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# Frege, his Logic and his Philosophy*
# Interview with Michael Beaney

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**Abstract:** The interview begins with an outline of Gottlob Frege's life, academic career and the reception of his ideas by later philosophers such as Bertrand Russell and Ludwig Wittgenstein. Frege's main contributions to logic and philosophy are summarized, and the key ideas of his three main books —— *Begriffsschrift*, *Die Grundlagen der Arithmetik* and *Grundgesetze der Arithmetik* —— are explained. Particular attention is paid to Frege's fundamental claim that "a statement of number contains an assertion about a concept" and to the 'Cantor-Hume Principle', which play a central role in his logicist project —— the attempt to show that arithmetic can be reduced to logic. Also discussed are three of Frege's important essays, which elucidate his view of concepts as functions and his distinctions between concept and object and between '*Sinn*' and '*Bedeutung*'. Here particular attention is paid to problems concerning his notion of an extension of a concept (or more generally, his notion of a value-range of a function), the translation of '*Bedeutung*', and the application of the distinction between *Sinn* and *Bedeutung* to different kinds of linguistic expressions. Recent developments of Frege's ideas by, for example, 'neo-logicists' are mentioned, and there is also discussion of Frege's conception of 'thoughts'. The interview ends with some remarks on certain influences on Frege, such as from neo-Kantianism, and some suggestions for further reading.

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## 1. Introduction: Frege's life, academic career and influence

**CB**[1]**:** Prof. Michael Beaney, I'm very glad to have the opportunity to interview you. You are a well-known Frege scholar. You edited *The Frege Reader* (1997) and co-edited *Gottlob Frege: Critical Assessments of Leading Philosophers* (2005), which are very helpful for teaching

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and research on Frege. I once used *The Frege Reader* as a textbook for my graduate class. Since there are few biographical accounts of Frege, Chinese readers are not so clear about Frege as a person and also as a logician and philosopher. Frege may be slightly more familiar to Japanese readers, but in Japan, too, he is still relatively unknown. Is that not right, Prof. Nakatogawa?

**KN[2]:** Yes, that is true. Frege's writings have indeed been translated into Japanese, but discussion of Frege's philosophy is still relatively rare.

**CB:** In the light of this, then, Prof. Beaney, could you first sketch the contours of Frege's life, especially his character and his academic career?

**MB[3]:** Prof. Chen, I'm glad to know that the books I have edited have been useful, and I'm very happy to talk to both you and Prof. Nakatogawa about Frege, whose work, in my opinion, lies at the basis of what we now call 'analytic philosophy', the dominant tradition in philosophy in the English-speaking world.

Gottlob Frege was born in Wismar, a town on the Baltic coast in northern Germany, on 8 November 1848. His parents were teachers, and he was the first of two sons. He was baptised into the Lutheran church, and remained a Lutheran for the rest of his life, although I am not sure how deeply religious he was, and I do not think, in any case, that it had any significant influence on his logical work. Little is known about his younger brother (born in 1852), but we know a bit more about his parents. His father (born in 1809) was principal of a private girls' school, and his mother (born in 1815) was a teacher there. His father died from typhus in 1866, however, and his mother then took over as principal. She was clearly very close to Frege. When she retired from her job in 1876, she left Wismar two years later to live with Frege, and remained with him until she died in 1898. It was with her money that Frege was able to buy a house in 1887, as he never earned enough as a professor. Frege attended the grammar school (*Gymnasium*) in Wismar from 1864 to 1869, where he seems to have had an average education.

In 1869 he entered the University of Jena, in east Germany, and took courses in mathematics, physics, chemistry and philosophy (with Kuno Fischer on Kant), before transferring to the University of Göttingen, at that time one of the leading centres of mathematics, where he took further courses in mathematics, physics and philosophy (this time with Hermann Lotze on philosophy of religion). In 1873 he was awarded his doctorate with a dissertation entitled 'On a Geometrical Representation of Imaginary Forms in the Plane'. He immediately wrote his *Habilitationsschrift*, on 'Methods of Calculation based on an Extension of the Concept of Magnitude'. This was required to gain a university lecturing post, and Frege completed it in just a few months — an unusually short time. On the recommendation of Ernst Abbe, his mentor at Jena, he was then appointed to teach analytic geometry and the theory of functions in the Department of Mathematics at Jena, covering for Carl Snell, who had become ill. Snell's illness was the reason why Frege had been encouraged to do his *Habilitationsschrift* as quickly as he could. Frege was to remain at Jena until he retired in 1917. He was promoted to *ausserordentlicher Professor* (roughly equivalent to Associate Professor, although with only a small salary) in

1879, and to *ordentlicher Honorarprofessor* (Full Professor, although only an honorary post) in 1896. Frege never attained a properly salaried university professorship, and was reliant on money given to him by the Carl Zeiss Foundation, which Abbe had set up in 1889. Abbe had worked closely with Zeiss in the optics industry in Jena, and it was the success of this industry that enabled Frege to pursue his research.

The three books that Frege published in his lifetime were *Begriffsschrift* (*Conceptual Notation*) in 1879, *Die Grundlagen der Arithmetik* (*The Foundations of Arithmetic*) in 1884, and *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*), the first volume of which appeared in 1893 and the second volume in 1903. Frege's main aim in these books was to demonstrate the logicist thesis that arithmetic is reducible to logic. In *Begriffsschrift* he gave his first exposition of the logical system by means of which arithmetic was to be reduced. In the *Grundlagen* he offered an informal account of his logicist project, criticizing other views about arithmetic, such as those of Kant and Mill. In the *Grundgesetze* he refined his logical system and attempted to demonstrate formally his logicist thesis. In 1902, however, as the second volume was going to press, he received a letter from Bertrand Russell informing him of a contradiction in his system — the contradiction we know now as Russell's paradox. Although Frege hastily wrote an appendix attempting to respond to the paradox, he soon realized that the response didn't work, and was led to abandon his logicist project. He continued to develop his philosophical ideas, however, and to correspond with other mathematicians and philosophers, and published a number of influential papers.

There is no doubt that Russell's paradox dealt a terrible blow to Frege, and this came at a difficult time in his personal life. Frege had married Margarete Lieseberg (born 1856) from Wismar in 1887, when they moved into a newly built house at 29 Forstweg in Jena with Frege's mother. But Frege's wife clearly had some illness, about which we know nothing, and this made it difficult to care for Frege's mother in the last few years of her life. She moved to a nursing home in 1896, and died two years later. But then, in 1904, Margarete died, too, and Frege was left with just his housekeeper Meta Arndt (born in 1879) for company. In 1908, however, he adopted a son, Alfred (born in 1903), who became his heir in 1921. Frege and Margarete had had no children themselves, but it seems that Frege liked children (and dogs), and was apparently a good father to Alfred.

There is no doubt, too, that Frege's ideas were not understood, let alone accepted, when they were first formulated, and this must have been very frustrating, especially as Frege writes with a clarity second to none in German-language philosophy, in my opinion. From the 1890s onwards his writings increasingly show signs of bitterness as he subjected the views of his contemporaries, including those of some of his colleagues at Jena, to devastating criticism. Rudolf Carnap, who attended some of Frege's lectures between 1910 and 1914, reports that Frege looked old beyond his years, was extremely shy and introverted, and rarely turned to the audience when he lectured. His ideas nevertheless made a deep impression on Carnap, as they did on Ludwig Wittgenstein, who met Frege on three occasions to discuss philosophy between 1911 and 1913. It was Frege who recommended that Wittgenstein study with Russell, and despite his criticisms of Frege's ideas, Wittgenstein held Frege in the highest regard throughout his life. Frege may have had few students, but two of them became two of the greatest philosophers of the twentieth century. If he had lived longer, I'm sure he would have died happier.

Frege's professional disillusionment and the sadness in his personal life gradually took a toll on his health, and in the last few years of his career he was able to do less and less teaching. When he finally retired in 1917, he moved back to his homeland on the Baltic Coast, able to buy a house at a time of economic collapse with a gift of money from Wittgenstein. He published the three essays that compose his 'Logical Investigations', but he also kept a diary in which he expresses some unpleasant right-wing and anti-Semitic views. Such views were not uncommon in Germany in the aftermath of the First World War, but I nevertheless find it very sad that someone of Frege's extraordinary intellectual calibre should have sunk to this in the twilight of his life. He died on 26 July at the age of 77.

**CB:** As you say, Frege's ideas were not appreciated at the time he wrote, even by his closest colleagues, and he was a relatively unknown figure outside Jena. But things changed dramatically in the twentieth century. Frege becomes regarded as the founder of modern logic, that is, mathematical logic, and also as the father of analytic philosophy. Prof. Beaney, I'd like to know, to what or to whom do you attribute this change? Husserl? Russell? Wittgenstein? Carnap? Dummett? Or someone or something else?

**MB:** Yes, you're right that the transition in our appreciation of Frege in the twentieth century has been dramatic. Edmund Husserl knew Frege's work, which arguably convinced Husserl of the error of his early psychologism, and they corresponded in 1891 and again in 1906. But I don't think that many people were drawn to Frege through Husserl — although Martin Heidegger discusses Frege in some of his writings. Russell claimed that he was responsible for drawing attention to Frege's work in 1903. It is true that Appendix A of Russell's *Principles of Mathematics* contains the first substantial discussion of Frege's ideas in English. But it was the Italian mathematician Giuseppe Peano who had directed Russell to Frege, and there were other German logicians and philosophers who had also discussed Frege's ideas earlier, such as Ernst Schröder and Benno Kerry. Carnap was profoundly influenced by Frege, but he admits himself that this only happened when he read his works more carefully after the First World War rather than at the time he attended Frege's lectures. I have already spoken about Wittgenstein's regard for Frege, whom he thanks for his 'great works' in the preface to the *Tractatus Logico-Philosophicus*, and there is no doubt that as people sought to understand Wittgenstein's philosophy, especially the ideas of the *Tractatus*, they were inevitably drawn to Frege. Indeed, I can say that this was exactly how I became interested in Frege — in trying to understand Wittgenstein as a student in Oxford in the 1980s. As far as Michael Dummett is concerned, his pioneering book, *Frege: Philosophy of Language*, published in 1973, was hugely significant in the blossoming of interest in Frege that occurred from the 1970s. But one should mention, too, the translation of Frege's *Grundlagen* that J. L. Austin published in 1950, and the selection of Frege's writings that was translated by Peter Geach and Max Black and published in 1951. Wittgenstein advised Geach and Black on their selection and even lent them his own copies of some of Frege's works. There is also the publication of Frege's *Nachgelassene Schriften* in 1969 (translated into English as *Posthumous Writings* in 1979) and his *Wissenschaftlicher Briefwechsel* in 1976 (translated into

English as *Philosophical and Mathematical Correspondence* in 1980) to note, indicating recognition of Frege's importance in Germany before Dummett's influence in the English-speaking world. So I think there is a complex story to tell about the gradual appreciation of Frege's philosophy, a story that itself sheds interesting light on Frege's ideas.

**KN:** Would you say that there was anything in Frege's character, behaviour or ideas themselves that contributed to the lack of recognition in his lifetime and to his growing influence after his death?

**MB:** Well, that's a good question. As far as his character is concerned, we might compare Frege with Russell, who in many ways was his exact opposite, despite the fact that they were both, for much of their careers, concerned to demonstrate logicism. Born an aristocrat, Russell strode the world stage throughout his working life with supreme confidence. Russell had extraordinary energy and wrote on a huge range of topics, from mathematical logic and metaphysics to marriage and sex. I'm sure that if Frege had been more outgoing and well-connected, he would have had greater influence in his lifetime. He rarely attended conferences, for example, or gave talks outside Jena. While Russell pursued love affairs, Frege walked his dog. There is also the nature of Frege's logical notation itself — his 'Begriffsschrift'. This never caught on among logicians, and is harder to learn and write than modern notation. Indeed, it may amuse you to hear that Frege was criticized by one of his first reviewers (Schröder) for indulging "in the Japanese practice of writing vertically". The two-dimensional nature of his notation certainly hindered people from appreciating Frege's quantificational logic, and it was only when this logic was developed by Peano and Russell that it took on the form with which we are familiar today. The failure of Frege's own logicism may also have led philosophers to think that there was little of value in Frege's work - nothing that hadn't been improved on by Russell and Wittgenstein. It was only when philosophers started investigating Russell's and Wittgenstein's philosophies in proper detail that they realized the importance of Frege's ideas and were led to explore those ideas in their own right, which opened up alternative possibilities of development and understanding than those pursued by Russell and Wittgenstein.

## 2. Frege's main contributions to logic and philosophy

**KN:** How would you summarize Frege's main contributions to logic and philosophy?

**MB:** Frege is rightly regarded as the founder of modern logic, giving the first exposition of quantificational logic in his *Begriffsschrift* of 1879. He also sought to justify that logic by clarifying its use of function-argument analysis — based on an ontological distinction between function and object, "founded deep in the nature of things", as he put it in his essay 'Function and Concept'. Quantificational logic was far more powerful than traditional logic, and with this new logic at his disposal, Frege was able to formalize arithmetical propositions in a way that hadn't been possible before. For the first time in history, logicism thus became a feasible project and in pursuing this project, Frege also made an important contribution to the philosophy of mathematics. Furthermore, in thinking through the philosophical implications of both his logic and his logicism, Frege was led to draw a number of further distinctions, such as that between *Sinn* and *Bedeutung*, which have been enormously influential in the development of modern philosophy of language and mind.

**CB:** As you have said, Frege is a logicist: he tries to reduce mathematics to logic. I think many readers would like to know exactly what his logicism is. And what does Frege mean by 'logic'? What similarities and differences are there between Frege's, Kant's and contemporary conceptions of logic? Prof. Beaney, could you clarify these questions for us?

**MB:** Well, the first thing to note is that Frege is a logicist about arithmetic, not geometry. That is, he thinks that arithmetic can be reduced to logic, but he does not — unlike Russell — think that geometry can also be reduced to logic (for example, via analytic geometry). He thinks geometrical truths are spatial truths, which we recognize through 'intuition' ('*Anschauung*', as Kant called it), whereas arithmetical truths are conceptual truths, which we recognize by 'reason' ('*Vernunft*'). In Kant's terminology, Frege held that geometrical truths are synthetic a priori, while arithmetical (and logical) truths are analytic a priori. Unlike Kant, however, Frege did not think that analyticity implied triviality. According to Kant, logical truths are analytic and thus trivial, but in claiming that arithmetic could be reduced to logic, Frege did not conclude that arithmetical truths are therefore trivial. On the contrary, he was concerned to show how logical truths can advance our knowledge.

Frege is often regarded as having had a 'universalist' conception of logic, according to which logical laws are universal truths that are applicable to everything, in other words, to whatever can be conceived. He did not hold a 'schematic' conception, according to which logic is the study of logical forms or schemata. Nor did he distinguish between logic and metalogic, though the extent to which he nevertheless (implicitly) pursued metatheoretical investigations is hotly debated among Frege scholars. One aspect of Frege's universalist conception is his doctrine that every concept must be defined for all objects. He regarded vague concepts as outside the domain of logic and therefore deficient. Logicians today tend to disagree: they might think, for example, that vague concepts and the problems they cause — such as the sorites paradox — just show the need to develop a different logic, such as intuitionistic logic or fuzzy logic, to deal with them.

## 3. Frege's *Begriffsschrift*

**CB:** Let us now talk about Frege's first book *Begriffsschrift* (1879). What did Frege achieve in this book? Could you summarize this achievement for us?

**MB:** In his *Begriffsschrift* Frege gave the first exposition in history of predicate logic, introducing a notation for quantification, and also offered an axiomatization of propositional logic. He showed how mathematical propositions could be formalized and succeeded in giving a purely logical analysis of mathematical induction. He later developed his logical system further, but what he achieved in this short book was truly remarkable, although it took a long time for logicians to appreciate it.

**KN:** What differences are there between Frege's logical system and traditional Aristotelian logic?

**MB:** Aristotelian logic had had great difficulty in analysing statements of multiple generality. In inventing quantificational logic, Frege showed how these could be easily formalized. He also showed how the two traditional parts of logic, syllogistic theory and propositional logic, could be integrated into one comprehensive system, a system that is far more powerful than anything remotely dreamt of in Aristotelian philosophy.

**KN:** What contribution did the *Begriffsschrift* make to Frege's logicist project? What remained to be done?

**MB:** The *Begriffsschrift* provided the basic logical system by means of which arithmetical propositions could be formalized and proved, and his logical analysis of mathematical induction — through his logical definition of a one-one relation — was also important. What remained to be done was to provide logical definitions of *all* arithmetical concepts, including number terms themselves, which is the task that Frege undertook in his second book, *Die Grundlagen der Arithmetik* (1884).

## 4. Frege's *Grundlagen*

**CB:** Very well, so let us now turn to that book, which many people regard as Frege's masterpiece. Dummett claims that the *Grundlagen* may justly be called "the first work of analytical philosophy". Prof. Beaney, do you agree with this, and could you outline the main achievements of this book?

**MB:** I do agree that the *Grundlagen* is a masterpiece and can indeed be regarded as the first work of analytic philosophy. In the first half of the book Frege provides some powerful objections to earlier accounts of arithmetic, such as Kant's theory of arithmetic as synthetic a priori and Mill's empiricist view. In the second half he sketches his own account, defining numbers as extensions of concepts, and drawing on his *Begriffsschrift*, defining the successor relation. In effect, he shows how to derive the Dedekind-Peano axioms, which we now take as defining the natural number sequence.

**CB:** Frege lays down three principles to adhere to in the introduction to the *Grundlagen*. The first is the anti-psychologism principle: "There must be a sharp separation of the psychological from the logical, the subjective from the objective". My questions are the following. What exactly is the psychologism to which Frege objects? What are his main arguments against psychologism? How do you evaluate his anti-psychologism and its influence?

**MB:** The psychologism to which Frege objects is essentially the view that logical laws are psychological laws of thought, understood as descriptions of the way we actually think. For Frege, this does not do justice to the *normative* character of logic. Logic tells us how we *ought to* think, not how we actually think. Frege says much more about all this in the preface to the *Grundgesetze*. But his main argument is straightforward. Just as we don't say that something is morally right just because the majority of people believe it is right, or act as if it is right, so we don't say that an argument is logically valid just because the majority of people think it is valid, or reason as if it is valid. On this matter, in my view, Frege is absolutely right. Of course, if he is, then I can feel happy saying this even if everyone else disagrees! As it happens, many philosophers do agree, although the issue of whether normativity can be explained naturalistically is fiercely contested at the moment.

**CB:** The second of Frege's three principles is the context principle: "The meaning of a word must be asked for in the context of a proposition, not in isolation". What role does this principle play in Frege's philosophy of mathematics, and what is its relation (if any) to the anti-psychologism principle?

**MB:** In the main argument of the *Grundlagen*, Frege appeals to the context principle in explaining how it is possible to apprehend numbers, given that they are abstract objects, that is, objects that are not located in space and time. Frege's answer is ingenious: we apprehend numbers by grasping the meanings of number terms, which we do by understanding the senses of sentences in which number terms appear. This gives us a clue as to the relation between the context principle and the anti-psychologism principle. For if we thought that grasping the meaning of a term were a matter of having an appropriate 'idea', understood psychologically, then we might have difficulty in finding ideas corresponding to numbers. But according to Frege, to grasp the meaning of a number term, it is enough to understand the sense of a sentence in which the number term appears, and this sense is to be explained logically, not psychologically.

**CB:** Prof. Beaney, we will come to Frege's third principle in a minute. But could you first clarify Frege's claim that "a statement of number contains an assertion about a concept"? This claim is very important in Frege's philosophy, is it not? Could you explain its significance?

**MB:** Yes, you are quite right about the importance of this claim: it lies at the very heart of Frege's account. Let us take one of Frege's own examples, the proposition that Jupiter has four moons. We might be tempted to interpret this proposition as predicating of Jupiter the property of having four moons, but we might then find it hard to analyse the property of having four moons. According to Frege, however, the proposition is to be understood as saying something not about an object but about a concept: it does not predicate of the object Jupiter the property of having four moons; instead, it predicates of the concept *moon of Jupiter* the property *has four instances*. The property *has four instances* is a second-level property, that is, a property that holds of a first-level property (in this case the property of being a moon of Jupiter), and the key thing about the property *has four instances* is that it can be defined purely logically.

In teaching and writing about Frege, I often explain the significance of this analysis by taking the example of a negative existential statement, such as 'Unicorns do not exist'. Here again we might be tempted to construe this as saying something about objects — attributing the property of non-existence to unicorns. But what, then, *are* these unicorns? Must they not somehow exist — or 'subsist' in some non-actual world — in order for there to be a subject of our proposition? On the Fregean account, however, to deny that something exists is to say that the relevant *concept* has no instances: there is no need to posit any mysterious *object*. To say that unicorns do not exist is to say that the concept *unicorn* is not instantiated, which can be easily formalized in predicate logic as '$\neg (\exists x)\ Fx$', where '$Fx$' represents '$x$ is a unicorn'.

Similarly, to say that God exists is to say that the concept *God* is instantiated, i. e., to deny that the concept has no (i. e. zero) instances. (If we wanted to say that there is one and only one God, we would have to deny as well that the concept has two or more instances.) On this view, existence is no longer seen as a first-level property, but instead, existential statements are analysed in terms of the second-level property *is instantiated*, represented by means of the existential quantifier. As Frege notes in section 53 of the *Grundlagen*, this offers a neat diagnosis of what is wrong with the traditional ontological argument. Frege's analysis of number and existential statements is thus a wonderful example of the power of logical analysis.

## 5. The Cantor-Hume Principle and the Julius Caesar problem

**CB:** Dummett has written that section 62 of the *Grundlagen* "is arguably the most pregnant philosophical paragraph ever written". It introduces what has been called 'Hume's Principle', about which there has been a lot of debate in recent years. It connects, too, with the so-called 'Julius Caesar problem'. So my questions are these. What exactly is Hume's Principle? What role does it play in Frege's logicist project? Why is it controversial? How is it related to the Julius Caesar problem, and what was Frege's solution to this problem? What is the latest thinking about these matters? Is Hume's Principle seen as analytic, for example? Prof. Beaney, could you shed some light on these questions for us?

**MB:** Well, Prof. Chen, we are indeed now at the very heart of Frege's philosophy, and you have asked some key questions about which there has indeed been much debate. Again, let me agree with Dummett about the importance of section 62 of the *Grundlagen*, although Dummett has also suggested that it is in section 62 that the linguistic turn first took place in philosophy. I think this is an exaggeration, for reasons that might already be clear from what I have just said. For Frege's analysis of number statements also exemplifies what people have in mind in talking of the linguistic turn — the idea, that is, that philosophical problems can be solved by transforming one proposition into another, by showing what a misleading proposition 'really means' by paraphrasing it. Number statements are 'really' about concepts, not objects, according to Frege. But as exaggerations go, Dummett's suggestion serves a useful purpose: it highlights another important example of what I myself have called 'interpretive' or 'transformative' analysis. Before answering your questions, I should also note that I prefer to talk of the 'Cantor-Hume Principle', since this does better justice to Cantor's role in the story. Hume was only thinking of the finite case, and it was Cantor who first used it as an explicit principle.

The Cantor-Hume Principle asserts the equivalence between the following two propositions, which I shall label (Na) and (Nb):

(Na) The concept $F$ is equinumerous to the concept $G$.

(Nb) The number of $F$s is identical with the number of $G$s.

Let us start with (Na). To say that two concepts $F$ and $G$ are equinumerous (*gleichzahlig*) is to say that the objects that fall under the concept $F$ can be one-one correlated with the objects that fall under the concept $G$, in other words, that there are as many $F$s as $G$s. And this is just to say the number of $F$s is the same as the number of $G$s, which is what (Nb) says. So (Na) and (Nb) are indeed equivalent.

The role of the Cantor-Hume Principle can now be explained as follows. Recall that, on Frege's view, we apprehend numbers by grasping the meaning of number terms — terms such as 'the number of $F$s'. We do this — according to the context principle - by grasping the sense of sentences in which the number terms appear — sentences such as (Nb). But what is the sense of (Nb)? By the Cantor-Hume Principle, (Nb) is equivalent to (Na). So we grasp the sense of (Nb) by grasping the sense of (Na) and accepting the principle. Furthermore, the crucial point about (Na) is that it can be defined purely logically, since one-one correlation can be defined logically. So if we put the context principle and the Cantor-Hume Principle together, then it looks as if our knowledge of logic is enough to explain our apprehension of numbers.

This seems too good to be true, and in the *Grundlagen*, Frege himself immediately goes on to raise an objection. This is the Julius Caesar problem, which, in essence, can be stated as follows. While the Cantor-Hume Principle allows us to tell when two numbers are the same or not as long as they are given to us as numbers (they are the same if the relevant concepts are equinumerous), it does not tell us whether an object given to us as a number is the same as an object not given to us as a number, for example, Julius Caesar. For all we know — or at any rate, if all we know is the Cantor-Hume Principle — the number 7 might actually be Julius Caesar — or Mao Tse-tung. The Principle does not, in other words, lay down a sufficient criterion of identity for numbers.

It is for this reason that Frege goes on to define numbers, not implicitly, via the context principle, but explicitly, by identifying them with appropriate extensions of concepts — extensions of concepts, that is, that can be defined purely logically. The number 0, for example, is defined in terms of the concept *not identical with itself*, which can be represented logically as '$x \neq x$'. Since nothing is not identical with itself, the number of things that fall under this concept is the number 0. In fact, the number 0 is the number of things that fall under *any* concept that is 'equinumerous' (to use Frege's term) to the concept *not identical with itself*; and this leads to the explicit definition of the number 0 as the extension of the concept 'equinumerous to the concept *not identical with itself*'. (I explain the details of Frege's account in *The Frege Reader*, pp. 116–20, to which I can refer anyone who would like to know more.) Having provided his explicit definitions, Frege can then *derive* the Cantor-Hume Principle, which can in turn be used to derive the Dedekind-Peano axioms.

The Cantor-Hume Principle is an example of what is now widely called an 'abstraction principle', which seeks to define abstract objects of a certain kind (such as numbers) in terms of some equivalence relation holding between objects of some other (more basic) kind. Another example which Frege himself gives is defining directions in terms of parallelism. Compare what I stated as (Na) and (Nb) with the following:

(Da) Line *a* is parallel to line *b*.

(Db) The direction of line *a* is identical with the direction of line *b*.

If two lines are parallel, then their directions are the same, and vice versa. So (Da) and (Db) are equivalent, and we can define directions in an analogous way to defining numbers.

There has been a great deal of debate about abstraction principles over the last few years. Which ones are good and which ones are bad? The Cantor-Hume Principle seems to be a good one. If so, then why cannot we simply take it as an axiom? We would have to answer the Julius Caesar problem, but what if we just took the Principle as constitutive of our knowledge of numbers? Would this be treating it as in effect 'analytic'? Can we regard it as logical? If so, then should we agree that Frege was essentially right in his logicism? These are some of the questions that philosophers of mathematics today are vigorously debating.[4)]

**CB:** Prof. Beaney, all this is very interesting. I am not a specialist in philosophy of mathematics myself, but it seems to me that you are saying that Frege's philosophy of mathematics is still very much alive. At the end of the *Grundlagen*, Frege concludes that "from all that has gone before, the analytic and *a priori* nature of arithmetical truths has thus emerged as highly probable". Do you think Frege was right here, bearing in mind the attack that Quine later made on analyticity? Is this connected with what is now called 'Frege's theorem'? Could you briefly explain that theorem to us?

**MB:** Certainly, I think that there is much of value in Frege's philosophy. But let me say one thing about analyticity. Interestingly, after the *Grundlagen*, Frege himself never talks again of the analyticity of arithmetic; what he claims is that arithmetic can be 'reduced' to logic. My suspicion is that he soon came to have doubts about the notion of analyticity, long before Quine. So the key issue, in my view, is whether arithmetic can be reduced to logic, and that raises the question about the nature of logic, about which I have already said there has been much recent debate. Frege's logic is a system of second-order logic, that is, as allowing quantification over functions as well as objects. But Quine, for example, notoriously denied that second-order logic was properly logic. I myself think it is, but the important questions concern what you can do with what kind of logic, not what is the 'correct' logic in any absolute sense. So this does indeed relate to what is now called 'Frege's theorem', which states that the Dedekind-Peano axioms can be deduced, within second-order logic, from the Cantor-Hume Principle. This is what Frege showed in the *Grundlagen*. So if we agree that second-order logic is logic, and also (more controversially) that the Cantor-Hume Principle is a logical principle, then we do have a version of logicism. So logicism is indeed a live issue.

## 6. Functions, concepts and objects

**CB:** Now let us discuss Frege's important paper 'Function and Concept' (1891). You have recently published a paper, 'Frege's use of function-argument analysis and his introduction of truth-values as objects' (2007), in which you discuss this paper. So could you briefly explain Frege's key ideas here — and in particular, his claim that "a concept is a function whose value is always a truth-value"?

**MB:** Okay, but let me briefly say something, first, about the period between 1884, when the *Grundlagen* was published, and 1893, when the first volume of *Grundgesetze* appeared. Frege refined his logical system in this period, so that he could actually prove formally what he had only sketched informally in the *Grundlagen*. In particular, he introduces a notation for extensions of concepts, in terms of which, as we have seen, he ends up defining numbers explicitly. He also developed and clarified some of the underlying philosophical ideas. This clarification is what he provides in three important papers published in 1891–92, 'Function and Concept', 'On *Sinn* and *Bedeutung*' and 'On Concept and Object'.

In 'Function and Concept', he explains how he has extended the notion of a function from mathematics into logic, by construing concepts, in particular, as functions. In the *Begriffschrift* he had thought of concepts as functions that map objects onto what he called 'conceptual contents'. So take the simple proposition 'Chen Bo is a philosopher'. Chen Bo is the object, if you don't mind my calling you an object, and the concept *philosopher* is what the proposition says applies to you. Frege thinks of the concept expression as what is yielded when we omit a name, such as 'Chen Bo', from the sentence, yielding 'is a philosopher', or more accurately, '( ) is a philosopher', since Frege wants to make clear the 'gap' which the name of an object must fill to complete a sentence. This sentence represents a 'conceptual content' — in this case, the 'circumstance', as we might describe it, of your being a philosopher — a circumstance with which I'm sure you're very happy.

For reasons that I explain in my paper, however, Frege comes to reject his earlier notion of content, and reconsiders what the value is of a concept, understood as a function. In 'Function and Concept' he argues that it must be taken as a truth-value. So this is how we arrive at the claim that a concept is a function whose value is always a truth-value, which Frege understands as itself an object. Now I think that Frege's argument for this is problematic, and many people have criticized this view, from Russell and Wittgenstein onwards. But it did allow Frege to simplify his logical system: truth-values could be treated as objects just like any other object, to which concepts and other functions could be applied. Just as numerical functions are mappings from numbers to other numbers, so concepts are mappings from objects to other objects, although in this case, there are only two objects which are the value of concepts, namely, 'the True' and 'the False', as Frege calls them.

**CB:** Perhaps this is then the point at which to ask about one of Frege's other papers from this period, 'On Concept and Object' (1892), and bring in, too, the third of the three principles that Frege lays down in the *Grundlagen*, which we haven't yet discussed. This is the principle that "The distinction between concept and object must be kept in mind". Can you explain this distinction for us? You have already talked of 'extensions of concepts', and one also finds in Frege's work talk of 'value-ranges'? So what are these? Are they objects or concepts, or neither? What kinds of entity are 'objects' and 'concepts'?

**MB:** Well, we have seen how Frege analyses a simple sentence such as 'Chen Bo is a philosopher' into the name 'Chen Bo' and the concept expression '( ) is a philosopher'. The name stands for an object, and the concept expression stands for a concept. This analysis into function and argument, with concepts regarded as a type of function, taking objects as arguments (to yield truth-values as values), is absolutely fundamental for Frege. As I remarked earlier, he saw this distinction between concept and object — or more generally, between function and object — as "founded deep in the nature of things". He characterized objects as 'saturated' entities, and concepts — and other functions — as 'unsaturated' entities, the unsaturatedness reflecting the 'gap' in concept expressions of which I have just spoken. The distinction between function and object is thus both exclusive and exhaustive. All entities are either saturated or unsaturated. If saturated, they are objects, and cannot be functions; if unsaturated, they are functions, and cannot be objects.

What, then, are extensions of concepts? Traditionally, the extension of a concept has been understood as the set of objects that fall under the concept. Frege has a slightly more technical conception, but still in keeping with this basic idea. For Frege, the extension of a concept is the set of ordered pairings of each object with one or other of the two truth-values. So the extension of the concept *philosopher* is the set of ordered pairs ⟨Chen Bo, the True⟩, ⟨Koji Nakatogawa, the True⟩, ⟨Julius Caesar, the False⟩, ⟨the Moon, the False⟩, and so on, where all objects are paired with either the True or the False. Frege's notion of the 'value-range of a function' is just a generalization of this idea of an extension of a concept. It is the set of ordered pairings of arguments with values, each function being defined by a unique such set.

Are extensions of concepts and value-ranges of functions objects or functions? Well, we talk about '*the* extension of the concept *philosopher*', for example, which indicates, according to Frege, that it is an object. We can apply concepts to such an object, in saying that it has more than one member, and its name shows it to be 'saturated'. Admittedly, extensions of concepts and value-ranges are abstract objects, not ordinary spatio-temporal objects, but they are nevertheless objects, not concepts.

**KN:** In explaining his views in 'On Concept and Object', Frege also makes the claim that 'The concept *horse* is not a concept'. This sounds very strange and even paradoxical. What led Frege to make this claim? And if it is indeed a paradox, then how can it be resolved?

**MB:** Yes, this is indeed a paradox, and Frege never provided an adequate answer to it, though we can suggest one on his behalf. He was led to the paradox by his absolute distinction between concept and object, and his doctrine that any expression of the form 'the *F*' stands for an object (if anything) rather than a concept. If we were asked to give an example of a concept, we might say 'The concept *horse* is a concept'. But on Frege's view, 'the concept *horse*' indicates an object, and since objects cannot be concepts, 'the concept *horse*' does *not* stand for a concept. So we wouldn't have given a good example! We do seem to have got ourselves into a bit of a muddle. Frege talks of the inadequacy of ordinary language here, which can sometimes mislead us into saying what is not, strictly speaking, correct. But we can do better than this - following a suggestion that Dummett first made. Recall that a concept, for Frege, must be defined for all objects. So we can think of concepts as dividing all objects into those that fall under it and those that do not. So what we are trying to say in saying 'The concept *horse* is a concept' (falsely, according to Frege) is just that everything is either a horse or not a horse. And this can be readily formalized in predicate logic as '$(\forall x)$ $(Hx \vee \neg Hx)$', where '$Hx$' represents '$x$ is a horse'. Frege would be perfectly happy with this. Just as we saw in the case of 'Unicorns do not exist', which can mislead us into thinking that unicorns must exist in some sense, we can rephrase the problematic proposition to make clear what is 'really' being said. This is another example of what I call 'interpretive' or 'transformative' analysis — resolving philosophical problems by rephrasing the propositions that have generated puzzlement. Although Frege did not resolve the paradox of the concept *horse* himself, he provided the tools and the general strategy by means of which to do so.

## 7. *Sinn* and *Bedeutung*

**CB:** Now let us turn to Frege's paper 'On *Sinn* and *Bedeutung*' (1892), which is his most famous and influential work. In your introduction to *The Frege Reader*, in which this paper is reprinted, you wrote a section on the translation of '*Bedeutung*', which has been a controversial issue in the literature on Frege. I found this very informative and asked one of my graduate students to translate it into Chinese. As you know, this has now been published in the journal *World Philosophy* (March 2008). So could you just briefly tell us what your own decision was as to how to translate '*Bedeutung*'?

**MB:** Yes, I can certainly tell you what my decision was — which was actually to leave '*Bedeutung*' untranslated! But before I say something to defend this decision, let me take this opportunity of thanking you for your work in having what I wrote published in Chinese translation. I explain the history of the translation into English of '*Bedeutung*', as it is used in Frege's work; so I refer any Chinese reader interested in this history to the relevant issue of *World Philosophy*. '*Bedeutung*' has been variously translated as 'reference', 'meaning', 'denotation', 'significance', 'indication' and 'nominatum'; and there are arguments — some good, some bad — for and against each of these. Precisely because of this, and all the controversy there has been, I thought it best in an edition of Frege's writings intended for wide use, to leave it untranslated. That way readers would be able to make up their own mind as to how it should be translated in each context, all the occurrences of the key word '*Bedeutung*' being highlighted by its non-translation. If I had to choose a single English word myself to translate '*Bedeutung*', I would pick 'reference'. But that works better in some places than in others, so in writing about Frege I tend to use the German term untranslated, explaining what it means in any given context wherever appropriate.

Perhaps at this point, Prof. Chen, I could ask you a question, since you recognize the philosophical importance of issues of translation. How has '*Bedeutung*' been translated into Chinese? Are there also a range of possible words that might be used — corresponding, perhaps, to English words such as 'reference' and 'meaning'? I would be interested to know if Chinese translators have had similar problems in translating '*Bedeutung*' — or indeed, any other key term in Frege's work.

**CB:** In China, there has not been much work on Frege so far, although interest has been growing rapidly recently. One Chinese scholar, Prof. Wang Lu, edited and translated into Chinese *The Selected Philosophical Papers of Gottlob Frege* (1994). He also translated *Die Grundlagen der Arithmetik* (*The Foundations of Arithmetic*). Hans Sluga's book, *Gottlob Frege* (1980), has also been translated into Chinese by another scholar. Chinese scholars usually accept English translations, that is, with '*Sinn*' translated as 'sense', translated as the Chinese word '含义', and '*Bedeutung*' translated as 'reference', translated as the Chinese word '所指'. Sometimes '*Sinn*' is also translated as '意义' ('meaning'), and '*Bedeutung*' as '指称' ('reference'). In recent years, influenced by the debate about English translations of '*Bedeutung*', some Chinese scholars have insisted that '*Sinn*' should be translated as '意义' ('meaning') and '*Bedeutung*' should be translated as '意谓' ('significance'). But I disagree. I found your comments and suggestions about the translation of '*Bedeutung*' quite reasonable.

**MB:** Prof. Nakatogawa, perhaps I could ask you the same questions. How has '*Bedeutung*' been translated into Japanese? Are there words in Japanese that correspond to English words such as 'reference' and 'meaning'?

**KN:** At first, Japanese scholars translated '*Sinn*' as '意味' (in Chinese characters) or 'いみ' (in hiragana, Japanese phonetic symbols), transliterated as 'i mi'. '*Bedeutung*' was at first translated as '指示' or 'しじ' — 'shi ji'. The English translation of 'i mi' is usually 'sense' or 'meaning', while the English translation of 'shi ji' is 'reference'. Researchers who came to Frege studies later, however, tend to use 'i mi' for '*Bedeutung*', and '意義' or 'いぎ' — 'i gi' — for '*Sinn*'. There are some scholars who suggest the use of another phonetic system, katakana, employing 'イミ' — 'i mi' — for 'sense'. Each translation emphasizes different aspects of *Sinn* and *Bedeutung*, and reflects different interpretations of Frege's texts. Most Japanese Frege scholars think that '*Bedeutung*' is a common word in German, used for just 'meaning'. If 'i mi' ('meaning') is used for '*Bedeutung*', the referring function of the word seems to be weakened.

**MB:** Well, it's interesting that there are indeed similar issues in translating Frege's terms into both Chinese and Japanese. It is easy to forget that translating involves interpreting, and we need to be very careful in reading philosophical works in translation. Translating a philosophical work itself requires philosophical skills, and I'm afraid that there are a lot of philosophers who do not appreciate this. I think reading and translating philosophy written in a different language from one's own is an essential skill for a philosopher to acquire: one learns to be very sensitive to the exact words being used to explain an idea or formulate an argument.

**CB:** With this in mind, then, how would you explain Frege's fundamental distinction between *Sinn* and *Bedeutung*? What motivated that distinction? What has it to do with Frege's logicist project? And what problems does it raise?

**MB:** There are a number of motivations for the distinction. Frege came to think that all three types of linguistic expression - names, concept words (or function terms, more generally) and sentences all had both a *Sinn* and a *Bedeutung*. But the reasons are not the same in each case (and some reasons are better than others). The simplest way to explain the distinction, though, as Frege himself does at the beginning of his famous essay, is by considering the informativeness of identity statements. A statement of the form '$a = a$' (e. g. '7 = 7') is trivially true while a statement of the form '$a = b$' (e. g. '4 + 3 = 5 + 2'), assuming it is true, can tell us something. In his early work, Frege had used the terms '*Inhalt*' ('content') '*Sinn*' and '*Bedeutung*' more or else synonymously. But if the 'content' of '$a$' and '$b$' is the same, then how can a statement of the form '$a = b$' be informative? Frege's answer is to distinguish between '*Sinn*' and '*Bedeutung*' (introducing a division *within* his earlier notion of 'content'). The *Bedeutung* of a name is the object referred to, while the *Sinn* indicates the *way* in which this object is referred to. (You will see that I have used the English verb 'refer' to explain Frege's idea here. We can then speak of the object referred to as the 'reference' or (perhaps better) 'referent' of the name.)

So consider Frege's own famous example of an informative identity statement, 'The morning star is the evening star'. The two names 'the morning star' and 'the evening star' refer to the same object, namely, Venus (which, of course, is actually a planet, not a star): they have the same *Bedeutung*. This is why the identity statement is correct. But they have different *senses* —— and I will now use the word 'sense' as the translation of '*Sinn*', as I think that is fairly unproblematic. 'The morning star' refers to Venus as it appears in the morning and 'the evening star' refers to Venus as it appears in the evening. We can 'sense' Venus in different ways, and this is why we have the two expressions to reflect these different ways. As Frege notes, it was an empirical discovery that the bright star we see in the morning is the same as the bright star we see in the evening.

Frege drew the same distinction for concept words. The concept words 'equilateral triangle' and 'equiangular triangle', for example, have the same *Bedeutung*, according to Frege, which is the concept itself. They also have the same extension, which is the set of objects that fall under this concept (roughly speaking, since we would need to talk of pairings, on Frege's view, as we have seen). But Frege nevertheless distinguishes between the concept itself (an 'unsaturated' entity) and the extension of the concept (which is an object). While they refer to the same concept, however, they do so in different ways, one reflecting the idea of equilaterality and the other reflecting the idea of equiangularity. Thus, even though all equilateral triangles are equiangular triangles, and all equiangular triangles are equilateral triangles, the two concept words have different senses. The distinction between *Sinn* and *Bedeutung* is less intuitive here, I think, since we might be inclined to say that there are different concepts here, understanding concepts intensionally rather than extensionally. Frege can be defended on this, but here is one issue about which there has been disagreement.

He also drew the same distinction for sentences, and this has proved particularly controversial. According to Frege, the *Bedeutung* of a sentence is a truth-value, and its sense is the thought it expresses. So the *Bedeutung* of 'The morning star is the evening star' is the True, and its sense is the thought (roughly) that the bright star we see in the morning is the same as the bright star we see in the evening. We have already talked of Frege's doctrine that concepts are functions that map objects onto one of the two truth-values. We have also seen that the result of completing a concept word with a name is a sentence. So if an object is the *Bedeutung* of a name, and a concept is the *Bedeutung* of a concept word, then the *Bedeutung* of the sentence that results from completing a concept word with a name is the value of the relevant concept for the relevant object, in other words, one of the two truth-values. We can think of these truth-values in many different ways, each of these ways being reflected in a corresponding sentence whose sense *is* the relevant thought.

It seems odd to talk of the 'meaning' of a sentence as a truth-value, which is one of the reasons why I do not favour 'meaning' as the translation of '*Bedeutung*' (though it has to be said that it also sounds odd in German). But 'reference' seems not to be that much better, which is why people have suggested 'significance' instead. But whatever its merit in the case of sentences, 'significance' is highly inappropriate in the case of names, so this term should be rejected, too. We thus see why '*Bedeutung*' has proved so hard to translate: nothing we choose works properly and equally well for all three cases - names, function terms and sentences.

The fact that Frege draws the distinction for all three cases suggests that the cases are analogous. But many people have argued that this is misleading. Given that truth-values are objects, according to Frege, sentences are a kind of name. This view has been particularly criticized. There are important differences between our apprehension of objects such as Venus, as reflected in our use of names such as 'the morning star' and 'the evening star', and our apprehension of truth-values, which we somehow grasp in thinking thoughts, expressed in our use of sentences.

The distinction between *Sinn* and *Bedeutung* seems best motivated in the case of names. But there are problems here, too. Frege allows that there can be senses without referents, and indeed, this might be taken as another motivation for the distinction. Frege talks in a few places of 'sense' as a 'mode of presentation' of a referent; but if there is no referent, then how can there be a mode of presentation of it? More frequently, Frege talks of 'sense' as a 'mode of determination' of a referent, which is better, since it may turn out that there is nothing determined. But if a name has no referent, then any sentence in which the name appears will lack a truth-value, since the *Bedeutung* of a sentence is determined by the *Bedeutung* of its parts. So it looks as if we have ruled out there being any fictional truths, such as 'Harry Potter is a boy wizard'. At any rate, for a Fregean philosophy of language to be sustained, we need to give some account of fiction.

Philosophers also now distinguish between names, such as 'Venus', and definite descriptions, such as 'the morning star'. Frege treated them as the same, but there are very good reasons for holding that their linguistic roles are quite different. One way to express a difference is to say that names must have reference (if they are to 'mean' anything at all) but not, perhaps, sense, while definite descriptions must have sense but not necessarily reference. This is only one of a number of possible views on the matter. All these issues — and many others that Frege's ideas have raised — are being vigorously debated today.

You also asked about the connection between Frege's distinction and his logicist project. There is a lot that can be said about this as well. But let me just note here one central connection. Recall that Frege wanted to show how arithmetical truths can be informative, i. e. non-trivial, even if they can be reduced to logical truths. The distinction between *Sinn* and *Bedeutung* allows this. Take the example of '7 + 5 = 12'. According to Frege, this is a logical truth, but it is not trivial, unlike '7 = 7', because '7 + 5' and '12' have different senses, just like 'the morning star' and 'the evening star'.

## 8. Frege's *Grundgesetze*, Russell's paradox and extensions of concepts

**CB:** This brings us back nicely to Frege's logicist project. So let us now talk about Frege's third book, *Grundgesetze der Arithmetik* (vol. 1, 1893; vol. 2, 1903). My questions are similar to those I asked about the *Begriffsschrift*. What was Frege's aim in this book? What did he actually achieve?

**MB:** Frege's aim was to demonstrate formally what he had only sketched informally in the *Grundlagen* — that arithmetic could be reduced to logic. He gave a more sophisticated exposition of his logical system, and in the second volume, he also provided powerful objections to existing views of the real numbers, just as in the *Grundlagen* he had criticized previous views of the natural numbers. He attacked psychologism and formalism — very convincingly, in my opinion — as well as the ideas of his contemporaries such as Cantor and Dedekind. He planned a third volume, to complete his logicist project, but as you know, a letter he received from Russell in June 1902, as the second volume was in press, dealt a devastatingly blow to his work.

**CB:** Yes, indeed, Russell's letter to Frege is very famous, and I wanted to ask you about this as well. What is the paradox that Russell discovered in Frege's system? Can you explain it to us, and how it arose in Frege's system? What was Frege's reaction? What solutions to the paradox are possible?

**MB:** The paradox arose from Frege's appeal to extensions of concepts. Recall that Frege ended up in the *Grundlagen* defining numbers in terms of extensions of concepts. But what are extensions of concepts? Frege's understanding of these was, in effect, encapsulated in a new axiom that he laid down in *Grundgesetze der Arithmetik*. This was his infamous Axiom V, asserting the equivalence between the following two propositions:

(Va) The function *F* has the same value for each argument as the function *G*.

(Vb) The value-range (*Wertverlauf*) of the function *F* is identical with the value-range of the function *G*.

As we have seen, Frege's notion of a value-range of a function is a generalization of the idea of an extension of a concept. (Va) and (Vb) thus yield the following as a special case:

(Ca) The concept F applies to the same objects as the concept G (i. e., whatever falls under concept F falls under concept G, and vice versa).

(Cb) The extension of the concept F is identical with the extension of the concept G.

Let us note straightaway the analogy between (Ca) and (Cb) — or more generally, (Va) and (Vb) — and (Na) and (Nb), the equivalence between which is asserted by the Cantor-Hume Principle. Axiom V, in other words, has exactly the same form as the Cantor-Hume Principle. As Frege saw it, Axiom V ensures that every (legitimate) concept has an extension (or more generally, that every function has a value-range) in just the same way as the Cantor-Hume Principle guarantees that number terms have a *Bedeutung*.

Russell's paradox, as it arises in Frege's system, can now be stated as follows. If every concept is defined for all objects (as Frege held, as we have seen), then every concept can be thought of as dividing all objects into those that do, and those that do not, fall under it. If extensions of concepts are objects (as Frege had assumed they were, just like numbers), then extensions themselves can be divided into those that fall under the concept whose extension they are (for example, the extension of the concept *is an extension*) and those that do not (for example, the extension of the concept *is a horse*). But now consider the concept *is the extension of a concept under which it does not fall*. Does the extension of *this* concept fall under the concept or not? If it does, then it does not, and if it does not, then it does. We have arrived at a contradiction: this is Russell's paradox.

Consider now the case in which the concept $F$ and the concept $G$ are one and the same. Then they have the same extension, so that (Cb) is true. But if this concept is the concept *is the extension of a concept under which it does not fall*, then it is not the case that anything that falls under this concept (the concept $F$) falls under this concept (the concept $G$), as the counterexample of its own extension shows, so that (Ca) is false. Axiom V, which asserts the equivalence between (Ca) and (Cb), is therefore false. Axiom V, which Frege had wanted to treat as a logical law, far from being a logical truth, isn't even a truth at all!

What has gone wrong? In my view, what is responsible for the paradox is the assumption that extensions of concepts are objects of the same kind — or on the same level — as the objects that fall under those concepts. As I said earlier, principles such as the Cantor-Hume Principle are now called 'abstraction principles', and Axiom V is also an abstraction principle. While the Cantor-Hume Principle might be thought a good abstraction principle, however, Axiom V would seem to be a bad one. What is especially problematic in Axiom V, at least as Frege understood it, is the assumption that the extensions of concepts — or value-ranges — thereby implicitly defined are already in the domain of objects over which the equivalence relation stated in (Ca) — or (Va) — is taken to hold.

Talk of 'abstraction principles' suggests an obvious answer: extensions of concepts should be seen as *abstracted out of* the relevant equivalence relation: whether one sees them as genuine objects or not, they are certainly not objects already in the original domain. This was essentially the response that Russell gave in developing his theory of types as an answer to the paradox. There are base-level objects, first-level extensions of concepts, second-level extensions of concepts, and so on. Recognizing such a hierarchy of objects enables one to avoid the paradox, although the difficult question then becomes how to develop a theory of types that still makes logicism a viable position. Russell ended up having to introduce further axioms whose logical status was by no means clear. But that is another long story in the history of logic and analytic philosophy.

As to Frege, his initial response, which he gave in an appendix hastily written as the second volume of *Grundgesetze* was going to press, was simply to disallow concepts from applying to their own extensions, restricting Axiom V accordingly. But he soon saw that this response was inadequate: among other things, it seemed *ad hoc* rather than philosophically motivated. He recognized the possibility of Russell's response - treating extensions of concepts as 'improper objects', as Frege put it. But for Frege, the complexity of the resulting theory conflicted with his 'universalist' conception of logic, that is, with his view that logical principles ought to apply to all kinds of objects unrestrictedly; and he eventually came to abandon his logicism.

## 9. Neo-logicism and the development of Frege's ideas

**KN:** In recent years there has been something of a revival of logicism under the name of 'neo-logicism' or 'neo-Fregeanism'. How does this differ from Frege's and Russell's logicism, and who are its main advocates? What is your own view on this?

**MB:** We talked earlier about what is now called 'Frege's theorem', which states that the Dedekind-Peano axioms, which define arithmetic, can be deduced within second-order logic from the Cantor-Hume Principle. It is this result that guides neo-logicism. Neither Frege nor Russell was prepared to claim that the Cantor-Hume Principle was a fundamental logical principle: they thought that it, too, had to be derived from logical laws and definitions. But if we could somehow argue that it is indeed a fundamental logical principle, then a new form of logicism is possible. Two of the most important neo-logicists are Crispin Wright and Bob Hale. They have also tried to identify abstraction principles by means of which to define the real numbers, understood as 'measuring numbers' rather than 'counting numbers'. Frege thought that the real numbers and the natural numbers were quite different entities, and offered different logicist accounts. Wright and Hale have followed Frege in this. My own view of logicism, as I suggested before, is that it all depends on what you take as 'logic'. The question 'Is arithmetic reducible to logic?', it seems to me, does not permit a simple 'Yes' or 'No' answer.

**CB:** Neo-logicism is one way in which philosophers have developed Frege's ideas. In what other ways have his ideas been developed in recent years?

**MB:** Frege is now widely recognized as one of the founders of analytic philosophy, and in most areas of analytic philosophy, his ideas are referred to, discussed, criticized and developed. So a full answer would involve telling the whole history of analytic philosophy. Having said this, though, there is little doubt that Frege's ideas have been particularly influential in the philosophy of language and the philosophy of mind, with the distinction between *Sinn* and *Bedeutung* lying at the core of debate. To take just one example, there has been much recent discussion of indexicals - terms, such as 'I', 'you', 'here' and 'now', whose reference depends systematically on the context of use. But while the reference might be clear on any given occasion of use, what is the *sense* of such terms? Frege himself makes only a few remarks about indexicals, and it is controversial just how his theory of sense should be developed to accommodate indexicals.

## 10. Frege's conception of thoughts (*Gedanken*)

**CB:** In his later years, Frege talked of there being a 'third realm' in addition to the outer world and the inner world, a realm in which numbers, concepts, senses, thoughts, the True and the False, and other abstract objects exist. Prof. Beaney, could you explain this conception of a third realm?

**MB:** Yes, this conception is introduced in the series of papers that Frege wrote at the end of his life, collectively called 'Logical Investigations' — and in particular, in the first of these papers, entitled 'Thought'. The basic idea is very simple. We might agree that there are at least two kinds of things — physical things and mental things. Physical things, such as tables and chairs, trees and rocks, and other empirical objects, inhabit the outer world. Mental things, such as sensations, feelings and 'ideas', which Frege also counts as 'contents of consciousness', inhabit the inner world. But there also seems to be a third kind of thing, which includes those you have mentioned — numbers, thoughts, and so on. These are not physical things; we cannot literally 'perceive' them in the outer world. Yet nor are they merely 'private' things; they can be apprehended by more than one person. Take Pythagoras' theorem, for example. We can all share the thought that this is true, so such a thought cannot be an 'idea' (as Frege understood this), yet nor can it be literally 'perceived'. So a 'third realm' must be recognized, Frege argued, to house such non-physical, non-mental entities.

**CB:** Frankly speaking, I have found Frege's conception of thoughts very hard to understand. How does Frege see the relationship between thoughts and sentences? If thoughts inhabit a third realm, then how do we grasp them? What is the relationship between thoughts and their truth-values? How are thoughts individuated? What is their criterion of identity?

**MB:** Well, these are all difficult questions that have been fiercely debated by philosophers and not just by Frege scholars. I can only gesture at Frege's views here and some of the issues they raise. As we have seen, for Frege, all logically significant expressions have both a sense and a *Bedeutung*. In the case of sentences, the sense is the thought expressed and the *Bedeutung* is one of the two truth-values, the True or the False. Frege distinguishes between thinking, judging and asserting. Thinking is grasping a thought. Judging is acknowledging the truth of a thought; in judging, on Frege's view, we advance from the sense of a sentence to its *Bedeutung*. Asserting is manifesting a judgement by uttering the relevant sentence with assertoric force. Now there is a great deal to say about all this. Let me just point out here a problem that arises in Frege's conception of 'grasping' a thought. Talk of grasping a thought might seem harmless enough, and we might even agree that thoughts can be regarded as a kind of object, which such talk certainly encourages. In suggesting that thoughts inhabit a 'third realm', however, Frege also talks of them as 'timeless' entities. But as a *process*, thinking surely occurs in space and time. So then we are left with the problem of how a temporal process of thinking connects with timeless thoughts. Frege admits that talk of 'grasping' thoughts is metaphorical, and he recognizes the problem here at the very end of 'Thought'. But many people — like you, Prof. Chen - have rightly been mystified by what Frege says.

You also asked about the criterion of identity for thoughts. Discussing this a little may suggest a way to resolve the problem we have just noted. Frege is not entirely consistent on the issue — and it has generated a lot of debate[5) — but the best criterion that can be offered (in my view) is something on the following lines:

> Two sentences *A* and *B* (as used in a given context) express the same thought if and only if anyone who understands both sentences (as used in that context) can immediately recognize that *A* is true (or false) if they recognize that *B* is true (or false), and vice versa.

There are still problems with such a criterion, but the details are not relevant to the point I now want to make. Let us speak of two sentences being 'cognitively equivalent' when they stand in the relationship captured in the criterion. We can then formulate a contextual definition of 'thought' (as the sense of a sentence) in the same way as contextual definitions can be offered of terms for abstract objects such as numbers, directions, value-ranges and extensions of concepts. What we can say, in other words, is that the following two propositions are equivalent:

(Sa) Sentence *A* is cognitively equivalent to sentence *B*.

(Sb) The sense of *A* (the thought expressed by *A*) is identical with the sense of *B* (the thought expressed by *B*).

You will see that (Sa) and (Sb) have exactly the same form as (Na) and (Nb), (Da) and (Db), (Va) and (Vb), and (Ca) and (Cb), which we discussed earlier. A proposition asserting an identity between objects of a certain kind is taken as definable by means of a proposition asserting an equivalence relation between objects of some other kind. This offers a useful way to understand Frege's conception of thought. For you can see now why Frege regarded senses (thoughts) as objects — in just the same way as he regarded numbers, directions, value-ranges and extensions of concepts as objects. They can all be defined by means of what we would now call an abstraction principle.

Recognizing that what we have here is an 'abstraction' principle (though this is not how Frege would have put it) also holds the key to how I would revise Frege's conception of thoughts. We can indeed talk of thoughts as objects, in the same way as we can talk of numbers as objects. But such talk is to be explained in terms of the relevant equivalence relations. Just as we apprehend numbers by understanding the senses of sentences in which numbers terms appear, explained by our understanding of one-one correlation, so too we grasp thoughts by understanding the senses of sentences in which 'thought terms' appear, explained by our understanding of cognitive equivalence. I grasp thoughts to the extent that I can use and understand sentences that express those thoughts and recognize when sentences are cognitively equivalent, as exhibited in our practices of inference, rephrasal and substitution of terms for one another. As I would put it, then, thoughts are not objects existing independently of our practices of using sentences. Talk of a 'third realm', in my view, is misleading. Frege was right to emphasize that thoughts are 'objective', and we can agree that they are 'objects'. But they are objects *abstracted* from our practices of using sentences, not 'timeless' entities. So to conclude, Prof. Chen, you are quite right to be confused by what Frege says about thoughts, but I think there is a way of sifting out what is right in his account from what is misleading. Indeed, philosophers ever since have been trying to do just this, though I wouldn't say that there is yet agreement on the matter.

## 11. Neo-Kantianism

**KN:** Prof. Chen asked you about Frege's influence on later philosophers, and what you have just said highlights a further issue that has inspired subsequent work in the philosophy of language and mind. I would like to ask you about the influence of earlier philosophers on Frege himself. What are the latest thoughts and debates about this? In Japan, there is a great deal of interest in Neo-Kantianism, which influenced Japanese thought after 1868. What influence did Neo-Kantianism, in particular, have on Frege?

**MB:** In the early days of Frege scholarship there was little concern with the influences on Frege himself. Dummett notoriously claimed in the introduction to his 1973 book that Frege's logic was "born from Frege's brain unfertilized by external influences". We have come a long way in our understanding of Frege since then. There has been research done —— by Mark Wilson and Jamie Tappenden, among others —— on the mathematical background to Frege's work; and there has been much discussion of Frege's philosophical predecessors and contemporaries. Hans Sluga, in the book on Frege he published in 1980, was one of the first to emphasize the influence of such philosophers as Trendelenburg and Lotze; and in recent years, Gottfried Gabriel, who is Professor of Logic and Theory of Science at Frege's own university, the University of Jena, has been writing a series of papers on the influence of key German thinkers on Frege. We talked earlier about the central claim of Frege's *Grundlagen*, that a number statement contains an assertion about a concept. The basic idea here came from Herbart. The influence of Neo-Kantianism is even more pervasive, especially the Neo-Kantianism of the so-called South-West (or Baden) school. Frege's anti-psychologism, his view of the normative nature of logical laws and his theory of judgement can all be broadly located in the Neo-Kantian tradition. The value theory developed by Windelband and Rickert might also be singled out as having had an influence on Frege, reflected in his conception of the '*Bedeutung*' of a sentence as its 'truth-value' —— which has led Gabriel, in particular, to suggest that '*Bedeutung*' is best translated as 'significance'.

## 12. Suggestions for further reading

**CB:** There is clearly a great deal of work being done on all aspects of Frege's philosophy, both historical and systematic. So finally, Prof. Beaney, can we ask you to recommend some secondary literature on Frege to Chinese and Japanese readers who are interested in understanding Frege's philosophy further?

**MB:** A useful introductory book is *Frege Explained* by Joan Weiner (Open Court, Chicago, 2004), a revised and expanded version of her earlier *Frege* (Oxford University Press, 1999). My own book, *Frege: Making Sense* (Duckworth, London, 1996) is rather longer, but I would hope it provides a good introduction to all the main ideas of Frege's philosophy —— in particular, showing how his philosophy of language emerges from his logicist project. Although I wrote it more than ten years ago, and feel clearer about some things now, I still agree with most of it! The best book on Frege's philosophy of mathematics is Michael Dummett's *Frege: Philosophy of Mathematics* (Duckworth, London, 1991). This was the long-awaited sequel to his *Frege: Philosophy of Language* (Duckworth, London, 1973), the pioneering book I mentioned at the beginning of our interview. This first book saw Frege too much, I think, as a modern philosopher of language. Most scholars recognize now how much Frege's philosophy of language was subservient to his philosophy of mathematics, and Dummett gives an excellent account of the latter in his sequel.

Finally, let me also mention a four-volume collection of papers on Frege that I edited with Erich Reck of the University of California at Riverside, *Gottlob Frege: Critical Assessments of Leading Philosophers* (Routledge, London, 2005). The four volumes are on Frege's philosophy in context, his philosophy of logic, his philosophy of mathematics, and his philosophy of thought and language. Each volume has its own introduction, and the 67 papers we selected were all written or published in the twenty-year period from 1986 to 2005. The work I have just mentioned by Mark Wilson, Jamie Tappenden and Gottfried Gabriel, for example, can be found in these volumes. The collection would certainly give any reader a good sense of the work that is now being done on Frege.

**CB:** Prof. Beaney, that is very helpful, and thank you very much for agreeing to talk to us. I am sure Chinese and Japanese readers will have learnt a lot about Frege, his logic and his philosophy, and about some of the debates that his ideas have generated. We wish you all the best in your continued work on Frege and the history of analytic philosophy, and in all your other philosophical activities.

**MB:** Well, thank you both very much for your questions and for your own interest. I hope our interview will result in a far higher number being assigned to the concept *reader of Frege*.

## Notes

1) CB stands for Chen Bo.
2) KN stands for Koji Nakatogawa.
3) MB stands for Michael Beaney.
4) The second volume of Beaney and Reck (2005) contains articles relevant to the debate in philosophy of mathematics.
5) Readers interested in the debate in philosophy of language can find relevant articles in the fourth volume of Beaney and Reck (2005).

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## Editorial note

This article is based on discussions between the authors which took place both actually in the UK and virtually on the Internet between the summer of 2008 and January 2009. Although most of it was written by Professor Beaney, some parts reflect the discussions with Professor Nakatogawa, and the main structure of the interview was set by Professor Chen Bo.

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