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Various Types of Out-of-Plane Vibration of Arcs

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Abstract

The natural frequencies and the mode shapes of out-of-plane vibrations of elastic arcs are calculated numerically on the basis of the Timoshenko beam theory and other specialized theories in which either or both of the rotatory inertia and shear deformation are not taken into account, and of the equations in which one or two of the displacements of arcs are restrained to be zero. The results are compared with one another, and the characters of out-of-plane vibrations governed by these various type equations are studied.

1. Introduction

The vibration theory of a beam has great importance in many engineering applications such as in the design of machines and structures. Therefore, a considerable number of papers are available on the out-of-plane vibration of arcs or curved beams, as well as straight beams. The fundamental equations of arcs or curved beams have been presented together with the solution to them in the book of Love.[1] Takahashi,[2] Vorterra and Morell,[3,4] Chang and Volterra,[5] and Suzuki, Aida and Takahashi[6] studied the free vibration of arcs and curved beams on the basis of the classical beam theory in which the rotatory inertia and shear deformation are not taken into account. Recently, Rao,[7] Kirkhope,[8] Suzuki and Takahashi,[9] and Davis, Henshell and Warburton[10] have analyzed uniform rings and curved beams, and Irie, Yamada and Takahashi [11,12] have studied analytically the vibration of arcs and curved beams of variable cross-sections. These recent studies have been based upon the Timoshenko beam theory in which both of the rotatory inertia and shear deformation are taken into account.

This paper presents an analysis of various types of out-of-plane vibration of uniform arcs governed by the Timoshenko beam theory, by the specialized theories in which either or both of the rotatory inertia and shear deformation are not taken into account, and by the equations in which one or two of the displacements of arcs are restrained to be zero. Although the vibrations of arcs arising from the assumption that any of the displacements which are restrained to be zero are artificial and would

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not occur except in specialized situations, they would have some meanings to clarify the dynamic characters of arcs.

For this purpose, the equations of each type out-of-plane vibration of an arc are written in a matrix differential equation of the first-order by use of the transfer matrix of the arc. The transfer matrix is conveniently expressed as the series type solution to the equation, and the frequency equation of each vibration is derived by the boundary conditions. The natural frequencies (the eigenvalues of vibration) and the mode shapes of various types of out-of-plane vibration are calculated numerically by the respective frequency equation, and are compared with one another for studying the dynamical characters.

2. Timoshenko Equations and Specialized Equations of Out-of-Plane Vibrations

We consider a uniform arc of a radius of curvature of the neutral axis R. With the angular co-ordinate denoted by θ and with the opening angle by α, the X-, Y-

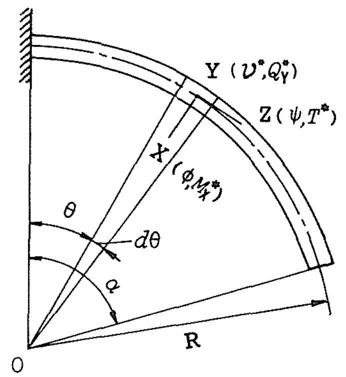


Fig. 1 Circular arc

and Z-axes are taken in radial, transverse and tangential directions, respectively, as shown in Fig. 1.

(TM) Timoshenko Equations

On the assumption that the shear center of the cross-section coincides with the centroid, the Timoshenko equations of free out-of-plane vibration of the arc are written as

$$\frac{dQ_Y^*}{Rd\theta}+\rho A\omega^2 v^*=0 \tag{1}$$

$$-\frac{dM_X^*}{Rd\theta}+\frac{T^*}{R}-Q_Y^*+\rho I_X\omega^2\varphi=0 \tag{2}$$

$$\frac{dT^*}{Rd\theta}+\frac{M_X^*}{R}+\rho J_Z\omega^2\psi=0 \tag{3}$$

where ρ is the mass per unit volume, A is the cross-sectional area, I_X and J_Z, respectively, are the second moment and polar moment of area of the arc, and ω is the circular frequency. The bending moment M_X^*, the torsional moment T^* and the shearing force Q_Y^*, respectively, are given by

$$M_X^*=\frac{EI_X}{R}\left(-\psi-\frac{d\varphi}{d\theta}\right),\qquad T^*=\frac{GC_Z}{R}\left(-\varphi+\frac{d\psi}{d\theta}\right) \tag{4, 5}$$

$$Q_Y^*=\varkappa GA\left(\varphi+\frac{dv^*}{Rd\theta}\right) \tag{6}$$

in terms of the transverse deflection v^*, the angle of rotation φ due to pure bending and the angle of torsion ψ. Here, the variables φ, v^*, ψ, M_X^*, Q_Y^* and T^* are taken to be of positive sign in the X-, Y- and Z-axes. The quantity E is Young's modulus, G is the shear modulus and C_Z is the St. Venant torsional constant of the cross-section. The parameter $\varkappa$ is the numerical factor depending upon the shape of cross-section, which is 0.85 for rectangular cross-section and 0.89 for circular cross-section for an arc of Poisson's ratio $\nu=0.3$.[13]

The boundary conditions of the arc are written as

$$\begin{aligned}&M_X^*=T^*=Q_Y^*=0 && \text{at free end}\\ &v^*=M_X^*=T^*=0 && \text{at hinged end}\\ &v^*=\psi=\varphi=0 && \text{at clamped end}\end{aligned} \tag{7}$$

For simplicity of the analysis, the following dimensionless variables are introduced:

$$v=\frac{1}{R}v^*,\quad (M_X,\ T)=\frac{R}{EI_X}(M_X^*\ T^*),\quad Q_Y=\frac{R^2}{EI_X}Q_Y^* \tag{8}$$

$$s_X^2=\frac{AR^2}{I_X},\quad s_Y^2=\frac{AR^2}{I_Y},\quad \mu=\frac{GC_Z}{EI_X} \tag{9}$$

and

$$\Lambda^2=\frac{\rho AR^4\omega^2}{EI_X} \tag{10}$$

The quantities s_X and s_Y are the slenderness ratios, μ is the rigidity ratio of the arc and Λ denotes a frequency parameter.

Equations (1)-(6) can be written in a matrix differential equation

$$\frac{d}{d\theta}\begin{Bmatrix}v\\ \psi\\ \varphi\\ M_X\\ T\\ Q_Y\end{Bmatrix}=\begin{bmatrix}0 & 0 & -1 & 0 & 0 & \frac{E}{\varkappa G}\frac{1}{s_X^2}\\ 0 & 0 & 1 & 0 & \frac{1}{\mu} & 0\\ 0 & -1 & 0 & -1 & 0 & 0\\ 0 & 0 & \Lambda^2\frac{1}{s_X^2} & 0 & 1 & -1\\ 0 & -\Lambda^2\left(\frac{1}{s_X^2}+\frac{1}{s_Y^2}\right) & 0 & -1 & 0 & 0\\ -\Lambda^2 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}v\\ \psi\\ \varphi\\ M_X\\ T\\ Q_Y\end{Bmatrix} \tag{11}$$

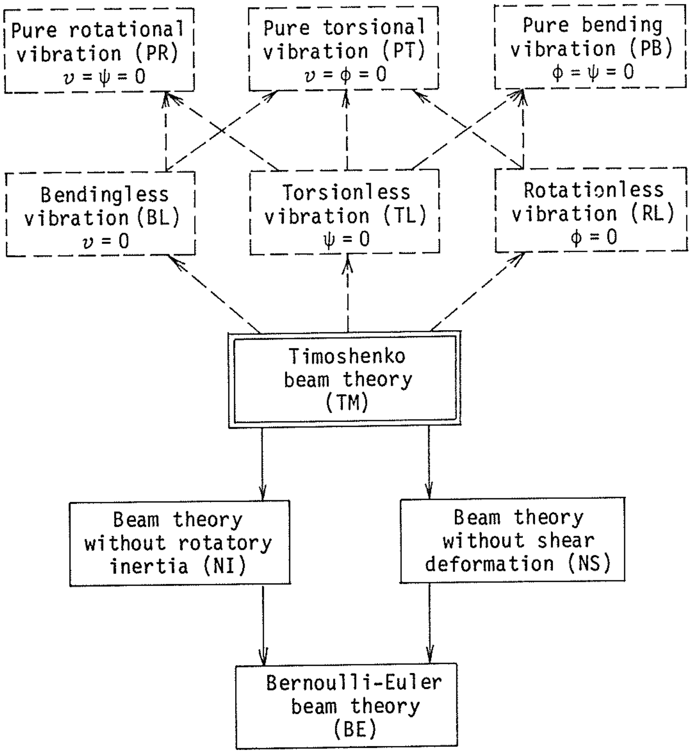
Equation (11) is also written as

$$\frac{d}{d\theta}\{z(\theta)\}=[M]\{z(\theta)\} \tag{12}$$

by using the state vector $\{z(\theta)\}=\{v\ \psi\ \varphi\ M_X\ T\ Q_Y\}^{\mathrm{T}}$ and the cross-symmetric coefficient matrix $[M]$ given in (11).

(NI) Equations without Rotatory Inertia Taken into Account

Table 1 Beam theories and specialized vibrations



When the rotatory inertia of the arc is not taken into account, the equations of out-of-plane vibration are given by (11) in which the element M_{43} of the matrix $[M]$ is taken as zero.

(NS) Equations without Shear Deformation Taken into Account

In this case, the equations of vibration are given by (11), in which the element M_{16} is taken as zero.

(BE) Bernoulli-Euler Equations (Classical Beam Theory)

In the classical beam theory in which both of the rotatory inertia and shear deformation are not taken into account, the equations of vibration are also given by (11) in which both of the elements M_{16} and M_{43} are taken as zero.

We consider other equations in which one or two of the deflection and the angles of rotation and torsion are restrained to be zero. Table 1 shows the relations among the Timoshenko equations and other specialized equations mentioned here.

(BL) Bendingless Vibration

When the transverse deflection v is restrained to be zero, the shearing force is expressed as $Q_Y=(\kappa G/E)s_X^2\varphi$ and other variables are governed by the equation which is obtained by removing the variables v and Q_Y from (11).

(TL) Torsionless Vibration

When the angle of torsion ψ in zero, the torsional moment is expressed as $T=$

$-\mu\varphi$ and other variables are governed by the equation obtained by removing the variables ψ and T.

(RL) Rotationless Vibration

When the angle of rotation φ is zero, the bending moment is expressed as $M_X = -\psi$ and other variables are governed by the four independent equations

$$\frac{d^2}{d\theta^2}\begin{Bmatrix} v \\ Q_Y \end{Bmatrix} + \frac{E}{\varkappa G}\frac{\Lambda^2}{s_X^2}\begin{Bmatrix} v \\ Q_Y \end{Bmatrix} = 0 \tag{13}$$

$$\frac{d^2}{d\theta^2}\begin{Bmatrix} \psi \\ T \end{Bmatrix} + \frac{1}{\mu}\left\{\Lambda^2\left(\frac{1}{s_X^2}+\frac{1}{s_Y^2}\right)-1\right\}\begin{Bmatrix} \psi \\ T \end{Bmatrix} = 0 \tag{14}$$

(PR) Pure Rotational Vibration

When both of the deflection and the angle of torsion are restrained to be zero, the torsional moment and shearing force, respectively, are written as $T = -\mu\varphi$, $Q_Y = (\varkappa G/E)s_X^2\varphi$ and other variables are governed by

$$\frac{d^2}{d\theta^2}\begin{Bmatrix} \varphi \\ M_X \end{Bmatrix} + \left\{-\mu - \left(\frac{\varkappa G}{E}\right)s_X^2 + \frac{\Lambda^2}{s_X^2}\right\}\begin{Bmatrix} \varphi \\ M_X \end{Bmatrix} = 0 \tag{15}$$

(PT) Pure Torsional Vibration

When the deflection and the angle of rotation are zero, the bending moment and shearing force, respectively, are $M_X = -\psi$, $Q_Y = 0$ and other variables are governed by (14).

(PB) Pure Bending Vibration

When the angles of rotation and torsion are zero, the bending moment and torsional moment are also zero, and other variables are govemned by (13).

3. Frequency Equations and Eigenvalues of Vibration

Here, the solutions to the above-mentioned equations are obtained by using the transfer matrix approach, from which the frequency equations of the arc are derived for each vibration.

3. 1 Frequency Equations of the TM-, NI-, NS- and BE-Vibration

The state vector $\{z(\theta)\}$ of (11) can be expressed as

$$\{z(\theta)\} = [T(\theta)]\{z(0)\} \qquad (\theta > 0) \tag{16}$$

by using the transfer matrix $[T(\theta)]$ of the arc. The substitution of (16) into (12) yields

$$\frac{d}{d\theta}[T(\theta)] = [M][T(\theta)] \tag{17}$$

The transfer matrix can be conveniently expressed as the power series type solution to (17),

$$[T(\theta)] = \exp([M]\theta) = [I] + \frac{1}{1!}[M]\theta + \frac{1}{2!}[M]^2\theta^2 + \cdots + \frac{1}{n!}[M]^n\theta^n + \cdots \tag{18}$$

Numerical difficulty arises in the calculation of $[T(\theta)]$ given by (18) if the opening angle α is too large. However, it can be overcome by subdividing the arc into five to ten small elements at the most and calculating the transfer matrices for each element. The entire structure matrix is obtained by assembling the matrices of these elements.

The substitution of (16) into a given set of the boundary conditions (7) yields the frequency equation of the arc with only the elements of the matrix $[T(\alpha)]$ necessary for the calculation. For example, the frequency equation of a clamped-clamped arc is written as

$$\begin{bmatrix} T_{14} & T_{15} & T_{16} \\ T_{24} & T_{25} & T_{26} \\ T_{34} & T_{35} & T_{36} \end{bmatrix}_{(\alpha)} \begin{Bmatrix} M_X \\ T \\ Q_Y \end{Bmatrix}_{(0)} = 0 \tag{19}$$

3. 2 Frequency Equations of the BL- and TL-Vibrations

The frequency equations of the BL- and TL-vibrations are expressed by the partitioned matrix equations obtained by removing more unnecessary elements from (19), though the elements of the matrix $[M]$ and hence the elements of the matrix $[T(\alpha)]$ have different values depending upon the type of vibration.

3. 3 Eigenvalues of the RL-, PR-, PT- and PB-Vibrations

The eigenvalues of the RL- and PB-vibrations are determined by

$$\Lambda = s_X \frac{n\pi}{\alpha}\sqrt{\frac{\varkappa G}{E}} \tag{20}$$

those of the RL- and PT-vibrations are

$$\Lambda = s_X \sqrt{\frac{1+(n\pi/\alpha)^2 \mu}{1+(I_Y/I_X)}} \tag{21}$$

and those of the PR-vibration are

$$\Lambda = s_X \sqrt{\left(\frac{n\pi}{\alpha}\right)^2 + \mu + \left(\frac{\varkappa G}{E}\right) s_X^2} \tag{22}$$

which are derived from the solutions to (13,14,15), respectively. The number n of these equations takes the specified values depending upon the boundary conditions. For example,

$$\begin{array}{ll} n=1,\ 2,\ 3,\cdots & \text{for clamped-clamped arcs} \\ n=1/2,\ 3/2,\ 5/2,\cdots & \text{for free-clamped arcs} \\ n=1,\ 2,\ 3,\cdots & \text{for hinged-hinged arcs} \end{array} \tag{23}$$

The number n takes $n=0,1,2,\cdots$ only for the RL- and PT-vibrations of hinged-hinged arcs.

4. Numerical Calculation and Discussion

In this section, the natural frequencies (the eigenvalues of vibration) and the mode shapes of various types of out-of-plane vibration are calculated numerically for clamped-clamped, free-clamped and hinged-hinged arcs of uniform circular cross-section. With the opening angle taken as $\alpha=120°$, all the figures show conveniently eigenvalues $\lambda=\Lambda/s_X$ versus the ratio $1/s_X$ $(=1/s_Y)$, since the dynamical characters are most affected by the slenderness ratio s_X or s_Y among the parameters of arcs.

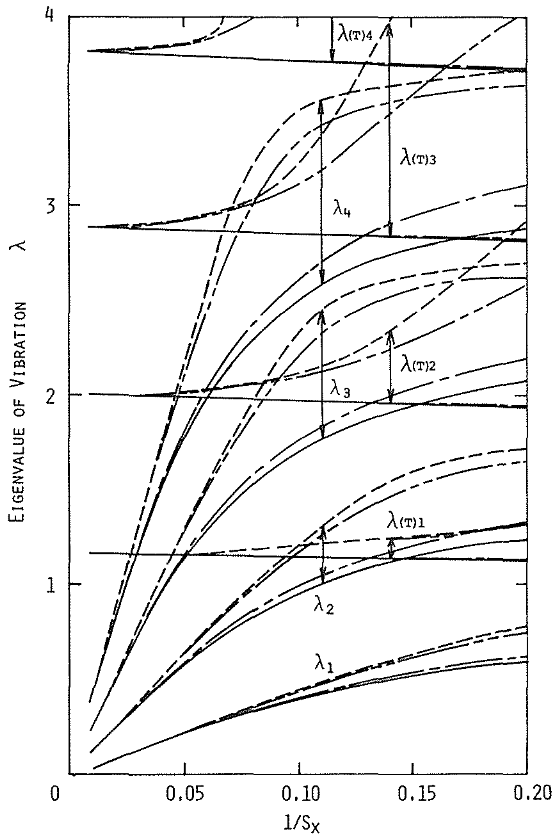


Fig. 2 Eigenvalues λ of out-of-plane vibration of clamped-clamped arcs : $\nu=0.3$, $\kappa=0.89$, $\alpha=120°$. ——————; (TM), —— - ———; (NI), —— - - ———; (NS), — — — — -; (BE)

Fig. 2 shows the first four eigenvalues λ of the TM-, NI-, NS- and BE-vibrations of clamped-clamped arcs. The eigenvalues of these vibrations become larger in that order for the TM-, NI-, NS- and BE-vibrations according to whether the rotatory inertia or shear deformation is taken into account or not. The eigenvalues of the NI-vibration are comparatively near to those of the TM-one, and the values of the NS-vibration are near to those of the BE-one. The difference among these eigenvalues is large in higher modes, and becomes larger with an increase of the ratio $1/s_X$ and hence with a decrease of the slenderness ratio s_X. In clamped-clamped arcs, torsion type vibrations ((T)-vibrations) with the frequencies $\lambda_{(T)1}$, $\lambda_{(T)2}$, $\cdots$ in which the angle of torsion is dominant among the displacements of arc arise besides the bending type vibrations in which the transverse deflection is dominant.[11),12)] The eigenvalues of the arcs become larger monotonically with a decrease of the ratio s_X, except for the TM (T)-and NI(T)-vibrations.

Fig. 3 shows the eigenvalues λ of the other six type vibrations of clamped-clamped

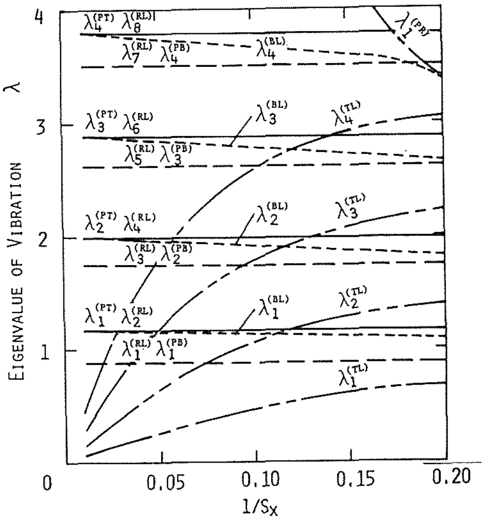


Fig. 3 Eigenvalues λ of out-of-plane vibration of clamped-clamped arcs: $\nu=0.3$, $\kappa=0.89$, $\alpha=120°$

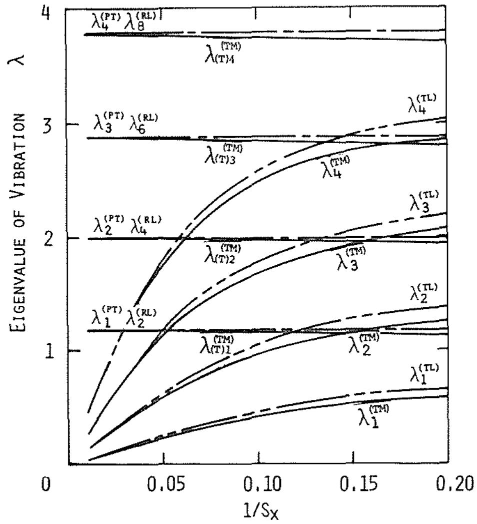
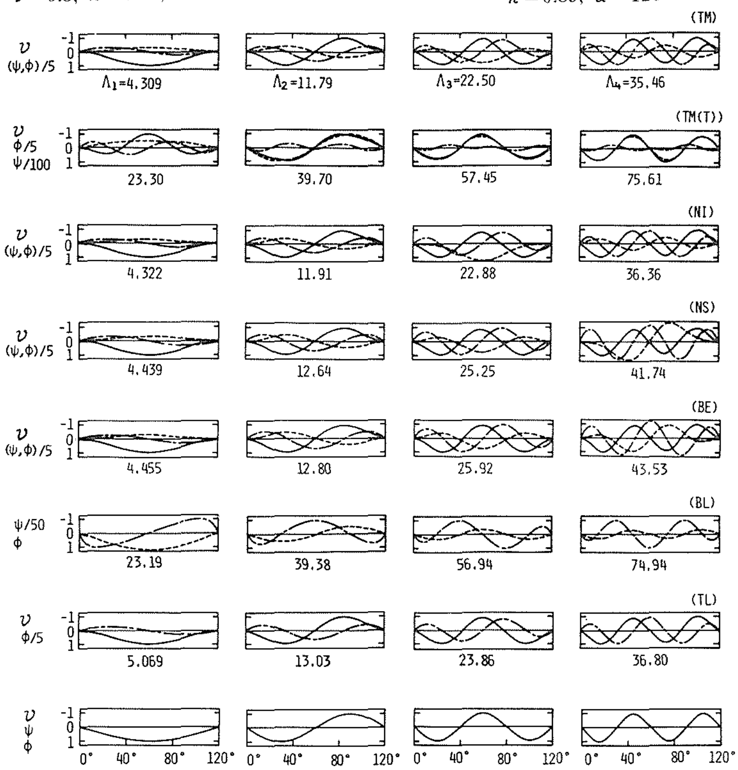


Fig. 4 Comparison of eigenvalues λ of clamped-clamped arcs: $\nu=0.3$, $\kappa=0.89$, $\alpha=120°$



17.52	35.03	52.55	70.06	(PB,RL)
23.37	39.81	57.58	75.75	(PT,RL)
236.1	241.8	250.9	263.2	(PR)

Fig. 5 Mode shapes of a clamped-clamped arc: $\nu=0.3$, $\kappa=0.89$, $s_X=s_t=20$, $\alpha=120°$. ——————; v, — — — —; ψ, ——— - ———; φ

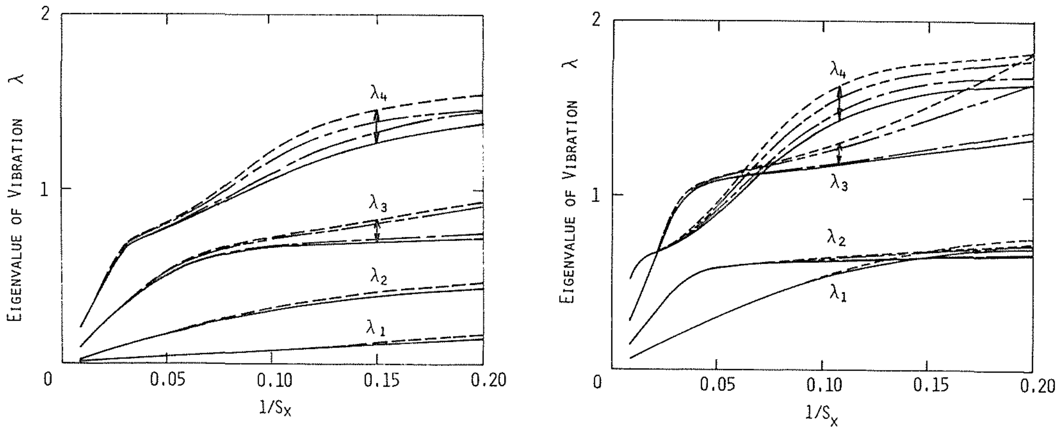


Fig. 6 Eigenvalues λ of out-of-plane vibration of arcs: $\nu=0.3$, $\kappa=0.89$, $\alpha=120°$. ———; (TM), —— - ——; (NI), —— - - ——; (NS), — — — —; (BE).
(a) Free-clamped arc, (b) hinged-hinged arc

arcs in which one or two of the deflection and the angles of rotation and torsion are restrained to be zero. The eigenvalues of the RL-, PT- and PB-vibrations are constant without being affected by the variation of the ratio $1/s_X$, as seen in (20) and (21). While, the values of the BL-vibration become slightly smaller and those of the TL-vibration become larger monotonically, with an increase of the ratio $1/s_X$. Though the values of the PR-vibration are very large, they become smaller with an increase of the ratio.

In Fig. 4, the eigenvalues λ of the TM- and TM (T)-vibrations based upon the Timoshenko theory are compared with those of the RL-, PT- and TL-vibrations for clamped-clamped arcs. The values of the bending type vibration (TM-vibration) are smaller than those of the TL-vibration, and the values of the torsion type vibration (TM(T)-vibration) are also slightly smaller than those of the RL- or PT-vibration. However, the differences between them become extremely small with a decrease of the ratio $1/s_X$.

Fig. 5 shows the eigenvalues Λ and the mode shapes of various types of out-of-plane vibration of a clamped-clamped arc. In clamped-clamped arcs, the transverse deflection and the angle of torsion are symmetrical and the angle of rotation is antisymmetrical with respect to the midpoint in the first and third modes. While, the angle of rotation is symmetrical and other displacements are antisymmetrical in the second and fourth modes. The figures of the second row show the mode shapes of the TM(T)-vibration in which the angle of torsion is dominant among the displacements.

Fig. 6 show the eigenvalues λ of the TM-, NI-, NS- and BE-vibrations of free-clamped and hinged-hinged arcs. In general, the eigenvalues of these arcs are smaller than those of clamped-clamped arcs, and the eigenvalues of the fourth mode change in a wave-like manner with the variation of the ratio $1/s_X$. Torsion type vibrations do not arise in free-clamped and hinged-hinged arcs. The out-of-plane vibrations of arcs with other boundary conditions are similar to those of free-clamped or hinged-hinged arcs in character, though there are differences to some extent among the eigenvalues of vibration.

The numerical computations presented here were carried out on a HITAC M-200H computer of the Hokkaido University Computing Center.

5. Conclusions

The various types of out-of-plane vibration of arcs governed by the Timoshenko beam equations and other specialized equations were studied by the transfer matrix method.

The equations of out-of-plane vibration of an arc have been expressed in a matrix differential equation of the first-order, being given a series type solution, from which the frequency equation was derived by the boundary conditions.

The eigenvalues of vibration and the mode shapes of clamped-clamped, free-clamped and hinged-hinged arcs were calculated numerically, from which the dynamic characters of arcs were studied.

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