Bulletin of the Faculty of Engineering,
Hokkaido University, No. 77 (1975)

A Geometrical Formulation of Asymmetric Features in Plasticity

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(Received March 31, 1975)

Abstract

A geometrical formulation of asymmetric features in plasticity is given in the present paper. A material body is assumed to be an aggregation of small material elements, each of which can deform and rotate freely, so that asymmetric features must be considered. The law of friction between antisymmetric parts of stress and strain is further assumed. The effect of dilatancy which plays an important role in soil mechanics, seismology, etc. can be explained in connection with the non-Riemannian theory of plasticity. An important recognition, which is the same conclusion in the epistemological analysis of asymmetric stress fields developed elsewhere, is obtained according to which a certain distribution of asymmetric stress entails the gradient of volumetric strain.

1. Introduction

It has been pointed out in regard to the non-Riemannian strain-incompatibility configuration[1] that stress is a transform of strain in terms of a specific coordinate system[2]. This enables us to treat asymmetric features such as asymmetric stresses and couple stresses, the source of which is given by the antisymmetric part of the stress[3].

If a material body is composed of an aggregation of small material elements, it can be considered that such a material is more likely to allow the existence of couple stress than other material. A typical asymmetric stress field may be observed in granular materials. In fact, it sometimes happens that the volumetric change is entailed by shear stress in coarse grain structure and is called dilatancy[4]. It is a kind of mutual action between grains originated from the antisymmetric part of stress, and it plays an important role in soil mechanics[5], and seismology[6], etc. Its geometrical background has been shown in the previous papers[7),8] in non-Riemannian terminology. However, the foregoing formulation has been made from the epistemological point of view which has it that the asymmetric stress is related to the strain and incompatibility with reference to specific coordinates and covers the known equilibrated stress field in periphracticity. We shall attempt in the following to obtain the same results in another way in which the law of friction between the antisymmetric part of stress and strain is taken into consideration.

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2. Material Space

A material body is assumed to be an aggregation of small material elements. The microscopic local configurations must be considered. A coordinate system is introduced in order to describe the deformed and rotated state of each material element. To each point of the material body, x^i $(i=1, 2, 3)$ is labelled with reference to a three-dimensional orthogonal coordinate system, and the material element, which supports the disturbances in the material, is denoted by dx^i.

Let the material element dx^i be released from the constraint of the surroundings to an unstrained one $(dx)^{i'}$. We shall call this procedure naturalization. If small disturbances alone are considered, we can assume a linear relation

$$(dx)^{i'} = A_i^{i'} dx^i \tag{2. 1}$$

or

$$dx^i = A_{i'}^i (dx)^{i'}, \tag{2. 2}$$

where the transformation tensor $A_i^{i'}$ and its inverse $A_{i'}^i$ are functions of x^j, and Einstein's summation convention is adopted here with regard to repeated indices. By the transformation (2. 1) or (2. 2) we have generally a non-holonomic coordinate system (i'), so that the non-holonomic object

$$\Omega_{k'}{}^{i'}{}_{j'} = A_{k'}^k A_{j'}^j \partial_{[k} A_{j]}^{i'} \tag{2. 3}$$

does not necessarily vanish, where

$$\partial_k = \frac{\partial}{\partial x^k}$$

and the parentheses [ ] indicate the alternating over indices, such as

$$\partial_{[k} A_{j]}^{i'} = \frac{1}{2}(\partial_k A_j^{i'} - \partial_j A_k^{i'}) .$$

Since the naturalized length of a material element $(dx)^{j'}$ is realized in the Euclidean space, it is given by

$$ds^2 = \delta_{j'i'} (dx)^{i'} (dx)^{j'} ,$$

where $\delta_{j'i'}$ is Kronecker delta. A metric of the material manifold is defined by the naturalized length of the material element

$$ds^2 = g_{ji} dx^i dx^j \tag{2. 4}$$

with the metric tensor

$$g_{ji} = \delta_{j'i'} A_i^{i'} A_j^{j'} . \tag{2. 5}$$

Let the transformation tensor be given by

$$A_i^{i'} = \delta_i^{i'} + \varepsilon_i^{i'} , \tag{2. 6}$$

which is only slightly deviated from the Euclidean condition. Substituting (2. 6) in (2. 5), we have

$$g_{ji} = \delta_{ji} + 2\varepsilon_{(ji)} , \tag{2. 7}$$

up to the first order in magnitude, where

$$\varepsilon_{ji} = \varepsilon_j^{j'}\delta_{j'i}\,, \tag{2.8}$$

which are also assumed to be small, represents the deformed and rotated state of the material element, and the parentheses ( ) indicate the mixing over indices, such as

$$\varepsilon_{(ji)} = \frac{1}{2}(\varepsilon_{ji} + \varepsilon_{ij})\,.$$

The quantity $\varepsilon_{(ji)}$ is an ordinary strain tensor. It should be noted that ε_{ji} defined by (2.8) is not assured to be symmetric with respect to the indices i and j.

A Euclidean connexion of plastic material manifold is introduced by the parallelism in the naturalized space[9]. Consider two vectors $v^i(x^k)$ and $v^i(x^k+dx^k)$ which are defined at two points x^k and x^k+dx^k in material manifold respectively, and let them be parallel to each other in the naturalized state. It holds

$$v^{i'}(x^k) = v^{i'}(x^k+dx^k)\,,$$

which leads to

$$dv^i = -\Gamma_{k\ j}^{\ i} v^j dx^k\,,$$

where $\Gamma_{k\ j}^{\ i}$ is the parameter of connection and

$$\Gamma_{k\ j}^{\ i} dx^k = A_{i'}^i dA_j^{i'}\,,$$
$$v^i(x^k+dx^k) = v^i + dv^i\,,$$
$$A_i^{i'}(x^k+dx^k) = A_i^{i'} + dA_i^{i'}\,. \tag{2.9}$$

It should be remarked that $dA_i^{i'}$ depends not only on the point x^k but also on the path dx^k of naturalization. However, if $dA_j^{i'}$ is uniquely defined as function of point, we have from (2.9)

$$\Gamma_{k\ j}^{\ i} = A_{i'}^i \partial_k A_j^{i'}\,. \tag{2.10}$$

In this case material space becomes a teleparallelism one, where the direction of a released material element is fixed. For small disturbances, we have from (2.6) and (2.10)

$$\Gamma_{kji} = \partial_k \varepsilon_{ji}\,, \tag{2.11}$$

where

$$\Gamma_{kji} = g_{ih}\Gamma_{k\ j}^{\ h}\,.$$

A material space is further characterized by non-Euclidean concepts called the torsion tensor and the Riemann-Christoffel curvature tensor. They are defined by

$$S_{kj}^{\cdot\cdot i} = \Gamma_{[k\ j]}^{\ \ i} \tag{2.12}$$

and

$$R_{kji}^{\cdot\cdot\cdot h} = 2(\partial_{[k}\Gamma_{j]\ i}^{\ h} + \Gamma_{[k|l|}^{\ \ h}\Gamma_{j]\ i}^{\ l}) \tag{2.13}$$

respectively, where indices to which the process of alternation is not to be applied can be isolated by vertical bars. A general kind of deformation includes both of them, and the former is responsible for the discrepancy of the location, and the latter for the change of the direction. In physical reality, they are the invariant expressions of the crystallographer's dislocation and disclination[10] (and/or incompatibility) respectively. For small disturbances, we have from (2. 6), (2. 11), (2. 12) and (2. 13)

$$S_{kji} = \partial_{[k}\varepsilon_{j]i} \tag{2. 14}$$

and

$$R_{kji}^{\cdot\cdot\cdot h} = 0\,.$$

3. Dual Regime

In the foregoing analysis a general kind of deformation such as strain and dislocation etc. are extended to asymmetry. H. Schaefer pointed out the analogy between the field structure of the incompatibility tensor and that of the stress tensor[11]. In the latter space, stress tensor plays the role of the Riemann-Christoffel curvature tensor of a Riemannian or non-Riemannian space, which are different from that of strain-incompatibility configuration. The metric tensor of the dual regime is Beltrami's three-dimensional stress function, and the torsion tensor is the so-called couple stress, the source of which originates from the antisymmetric part of stress. Many dual aspects are found in plasticity physics. Constitutive equation arise from this duality and dual yielding criterion in the strain space is considered to be responsible for the fatigue fracture[12].

4. Volumetric Strain

It is necessary to consider the microscopic rotation ω_{ji} of a material element in granular media, which is independently introduced of its neighbours. We shall assume, without loss of generality, the asymmetric tensor ω_{ji} to be antisymmetric, i. e.

$$\omega_{ji} = -\omega_{ij}\,, \tag{4. 1}$$

to represent the rotation of a material element. This is a relative movement to surroundings and need not be holonomic, so that it causes the discontinuity along the boundary of the material element, and the local coordinate axes fixed to each material element rotate by ω_{ji}.

Since the stress is a quantity in the regime of stress-function and stress, and the rotation ω_{ji}, which is associated with the asymmetric strain, is that of strain and incompatibility extended to asymmetry, we introduce here a linear coupling relation between the stress and the rotation ω_{ji}. We assume the law of friction

$$\sigma_{[ji]} = A_{[ji]}^{\cdot\cdot[kl]}\omega_{[lk]}\,, \tag{4. 2}$$

where σ_{ji} is the stress tensor and $A_{ji}^{\cdot\cdot kl}$ is the coefficient of friction. For isotropic material the latter is given by

$$A_{ji}^{\cdot\cdot kl} = A(\delta_j^l\delta_i^k - \delta_i^l\delta_j^k)\,, \qquad A>0\,. \tag{4. 3}$$

From (4. 1), (4. 2) and (4. 3) we have

$$\sigma_{[ji]} = A\omega_{[ji]}\,. \tag{4. 4}$$

On the other hand from (2. 14) we have

$$S_{kji} = \partial_{[k}\omega_{j]i} \qquad (4.5)$$

or considering (4. 1) we have

$$S_i = \frac{1}{2}\partial^j \omega_{ji}\,, \qquad (4.6)$$

where

$$\partial^j = \delta^{jk}\partial_k$$

and

$$S_i = S_{ij}^{\cdot\cdot j} \qquad (4.7)$$

in three dimensions.

It should be noted that the equation (4. 5) and (4. 6), which are not apparently tensor relations, hold even in any coordinate system, so long as we are concerned with small disturbances alone.

It follows from (4. 4) and (4. 6) that

$$S_i = \frac{1}{2A}\partial^j \sigma_{[ji]}\,, \qquad (4.8)$$

which has the same structure obtained in the previous paper[8] within a scalar factor. The equation (4. 8) means that the S_i is entailed, if and only if, there is a non-divergent asymmetric stress. The vector S_i is related to the gradient of the volumetric strain along the axis of x^i.

Acknowledgement

The author wishes to take this opportunity to thank Professor Michiaki Kawaguchi for his valuable advice and inspiring criticisms.

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