北海道大学工学部研究報告
第108号（昭和57年）

Bulletin of the Faculty of Engineering,
Hokkaido University, No. 108 (1982)

A New Distortion Measure for Transform Image Coding

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(Received December 26, 1981)

Abstract

A new distortion measure for transform image coding is presented. A purely statistical method for evaluating image blurs caused by the ommission of high spatial frequency components is developed. The blur measure defined as the correlation between quantization error and the image signal is combined with the conventional mean-square error criterion to define a new distortion measure, which is then utilized in the optimum bit allocation based on the rate-distortion theory. The new distortion measure suggests that signal components with small variances should not be omitted, as in conventional transform image coding, but should be retained with low signal-to-noise ratios.

Introduction

Transmission of pictorial data over a digital channel is a demanding task since the amount of data representing a given image is tremendously large. Therefore, search for efficient digital coding of pictorial information has been an active region of studies. Transform image coding[1] is considered to be one of the most efficient ways to utilize the statistical structure of image data for efficient coding. In transform coding, the image signal is first transformed by an orthogonal transform. The transform is chosen to reduce the correlation between adjacent picture elements or pixels. The decorrelated signal components are then individually quantized and assigned binary code words. At the receiving end of the channel, decoding and the inverse transform are required to reconstruct the image.

Signal energy in the transform domain tend to be packed into a small group of transform components. The rest of the signal components have relatively low energy and therefore can be omitted without a significant increase in the mean-square error. The disregarded components are naturally assigned only zero bits. This significantly contributes to transmission rate reduction. This is the fundamental mode of operation of statistical transform image coding. Transform image coding commonly employs mean-square error as the performance criterion. It it well known in literature, however, that the mean-square error does not honestly reflect subjective quality of the coded image. This has led many researchers to abandon statistical transform coding. Various adaptive transform coding techniques have been tested[2]. A major drawback of adaptive methods is that optimum design and a performance evaluation of a coding system are quite complicated because the designer must deal with a complex source model.

An alternative approach to more usable statistical transform image coding is discussed in

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this paper. We shall develop a new statistical performance measure that reflects subjective image quality better than the conventional mean-square distortion measure does. The new distortion measure will then be used in the optimum bit assignment for the transformed signal components. It will be shown that the components with small variances are not totally disregarded but quantized with low signal-to-noise ratios. This is in sharp contrast with the conventional transform coding technique where low energy components are completely disregarded.

In our earlier report[3)] it was argued that retaining low energy components improved image quality. The new distortion measure and subsequent optimum bit allocation proposed in this paper is a more systematic approach aiming at the same goal.

The mean-square error measure for transform image coding

Consider a signal source represented by a vector

$$x = (x_1 x_2 \cdots\cdots x_n)^t \tag{1}$$

where x_i are zero mean random variables with covariances

$$c_{ij} = E\{x_i \, x_j\} \qquad i, \ j = 1, 2, \cdots\cdots N$$

For $i=j$, $c_{ij}=c_{ii}$ are the variances of x_i. c_{ij}, $i \neq j$, represent statistical dependence between x_i and x_j. The correlation must be removed if it is required that each signal component should be individually quantized and coded. Otherwise coding efficiency would be unreasonably impaired.

The required decorrelation can be carried out by means of an appropriate orthogonal transform[4,5)]:

$$y \triangleq Tx \tag{2}$$

where T is the orthogonal matrix chosen for the decorrelation. The transform domain signal representation $y=(y_1 y_2 \cdots\cdots y_N)^t$ will be subsequently quantized and transmitted. Let $\hat{y} = (\hat{y}_1 \hat{y}_2 \cdots\cdots \hat{y}_N)^t$ be a quantized version of y. The quantizer error will be

$$y - \hat{y} \triangleq \xi \triangleq (\xi_1 \, \xi_2 \cdots\cdots \xi_N)^t$$

in vector form. The average mean-square error is given by

$$D \triangleq \frac{1}{N} < \xi, \ \xi > = \frac{1}{N} \sum_{i=1}^{N} E\{\xi_i^2\} = \frac{1}{N} \sum_{i=1}^{N} D_i \tag{3}$$

where $\quad D_i \triangleq E\{\xi_i^2\} \qquad i = 1, \ 2, \cdots\cdots\cdots\cdots, N$

and $\langle \cdot , \cdot \rangle$ denotes the inner product of the two vectors enclosed. If we take the inverse transform of $\hat{y}$, we obtain

$$\hat{x} = T^{-1} \hat{y} = T^t \hat{y} \tag{4}$$

as the signal domain representation of $\hat{y}$. This is an ppropriate representation of the signal vector reconstructed at the receiving end of the channel. From (3) and (4) it is easy to see that the average signal domain error is given by

$$D' \triangleq \frac{1}{N}\langle e, e\rangle = \frac{1}{N}\langle \xi, \xi\rangle = D \tag{5}$$

where

$$e \triangleq x - \hat{x}$$

Thus the average mean-square error is invariant under an orthogonal transform.

We have seen that the distortion defined by (3) is a convenient performance measure to be used with an orthogonal transform; we first transform the given signal vector into the orthogonal transform domain. Then we implement the required processing there. The signal vector containing the results will be inverse-transformed back into the signal domain with the invariant processing error(see (5)). The fact that the error is preserved is useful in the design of signal processors.

We shall now consider optimum quantization of the signal vector x, which can be efficiently carried out in the orthogonal transform domain; y defined by (2) is quantized in place of x. Assume that x_i, $i = 1, 2, \cdots\cdots, N$ are jointly Gaussian. Then y_i, $i=1, 2, \cdots\cdots, N$ are statistically independent Gaussian random variables. Let $R_i(D_i)$ be the number of required bits for quantizing y_i with the mean-squarre error D_i. For the Gaussian variables y_i, the bit rates are given by[6]

$$R_i(D_i) = \begin{cases} \frac{1}{2}\log_2 \frac{\sigma_i^2}{D_i}, & D_i < \sigma_i^2 \\ 0, \quad D_i \geqq \sigma_i^2 \end{cases} \tag{6}$$

$$i = 1, 2, \cdots\cdots, N$$

where

$$\sigma_i^2 = E\{y_i^2\}$$

What is meant by the above is that if a signal component has a variance greater than the allowed error D_i, it requires at least

$$\frac{1}{2}\log_2 \frac{\sigma_i^2}{D_i} \ [\text{bits}]$$

If the tolerated error D_i is greater than the signal variance σ_i^2, the component requires only zero bits. We simply discard it. At the receiving end it will be approximated by its expected value.

The actual mean-square error is given by

$$D_i = \sigma_i^2$$

The required number of bits averaged over all y_i, $i = 1, 2, \cdots\cdots, N$ is given by

$$R(D) = \frac{1}{N}\sum_{i=1}^{N} R_i(D_i) \tag{7}$$

where D is the mean-square error averaged over all y_i, as defined in (3):

$$D = \frac{1}{N} \sum_{i=1}^{N} D_i \tag{8}$$

Assume that we are given the average mean-square error D as in (8). We are faced with a question: How should we distribute D_i to y_i in order to minimize the average rate $R(D)$ assuming D is given. The question can be answered as follows. Since $R(D)$ is a function of $D_k < \sigma_k^2$, we set

$$\frac{\partial}{\partial D_k}\left[\frac{1}{N} \sum_{i=1}^{N} R_i(D_i) + s\left(D - \frac{1}{N} \sum_{i=1}^{N} D_i\right)\right] = 0 \qquad \text{for all } D_k < \sigma_k^2 \tag{9}$$

which reduce to

$$\frac{\partial}{\partial D_k} R_k(D_k) = s \qquad \text{for } D_k < \sigma_k^2 \tag{10}$$

From (6) and (10) we have

$$D_k = -\frac{1}{2s} \log_2 e \triangleq \theta \, (\text{constant}) \tag{11}$$

for $D_k < \sigma_k^2$, with θ a constant yet to be defined. The rest of D_k, i.e., $D_k \geqq \sigma_k^2$ are arbitrary. But actual D_k never exceed σ_k^2 as previously mentioned. Hence

$$D_k = \sigma_k^2 \tag{12}$$

for them. We concluded that D_k should be chosen so as to satisfy

$$D_k = \min\{\sigma_k^2, \theta\} \qquad k = 1, 2, \cdots\cdots, N \tag{13}$$

where the parameter θ will be determined so that D_k thus specified satisfy (8).

Equation (13) indicates that the signal components with large variances are quantized with an equal distortion θ and the rest are discarded. Under these conditions the average bit rate (7) is minimized. By the use of the mean-square distortion (8), the task of minimizing the required rate has been elegantly solved. The quantization rule (13) gives satisfactory results in coding experiments if D is set sufficiently low. But when the rate $R(D)$ is low, corresponding to large D, we encounter a serious problem. The parameter θ for such a low rate is large. Therefore many signal components with variances $\sigma_k^2 \leqq \theta$ are disregarded. A large class of images of practical interest can be modeled by a lowpass random process, the variances of which roll off at high spatial frequencies. Hence the omission of low variance signals causes the loss of high frequency signals and deteriorates image quality; the reconstructed image tends to be blurred. The edges of the original image will be most severely damaged. To improve the situation, it is obvious that we should somehow rescue the abandoned low variance terms. We develop in the next section a systematic way of doing this through modification of the distortion measure.

A statistical image blur measure and a modified distortion measure

In the previous section we have discussed the drawbacks of the quantization strategy based on the mean-square distortion. Although there is a circumstantial evidence that the loss of high frequency signals degrades the reconstructed image, we have not had a well defined model of the blurring. The word "frequency," for example, is commonly used in literature in connection with more general orthogonal transforms other than the Fourier transform. But the precise relation between image blurs and frequencies is not known. Indeed the quantization rule (13) has no explicit parameters associated with frequencies. The fact is that the frequency concept has no place in the description of the statistical structure of the signal source. The appropriate frequency domain can be determined only after the covariances are known.

If we look at the error images defined as differences between original images and blurred images, we notice that the error images have close resemblance to the originals; edges are seen in the error images. From this observation we define a statistical measure of image blurs as the correlation between quantization errors and original signals:

$$\text{Blur} = \langle e,\ x\rangle = E\{e'x\} \tag{14}$$

where e and x are vectors as defined previously. The inner product (14) reflects image blurs and is invariant through an orthogonal transform; let

$$\xi = Te \tag{15}$$

$$y = Tx \tag{16}$$

then

$$\langle e,\ x\rangle = \langle \xi,\ y\rangle \tag{17}$$

as we readily see from the orthogonality of the transform matrix T. Thus the blur measure (14) has the basic characteristic enjoyed by the conventional mean-square distortion measure.

Combining (5) and (14) we define a new distortion measure for transform image coding:

$$D = \frac{1}{N}[\langle e,\ e\rangle + \gamma\langle e,\ x\rangle] \tag{18}$$

or equivalently in the transform domain,

$$\begin{aligned} D &= \frac{1}{N}[\langle \xi,\ \xi\rangle + \gamma\langle \xi,\ y\rangle] \\ &= \frac{1}{N}[\sum_{i=1}^{N} E\{\xi_i^2\} + \gamma\sum_{i=1}^{N} E\{\xi_i y_i\}] \end{aligned} \tag{19}$$

where γ is a parameter controlling the contribution of the correlation part of the error measure.

As in (4) we denote

$$E\{\xi_i^2\} = D_i \tag{20}$$

It can be shown that the correlations can be written as

$$E\{\xi_i y_i\} = \sigma_i^2 g(D_i/\sigma_i^2) \qquad i = 1, 2, \cdots\cdots, N \tag{21}$$

where $g(x)$, $0 \leqq x \leqq 1$, is a monotone increasing function with

$$g(0) = 0, \quad g(1) = 1 \tag{22}$$

If $D_i = 0$ for some i, then

$$E\{\xi_i y_i\} = \sigma_i^2 g(0) = 0$$

while if $D_i = \sigma_i^2$, which is associated with the omission of y_i for some i,

$$E\{\xi_i y_i\} = \sigma_i^2 g(1) = \sigma_i^2$$

If we substitute (20) and (21) into (19), we obtain

$$D = \frac{1}{N}\sum_{i=1}^{N}\left[D_i + \gamma\sigma_i^2 g(D_i/\sigma_i^2)\right] = \frac{1}{N}\sum_{i=1}^{N} f_i(D_i) \tag{23}$$

where

$$f_i(D_i) = D_i + \gamma\sigma_i^2 g(D_i/\sigma_i^2) \qquad i = 1, 2, \cdots\cdots, N \tag{24}$$

It is known that a uniform quantizer closely approximates[7] the optimum quantizer if its output levels are entropy-coded. The output entropy of a properly designed uniform quantizer is only slightly greater than the theoretical minimum rate given in (6). Therefore we shall employ uniform quantizers in the subsequent discussions. Computational results show that $g(x)$ in (21) takes the form

$$g(x) \fallingdotseq x^n, \qquad n = 7 \tag{25}$$

Equation (24) reduces, with (25), to

$$f_i(D_i) = D_i + \gamma\sigma_i^2 (D_i / \sigma_i^2)^n \qquad i = 1, 2, \cdots\cdots, N \tag{26}$$

Repeating the steps that follow (8) with the new error measure (23), we can find optimum D_i, $i = 1, 2, \cdots\cdots, N$ that minimize the average transmission rate (7), assuming D in (23) is given. Omitting the details of the derivation we state the results; D_k should satisfy the following equations in d:

$$d \cdot f_k'(d) = \lambda \qquad \text{for all } D_k < \sigma_k^2 \tag{27}$$

where λ is a constant to be determined so that D_k satisfy (23), and $f_k(\cdot)$ are defined in (24). Denoting solutions of (27) by d_k, we have

$$D_k = d_k, \qquad \text{for } D_k < \sigma_i^2 \tag{28}$$

As in (13) we finally obtain

$$D_k = \min\{\sigma_i^2, d_k\} \qquad k = 1, 2, \cdots\cdots, N \tag{29}$$

Setting $\gamma=0$, we observe that (29) reduces to (13).

With $f_i(D_i)$ in (26), equation (27) yields

$$d+cd^n = \lambda \tag{30}$$

where

$$c = \gamma n\,(\sigma_k^2)^{1-n} \tag{31}$$

and n is asdefined in (25).

We shall find approximate solutions to (30):

Case 1. Small c.

Consider small c that satisfies

$$cd^n \ll d$$

or

$$d \ll c^{\frac{1}{1-n}} \tag{32}$$

Then (30) reduces to

$$d = \lambda$$

giving an obvious solution. Since d must satisfy (32), we require

$$d = \lambda \ll c^{\frac{1}{1-n}}$$

or

$$c \ll \lambda^{1-n} \tag{33}$$

In other words, by small c we mean such a c as in (33).

Case 2. Large c.

Consider a large c such that

$$d \ll cd^n \quad \text{or} \quad d \gg c^{\frac{1}{1-n}} \tag{34}$$

For this case (30) reduces to

$$cd^n = \lambda$$

giving an approximate solution to the equation:

$$d = (\lambda/c)^{\frac{1}{n}} \tag{35}$$

With (34) in mind we require

$$d = (\lambda/c)^{\frac{1}{n}} \gg c^{\frac{1}{1-n}}$$

or

$$c \gg \lambda^{1-n} \tag{36}$$

Thus we have approximate solutions to (30):

$$d_k = \begin{cases} \lambda, & c \ll \lambda^{1-n} \\ (\lambda / c)^{\frac{1}{n}}, & c \gg \lambda^{1-n} \end{cases} \tag{37}$$

The crss-over point of the above solutions is found by equating the two:

$$\lambda = (\lambda/c)^{\frac{1}{n}}$$

yielding

$$c = \lambda^{1-n} \tag{38}$$

We have thus found approximate slutins to (30) for c in the opposite extreme regions. We extend the both regions to the cross-over point defined by (38) and use the two solutions in the respective extended regions:

$$d_k = \begin{cases} \lambda, & c \leqq \lambda^{1-n} \\ (\lambda / c)^{\frac{1}{n}}, & c > \lambda^{1-n} \end{cases} \tag{39}$$

The above approximations are compared with more accurate, cmputatinal results in Fig. 1.

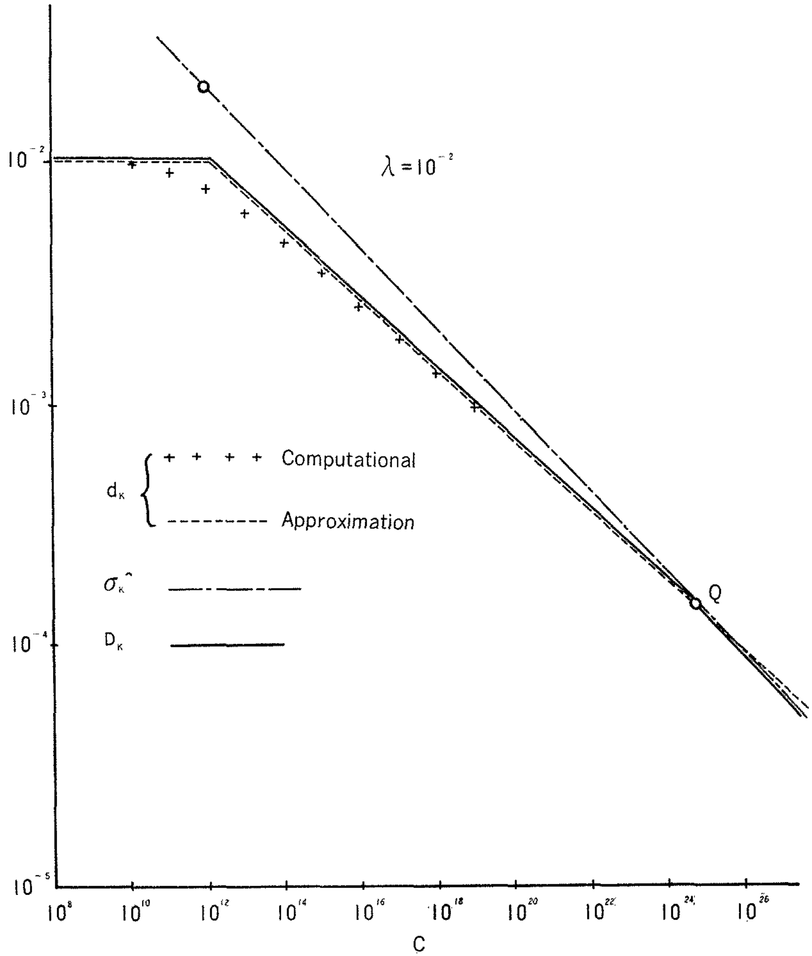

Fig. 1. Parametric representation d_k and σ_k^2

It is seen that the approximations are fairly good estimates of the solutins.

Equatin (31) defining c can be solved for σ_k^2

$$\sigma_k^2 = (c / n\gamma)^{\frac{1}{1-n}} \tag{40}$$

which is also shown in Fig. 1 as a function of c, with $\gamma=10$. γ around 10 seems to be optimum from coding experiments. We have obtained a parametric representation of the relation between d_k and σ_k^2. Recall that the optimum distortion assignment requires that D_k be chosen as in (29). D determined according to (29) is also illustrated in Fig. 1, where we are using the approximate curve for d_k. Signal components with variances smaller than the threshold indicated by Q are discarded and therefore $D_k=\sigma_k^2$ for them. The signal-to-noise ratios between P and Q are $2\sim1$.

If we replace c in (29) by (31) we have

$$d_k = \begin{cases} \lambda, & \sigma_k^2 \geqq \lambda \cdot \beta \\ [\lambda\,(\sigma_k^2)^{n-1} / \gamma n]^{\frac{1}{n}} & \sigma_k^2 < \lambda \cdot \beta \end{cases} \tag{41}$$

where

$$\beta = (\gamma n)^{\frac{1}{n-1}}, \quad \gamma = 10, \quad n = 7 \tag{42}$$

With (29) and (41) the proposed quantization strategy is now complete. If we let $\gamma \to 0$, equation (41) reduces to

$$d_k = \lambda \tag{43}$$

and (29) agrees with (13).

(a) (b)

(**a**) Original. (**b**) 0.74 bits/pixel.

Fig. 2. A coded image.

Signal components with large variances are treated similarly under the new and conventional distortion measures. The major difference resides in the treatment of low variance components, which are quantized with varying distortions in the proposed case; the distortion is made smaller as the variance decrases (see (41)), while low variance terms are completely disregarded in the conventional quantization scheme. Coding experiments are under way using the new quantization scheme. A sample coded image is shown in Fig. 2. We observe that the reconstruction has promising image quality at a remarkably low bit rate.

Concluding remarks

We have discussed the fundamental problem of the mean-square distortion measure, as applied to transform image coding; low energy signal components having high spatial frequencies tend to be discarded, contributing to blurs of the coded image. A statistical method to evaluate image blurs has been proposed. The blur measure, defined as the correlation between quantizer noise and signals, was combined with the conventional mean-square error to produce a generalized performance measure. It was then used to generate the optimum bit allocation for a given distortion.

In the new quantization strategy, low variance components, associated with high spatial frequencies, are retained with varying mean-square errors depending on the variances. The retained high frequency terms contribute to suppressing image blurs common to transform coding based on the mean-square error. A sample of coded images has been shown to illustrate the effectiveness of the proposed method. Complete evaluation of the new performance measure and subsequent bit allocation will be reported in a future paper.

Acknowledgement

This work was supported in part by a research grant from the Ministry ofEducation, No. 56550220.

References

1) P. A. Wintz, "Transform image coding," Proc. IEEE, 60, pp. 809～820, July 1972
2) A. Habibi, "Survey of adaptive image coding techniques," IEEE Trans. Commun., COM-25, pp. 1275～1289, November 1977
3) H.Kitajima, T. Shimono and T. Kurobe, "Hadamard transform image coding," Bulletin of the Faculty of Eng., Hokkaido Univ., No. 101, pp. 39～50, December 1980
4) N. Ahmend, T. Natarajan and K. R. Rao, "Discrete cosine transform," IEEE Trans. Comput., C-23, pp. 90～93, January 1974
5) H. Kitajima, "Symmetric cosine transform," IEEE Trans. Comput., C-29, pp. 317～323, April 1980
6) T. Berger, Rate distortion theory, Prentice Hall, 1971
7) T. J. Goblitz and J. L. Holsinger, "Analog source digitization: comparison of theory and practice," IEEE Trans. Inform. Theory, IT-13, pp. 323～326, April 1967.