北海道大学工学部研究報告
第81号（昭和51年）

Bulletin of the Faculty of Engineering,
Hokkaido University, No. 81 (1976)

Convective Instability in a Porous Medium Heated From Below

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(Received March 31, 1976)

Abstract

An analytical investigation is carried out to determine the conditions marking the onset of free convection in a horizontal porous layer. Consideration is given to a variety of thermal boundary conditions at the surfaces which bounded the porous layer. The thermal conditions adopted are related to the convective-radiative exchanges at the surfaces which include a fixed temperature and a fixed heat flux as a special case. It is demonstrated that the Rayleigh number marking the onset of free convection is the greatest for the boundary condition of fixed temperature and decreases monotonously as the condition of fixed heat flux is approached.

1. Introduction

The problem of the onset of free convection in a horizontal porous layer has been the subject of extensive study, both analytically and experimentally. The theoretical and experimental studies of the onset of free convection in a porous medium were made at an early date by Horton and Rogers[1] in connection with the distribution of NaCl in subterranean sand layers. Lapwood[2] treated a similar problem analytically in a more precise manner and established the criterion for the onset of free convection in such a layer, which was later confirmed experimentally by Katto and Masuoka[3].

In the aforementioned analytical studies, thermal boundary conditions applied at the upper and the lower surfaces of the porous layer are based on the supposition that these surfaces are in contact with materials of infinite thermal conductivity and heat capacity. In such a model, it follows that the temperatures at the surfaces are not perturbed even if the quiescent state breaks down.

Consideration of actual physical situations leads to the fact that the heretofore standard thermal condition of fixed temperature at the surfaces of the porous layer may be too restrictive. For example, when the upper surface is free, there will be a heat exchange between the free surface and the environment. In this case, if the heat transfer coefficient between the surface and the environment is finite, the surface temperature will easily be perturbed when the quiescent state breaks down. Moreover, if the heating at the lower surface is accomplished by passing an electric current through a thin metallic foil, the boundary condition at the lower surface may be more close to a fixed heat flux rather than a fixed temperature.

The purpose of the present study is to predict analytically the onset of free convection for a broad range of thermal boundary conditions which may be actu-

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ally of importance.

2. Analysis

Consideration is given to a horizontal porous medium bounded above and below by plain surfaces through which heat may flow into or out of the porous medium. The physical situation to be analyzed here is depicted schematically in Fig. 1. The horizontal extension of the porous medium is sufficiently great so that edge effects may be neglected. If the temperature within the porous medium were increased monotonously from the lower side under a rest-state condition, then lighter fluid would always be above heavier fluid in the porous medium and there would be static equilibrium without motion. The existence of such a state can be investigated by finding the conditions under which a given temperature distribution is stable even if a small disturbance is present.

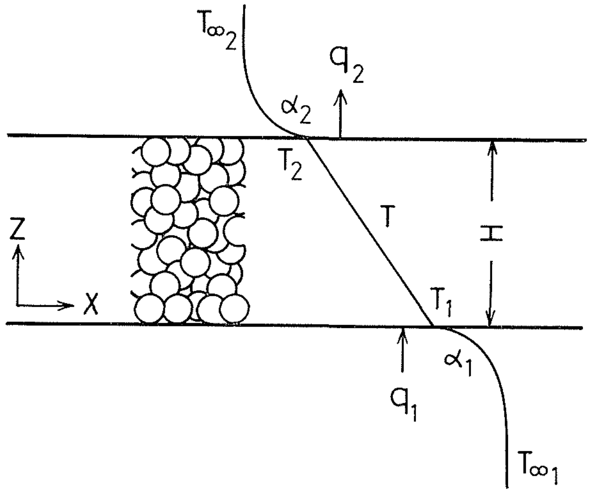

Fig. 1 Schematic diagram of the physical problem and co-ordinate system

The thickness of the porous medium will be represented by H, with z measuring distance vertically upward : $z=0$ corresponds to the lower surface of the porous medium and $z=H$ corresponds to the upper surface. The co-ordinates axes x and y lie in a horizontal plane. Under a rest-state quiescent condition, the temperature and the pressure distributions in the porous medium can be considered to depend only on z, as yielding

$$T_{rs}=T_1-\delta z \tag{1}$$

$$\frac{dp_{rs}}{dz}=-g\rho_1(1+\beta\delta z) \tag{2}$$

in which T denotes temperature, δ the temperature gradient, g the accerelation velocity, ρ the density and β the coefficient of thermal expansion. The suffix 1 indicates a value at the lower surface of the porous medium.

To study the conditions under which the rest-state is unstable, small disturbances are superposed on the rest-state temperature, velocity, and pressure fields. In general, the disturbances will depend on all of three co-ordinates and time, so that the resultant quantities (rest-state plus disturbance) will have a similar dependability. Correspondingly,

$$T(x, y, z, t)=T_{rs}(z)+T^*(x, y, z, t), \qquad p(x, y, z, t)=p_{rs}(z)+p^*(x, y, z, t) \tag{3a}$$

$$u=u^*(x, y, z, t), \qquad v=v^*(x, y, z, t), \qquad w=w^*(x, y, z, t) \tag{3b}$$

where u, v, and w are velocity components corresponding to x, y, and z respectively; t is time, and p is static pressure. The quantities T^*, u^*, v^* and w^* are taken to be sufficiently small so that their squares and products be negligible.

It is well known that the actual flow in a porous medium can be replaced by a hypothetical uniform flow of the same gross rate on an assumption of the space being homogeneous. For this flow the expression of resistance differs from that for normal fluid flow, and instead of shearing stress proportional to the velocity gradient, Darcy's law can be applied. The equations expressing continuity, motion and energy may now be written and the foregoing perturbations are introduced. Carrying out manipulations similar to those of Lapwood[2], it is possible to eliminate all perturbations except T^* and the governing equation is given as

$$\left(\frac{\partial}{\partial t}+\frac{g}{k}\right)\left(\frac{\partial}{\partial t}-\kappa\nabla^2\right)\nabla^2 T^*=g\delta\beta\nabla_1^2 T^* \tag{4}$$

in which k is the permeability of the porous medium, κ the thermal diffusivity and ∇_1^2 the two-dimensional Laplace operator $(\partial^2/\partial x^2+\partial^2/\partial y^2)$.

Let introduce the non-dimensional quantities $(\xi, \eta, \zeta)=(x, y, z)/H$, $\theta=T^*/(T_1-\delta H)$, and dimensionless time scale $\tau=t/(H^2/\nu)$. In order to show the region of interest in the present study, one considers the following general form of disturbance

$$\theta=\Theta(\zeta)exp\{i(a_\xi\xi+a_\eta\eta)+\phi\tau\} \tag{5}$$

where a_ξ and a_η are wave numbers in the ξ- and η-directions, respectively. Substitution of Eq. (5) into Eq. (4) yields

$$\left[(D^2-a^2)+\frac{k}{H^2}\frac{\phi}{Pr}(D^2-a^2)\right][(D^2-a^2)-\phi]\Theta=\frac{k}{H^2}a^2Ra\Theta \tag{6}$$

where $D=\partial/\partial\zeta$, $a=\sqrt{a_\xi^2+a_\eta^2}$, $Pr=\nu/\kappa$ and $Ra=g\delta\beta H^4/(\nu\kappa)$ which is a Rayleigh number.

3. Boundary conditions

For the impermeable boundaries applied at the upper and the lower surfaces of the porous medium, one has

$$(D^2-a^2)\Theta=0 \tag{7}$$

Sparrow et al.[4] extensively studied the effect of the thermal boundary conditions on thermal instability in a horizontal normal fluid layer, hence a detailed interpretation regarding their manipulations is not repeated here. According to Sparrow et al.[4], the general thermal boundary condition at the surface can be written in the following form

$$D\Theta+Bi\Theta=0 \tag{8}$$

where $Bi=\alpha H/\lambda$ in which α is the heat transfer coefficient and λ thermal conductivity.

4. Solution of the perturbation equation

Solution of Eq. (6) must be subjected to the boundary conditions of Eqs. (7) and (8). A numerical technique developed originally by Sparrow et al.[4] was adopted. It has been shown by Pellew and Southwell[5] that the threshold of instability is marked by $\phi=0$. A general solution of Eq. (6) with $\phi=0$ can be constructed in the form

$$\Theta(\zeta)=\sum_{i=0}^{3}X_iF^{(i)}(\zeta) \text{ and } F^{(i)}(\zeta)=\sum_{n=0}^{\infty}Y_n^{(i)}\zeta^n \tag{9}$$

where X_i is arbitrary and the series coefficient $Y_n^{(i)}$ obeys the following recurrent relationship for $n \geqq 4$

$$Y_n^{(i)}=(1/n!)\{\Pi_1(n-2)!Y_{n-2}^{(i)}-\Pi_2(n-4)!Y_{n-4}^{(i)}\} \tag{10}$$

where $\Pi_1=2a^2$ and $\Pi_2=a^4-(k/H^2)a^2Ra$. In addition, $Y_0^{(i)}$ to $Y_3^{(i)}$ are specified as

$$Y_n^{(i)}=\Delta_{ni} \quad (0 \leqq n \leqq 3) \tag{11}$$

wherein $\Delta_{n_i}=1$ for $n=i$ and $\Delta_{n_i}=0$ for $n=i$.

The constants $X_0, X_1, ..., X_3$ which appear in the solution for Θ are to be determined by using the boundary conditions. Substitution of Eq. (9) into Eqs. (7) and (8) yields four linear algebraic equations for X_0, X_1, X_2 and X_3. The solution of such an algebraic system is possible only if the determinant of the coefficients of X_i vanishes. The value of the deteminant depends upon four parameters; wave number a, Biot number Bi, constant k/H^2 and Rayleigh number Ra. Now suppose that k/H^2 and Bi be constant. Then, for every wave number a which one might select, there can be found a Rayleigh number that causes the determinant of the coefficients to be zero. Numerical computations to find the aforementioned Rayleigh number are carried out with the aid of FACOM 230–75 Digital Computer at the Computer Center of Hokkaido University.

5. Results and discussions

Lapwood[2] derived the criterion for the onset of free convection in the porous medium under the conditions that velocity at the upper and the lower bounding surfaces vanishes and the temperature at those surfaces can be fixed, yielding

$$\frac{k}{H^2}Ra=4\pi^2$$

where Rayleigh number $Ra=g\beta H^3\Delta T/(\nu\kappa)$ and β is the cubical expansion coefficient of fluid, and H the thickness of porous medium, ΔT the temperature difference between the two bounding surfaces, and ν the kinematic viscosity.

In Figs. 2 and 3 the neutral stability curves are drawn for different values of Biot number under the condition of a fixed temperature or a fixed heat flux at the lower surface in the form of $(k/H^2)Ra$ vs. a. It can be clearly noted from these figures that the tendency remains unaltered by increasing the Biot number since the shapes of the neutral stability curves are almost identical except for the case of $Bi=0$ in which the lower surface is at a fixed heat flux. As will be seen in Figs. 2 and 3, it is found that for a particular wave number a, the corresponding modified Rayleigh number $(k/H^2)Ra$ has a value which is smaller than that for any other wave number a. Correspondingly, for each Biot number, there is a minimum modified Rayleigh number which permits a solution of the perturbation equation. Below this modified Rayleigh number for each Biot number, a solution for the perturbation equation cannot be found and this indicates that the quiescent state may be stable. The thus determined minimum Rayleigh number corresponds to the criterion for the onset of free convection in the porous medium.

The modified Rayleigh numbers marking the onset of free convection are presented graphically in figure 4. The solid curve in this figure provides

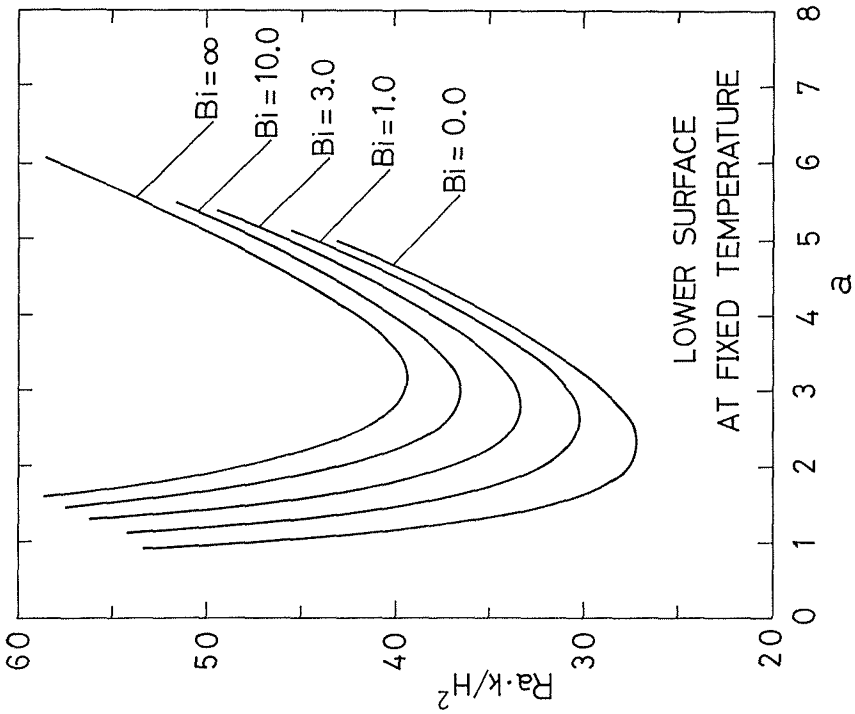

Fig. 2 Neutral stability curves where the lower surface is at a fixed temperature

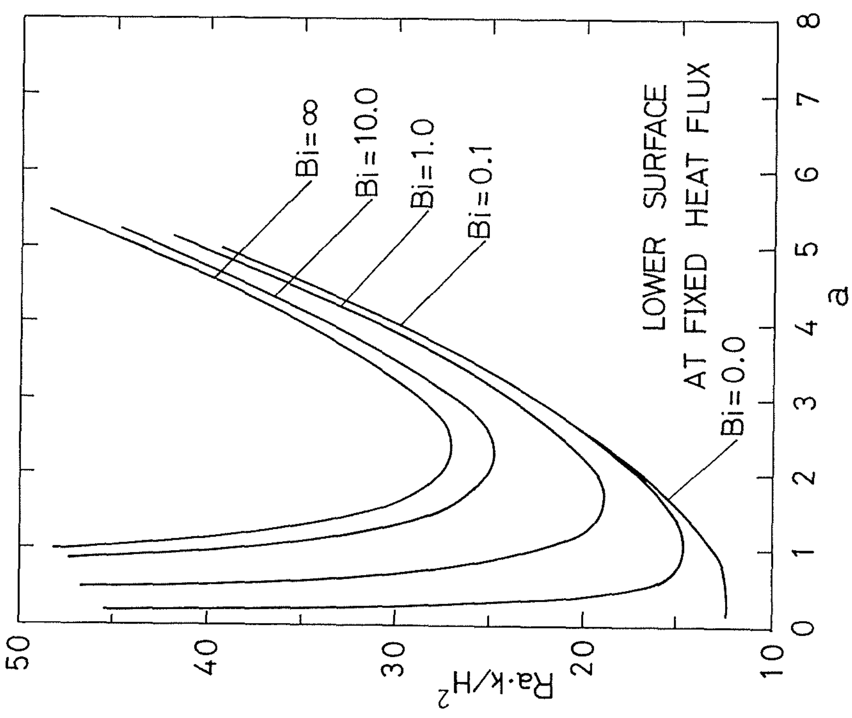

Fig. 3 Neutral stability curves where the lower surface is at a fixed heat flux

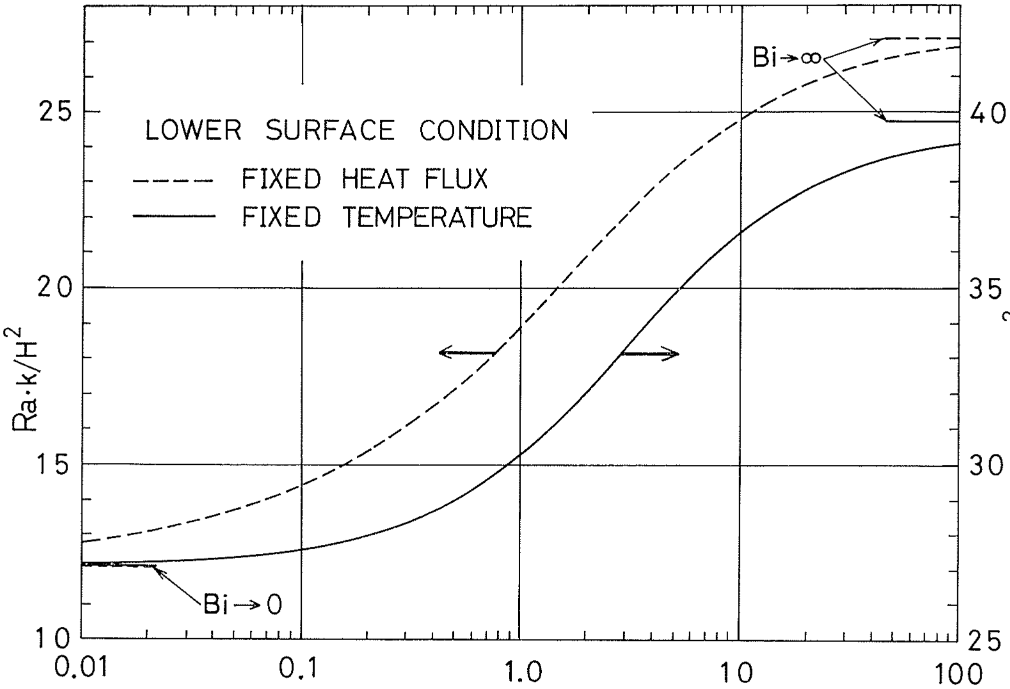

Fig. 4 Results for the Rayleigh number marking the onset of free convection

Table 1 Critical Rayleigh numbers for the case of fixed temperature at lower bounding surface

$\alpha H/\lambda$	a	$(k/H^2)Ra$
0*	2.323	27.098
0.01	2.330	27.142
0.03	2.342	27.230
0.1	2.365	27.526
0.3	2.436	28.287
1.0	2.598	30.269
3.0	2.818	33.275
10.0	3.010	36.539
30.0	3.090	38.298
100.0	3.126	39.097
∞**	3.142	39.478

* Corresponds to fixed surface heat flux
** Corresponds to fixed surface temperature

Table 2 Critical Rayleigh numbers for the case of fixed heat flux at lower bounding surface

$\alpha H/\lambda$	a	$(k/H^2)Ra$
0*	0.132	12.020
0.01	0.569	12.746
0.03	0.750	13.294
0.1	1.006	14.361
0.3	1.307	16.024
1.0	1.678	18.851
3.0	1.989	22.015
10.0	2.201	24.855
30.0	2.279	26.227
100.0	2.313	26.820
∞**	2.350	27.100

* Corresponds to fixed surface heat flux
** Corresponds to fixed surface temperature

information for the case in which the lower surface is kept at a fixed temperature and is referred to on the right-hand ordinate scale, while the dashed curve is for the case in which the lower surface is kept at a fixed heat flux and is referred to on the left-hand ordinate scale. The horizontal lines adjacent to each extremity of the curve indicate the critical Rayleigh numbers for the limiting cases of $\alpha H/\lambda \to 0$ and $\alpha H/\lambda \to \infty$.

A parallel presentation of the calculated results is made in table 1 and 2, wherein are also listed the a values corresponding to the critical Rayleigh numbers.

The cases designated by $\alpha H/\lambda=0$ and $\alpha H/\lambda=\infty$ correspond to fixed heat flux and to fixed temperature at the upper surface, respectively.

From an inspection of figure 4, it can be seen that the critical Rayleigh number increases monotonously with increasing Bi. Thus, the most stable situation corresponds to a fixed surface temperature. The critical Rayleigh number is most sensitive to Biot number in a range of $Bi=1.0\sim10.0$. The foregoing remarks can be applied regardless of whether the lower surface is kept at a fixed temperature or at a fixed heat flux. However, there is a marked difference between the numerical values of the critical Rayleigh number for these two cases. The critical Rayleigh numbers for the lower surface at a fixed temperature exceed those for the lower surface at a fixed heat flux by about 12, this difference being approximately uniform for Bi ranging from 0.5 to ∞.

It is of interest to inquire how the present calculated results compare with those of other investigators. Only two of the entries in tables 1 and 2 can be specially compared. For the case of a fixed temperature at both boundaries, Lapwood[2] found a critical Raylefgh number of $4\pi^2$, while that of the present calculated result is 39.478. On the other hand, for the case of a fixed heat flux at both boundaries, Nield[6] pointed out a critical Rayleigh number of 12, while that of the present one is 12.020. The agreement between the previous investigators' results and the present ones is thus seen to be excellent.

References

1) Horton, C. W. and Rogers, F. T.: "Convective Current Motion in a Porous Medium", J. Appl. Phys., Vol. 16, 1945, pp. 367–370.
2) Lapwood, E. R.: "Convection of a Fluid in a Porous Medium", Proc. Camb. Soc., Vol. 44, 1948, pp. 508–521.
3) Katto, Y. and Masuoka, T.: "Criterion for the Onset of Convective Flow in a Fluid in a Porous Medium", Int. J. Heat & Mass Transfer, Vol. 10, 1967, pp. 297–309.
4) Sparrow, E. M., Goldstein, R. J. and Jonsson, V. K.: "Thermal Instability in a Horizontal Fluid Layer: Effect of Boundary Conditions and Nonlinear Temperature Profile", J. Fluid Mech., Vol. 18, 1964, pp. 513–528.
5) Pellew, A. R. and Southwell, R. V.: "On Maintained Covective Motion in a Fluid Heated From Below", Proc. Roy. Soc., A, Vol. 176, 1940, pp. 312–343.
6) Nield, D. A.: "The Onset of Transient Convective Instability", J. Fluid Mech., Vol. 71, 1975, pp. 441–454.