.

*Theorem 4.3.* If $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)}$ and $T^{k_1\cdots k_p}{}_{l_1\cdots l_q(\nu)}$ are the components of excontravariant and excovariant hybrid extensor of range $M$ respectively, then the quantities

$$T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\omega)}, \qquad (0\leqq\omega\leqq K\leqq M)$$

$$\binom{M-K+\nu}{\nu}T^{k_1\cdots k_p}{}_{l_1\cdots l_q(M-K+\nu)}\equiv\mathfrak{B}_{[M-K]}T^{k_1\cdots k_p}{}_{l_1\cdots l_q(\nu)}{}^{*} \qquad (0\leqq\nu\leqq K\leqq M)$$

are the components of hybrid extensor of reduced range $0\leqq\omega\leqq K$ and $0\leqq\nu\leqq K$ respectively.

Applying these quantities to Theorem 3.2, 3.3, 3.4 and 3.5, we have

*Theorem 4.4.* If $T^{i_1\cdots i_r}{}_{j_1\cdots j_s}{}^{(\mu)}$ are the components of reduced range $0\leqq\mu\leqq K$ and $T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\nu)}$ are of range $0\leqq\nu\leqq L$ $(K\leqq L\leqq M)$, then the quantities

$$\sum_{\nu=0}^{\omega}\binom{\omega}{\nu}T^{i_1\cdots i_r}{}_{j_1\cdots l_s}{}^{(\omega-\nu)}T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\nu)}$$

are the components of excontravariant hybrid extensor of reduced range $0\leqq\omega\leqq K$.

*Theorem 4.5.* If $\mathfrak{B}_{[M-K]}T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\delta)}$ are the components of reduced range $0\leqq\delta\leqq K$ and $T^{k_1\cdots k_r}{}_{l_1\cdots l_q(\sigma)}$ are of range $M$, then the quantities

$$\sum_{\beta=0}^{K-\gamma}\binom{M}{\beta}^{-1}\binom{\gamma+\beta}{\beta}[\mathfrak{B}_{[M-K]}T^{i_1\cdots i_r}{}_{j_1\cdots j_s(\gamma+\beta)}]\,T^{k_1\cdots k_p}{}_{l_1\cdots l_q(M-\beta)}$$

are the components of excovariant hybrid extensor of reduced range $0\leqq\gamma\leqq K$.

* This notation is due to A. Kawaguchi's paper 4).

*Theorem 4.6.* If $\mathfrak{B}_{[M-K]} T^{i_1\cdots i_r}{}_{j_1\cdots j_{s(\gamma)}}$ are the components of reduced range $0 \leqq \gamma \leqq K$ and $T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\alpha)}$ are of range $0 \leqq \alpha \leqq L$ $(K \leqq L \leqq M)$, then the quantities

$$\sum_{\alpha=0}^{K-\beta} \binom{\beta+\alpha}{\alpha} [\mathfrak{B}_{[M-K]} T^{i_1\cdots i_r}{}_{j_1\cdots j_{s(\beta+\alpha)}}] T^{k_1\cdots k_p}{}_{l_1\cdots l_q}{}^{(\alpha)}$$

are the components of excovariant hybrid extensor of reduced range $0 \leqq \beta \leqq K$.

## 5. Conclusions

In order to consolidate the extensor analysis, an investigation of the existence of various unknown extensors is important at present.

The hybrid extensor analysis, proposed in this paper, is one of the available means to treat numerous varieties of quantities concerned with higher order differentials. And we have studied its properties in detail. Moreover, we have proved that the hybrid extensor contains the extensor, the crossed extensor and the generalized crossed extensor as a special case.

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