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An inverse problem for the one-dimensional wave equation in multilayer media

Sei NAGAYASU

Abstract

We consider half-line media which consist of many kinds of substances. We assume that the waves through this media are described by the one-dimensional wave equation. We can directly observe the data near the boundary point of the half-line, but we cannot directly observe the data of things away from the boundary point. In this situation, we try to identify these unknown things by creating an artificial explosion and observing on the boundary point the waves generated by the explosion. In the previous works related to this problem, only the speeds of the waves were treated, but we also take into account the impedances of the media in our setting.

1 Introduction

We consider half-line media which consist of many kinds of substances. We can directly observe the data near the boundary point of the half-line, but we cannot directly observe the data of things away from the boundary point. In this situation, we perform the following experiment in order to investigate them: We first create an artificial explosion at a point near the boundary point. Waves generated by this explosion travel in the media. Then we observe the waves at the boundary point, and guess the situation away from the boundary point.

This problem has been studied by Bartoloni-Lodovici-Zirilli [1], for example. However, from the experimental point of view, this result has some problem with respect to the formulation of the situation. Indeed, in [1], they deal with

\[ \frac{\partial^2 u}{\partial t^2}(t, x) = \frac{\partial}{\partial x} \left( S(x) \frac{\partial u}{\partial x}(t, x) \right), \quad t > 0, \ x > 0 \]
in order to express behavior of the waves inside the half-line, where $S(x)$ is a piecewise constant function. In this case, the interface or transmission conditions are determined by only the speeds of the waves. However this is not natural since the interface or transmission conditions depend on not only the speeds of the waves but also the impedances of the substances. Then we consider this problem in consideration of the impedances, and we try to reconstruct the unknown data concretely.

Now, we introduce the notations and formulate this problem. Put $h_0 := 0$. Let $h_k$ be a positive constant and $h_k > h_{k-1}$ for $k = 1, \ldots, N-1$. We call the interval $(h_{k-1}, h_k)$ Medium $k$ for $k = 1, \ldots, N-1$ and the interval $(h_{N-1}, \infty)$ Medium $N$. Let $a_k$ and $b_k$ be positive constants for $k = 1, \ldots, N$. The positive number $a_k$ describes the speed of the waves through Medium $k$, and $b_k$ the impedance of Medium $k$. Put $D_t := (1/i)(\partial/\partial t)$ and $D_x := (1/i)(\partial/\partial x)$, where $i$ is the imaginary unit. We define $P_k(D_t, D_x) = a_k^2 D_x^2 - D_t^2$ for $k = 1, \ldots, N$. Suppose $0 < y < h_1$.

We consider the following equations:

\begin{align}
P_1(D_t, D_x)u(t, x) &= \delta(t, x - y), \quad 0 < x < h_1, \quad (1) \\
P_k(D_t, D_x)u(t, x) &= 0, \quad h_{k-1} < x < h_k \quad (2 \leq k \leq N - 1), \quad (2) \\
P_N(D_t, D_x)u(t, x) &= 0, \quad h_{N-1} < x, \quad (3) \\
D_x u(t, x)|_{x=0} &= 0, \quad (4) \\
u(t, x)|_{x=h_{k-1}} &= u(t, x)|_{x=h_k} \quad (1 \leq k \leq N - 1), \quad (5) \\
a_k b_k D_x u(t, x)|_{x=h_{k-1}} &= a_{k+1} b_{k+1} D_x u(t, x)|_{x=h_k} \quad (1 \leq k \leq N - 1). \quad (6)
\end{align}

The equation (4) means the free boundary condition at the point $x = 0$. The equations (5) and (6) for $k$ express the conditions at the point $x = h_k$ which
is the joining of Medium \( k \) and Medium \( k + 1 \). The equation (5) describes the continuity of the displacement of the waves, and (6) the continuity of the stress. The equations (1)–(6) express the situation that the initial data is the delta function at the point \( y \) in Medium 1 at the time \( t = 0 \) with the boundary condition (4) and the interface or transmission conditions (5) and (6) at the joining point between Medium \( k \) and Medium \( k + 1 \). The following main result says that we can reconstruct the impedances \( b_{k+1} \) and the ratios \( (h_k - h_{k-1})/a_k \) of the width to the speeds of the waves by the observation data \( u(t, 0) \) when the data \( a_1, b_1 \) of Medium 1 are known.

**Main result.** Suppose that the constants \( a_1, b_1, y \) are known. Assume \( b_j \neq b_{j+1} \) for \( j = 1, \ldots, N - 1 \). Assume that the observation data \( v(t) := u(t, 0) \) are given on \([0, T)\), where \( u(t, x) \) denotes the solution of the equations (1)–(6). Then \( b_{k+1} \) and \( (h_k - h_{k-1})/a_k \) are reconstructed by the following process:

- **The first step:** Put \( v_1(t) := (1/a_1)H(t - y/a_1) - v(t) \), where \( H \) is the Heaviside function.

- **The \((k + 1)\)-st step \((k = 1, 2, \ldots)\):** If \( v_k(t) \equiv 0 \) then the process is finished. If \( v_k(t) \neq 0 \), then put \( t_k := \inf\{t \in [0, T) : v_k(t) \neq 0\} \), reconstruct the constants \( (h_k - h_{k-1})/a_k \) and \( b_{k+1} \) by

\[
\frac{h_k - h_{k-1}}{a_k} := \frac{1}{2} \left( t_k + \frac{y}{a_1} \right) - \sum_{j=1}^{k-1} \frac{h_j - h_{j-1}}{a_j},
\]

\[
b_{k+1} := \frac{2^{2k-2} \prod_{j=1}^{k-1} (b_j b_{j+1}) + v_k(t_k + 0)a_1 \prod_{j=1}^{k-1} (b_j + b_{j+1})^2}{2^{2k-2} \prod_{j=1}^{k-1} (b_j b_{j+1}) - v_k(t_k + 0)a_1 \prod_{j=1}^{k-1} (b_j + b_{j+1})^2}.
\]

Define \( v_{k+1}(t) \) by the known data and the reconstructed data, and go the next step.

We state the concrete way of determining \( v_{k+1}(t) \) in Theorem 13. We remark that we can reconstruct the impedances \( b_{k+1} \) but we cannot identify the speeds \( a_k \) themselves of the waves. This result is not obtained by \([1]\).

On the other hand, our main result is also the expansion of Nagayasu \([4]\) for the one-dimensional case. In \([4]\), the author considers the situation that
The $n$-dimensional case ($n \geq 2$).

The one-dimensional case.

Figure 2: The two-layer case.

the half-line consists of two layers, and determine the unknown data by using the observation data on the whole time. However, our main result says that we can reconstruct the unknown data by the observation data on the finite time, and how many data we can reconstruct is determined as to the observation time.

We remark that the one-dimensional case differs from the $n$-dimensional case ($n \geq 2$) in that the speeds themselves cannot or can be reconstructed. Indeed, we obtain the following result from [4] for example. We consider the two-layer case (see Figure 2), and assume that $a_1$ and $b_1$ are known. Let observation data be given. Then, we can identify $a_2$, $b_2$ and $h_1$ when the physical space dimension is greater than or equal to two. However, we can identify $b_2$ and $h_1/a_1$ (namely $h_1$ itself) but cannot identify $a_2$ when the physical space dimension is one.

Finally, we explain the plan of this paper. In Section 2, we construct the solution formula of the equations (1)–(6). In Section 3, we state our main result concretely and give its proof. In Appendix, we discuss the case that the impedance of the adjacent media may be equal, that is, $b_j = b_{j+1}$ may hold.

2 The solution formula

In this section, we construct the explicit solution formula in Medium 1 of the equations (1)–(6). In order to make the dependence of the solution on the
coefficients clearly, we denote the solution of (1)–(6) by
\[
u(t, x) = u_N(t, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y).
\]
In Section 2.1, we express it in the case of \(N = 1\). In Section 2.2, we construct it for \(N \geq 2\).

### 2.1 The solution formula for \(N = 1\)

The equations which we deal with are as following:
\[
P_1(D_t, D_x)u_1(t, x) = \delta(t, x - y), \quad x > 0,
\]
\[
D_xu_1(t, x)|_{x=0+0} = 0.
\]

By Matsumura [2], we find the solution
\[
u_1(t, x; a_1; b_1; \cdot; y) = \frac{1}{2a_1} H\left(t - \frac{|x-y|}{a_1}\right) + \frac{1}{2a_1} H\left(t - \frac{x+y}{a_1}\right).
\]

We remark that its Fourier-Laplace transform along \(\rho = \tau - im \log(2 + |\tau|)\) with respect to \(t\) is
\[
\hat{u}_1(\rho, x) = \frac{1}{2a_1 i \rho} \left\{ e^{-i\rho|x-y|/a_1} + e^{-i\rho(x+y)/a_1} \right\}, \quad x > 0.
\]

### 2.2 The solution formula for \(N \geq 2\)

We construct the solution of (1)–(6) by induction on \(N\). Then we first define
\[
F_k^{(N)}(t, x) = F_k^{(N)}(t, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)
\]
by
\[
F_k^{(N)}(t, x) = u_{N-1}(t, x) - u_{N}(t, x), \quad h_{k-1} < x < h_k \ (1 \leq k \leq N - 1),
\]
\[
F_N^{(N)}(t, x) = u_{N}(t, x), \quad h_{N-1} < x,
\]
where we write
\[
u_{N-1}(t, x) = u_{N-1}(t, x; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}; y),
\]
\[
u_{N}(t, x) = u_{N}(t, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)
\]
for short notation. The distribution \(F_k^{(N)}(t, x)\) expresses the behavior of the waves in Medium \(k\) which are affected by Medium \(N\). We remark that this
$F^{(N)}_k(t, x)$ and $F_1(t, x)$ in Matsumura [3] are different. By the definition of $F^{(N)}_k(t, x)$, the equations (1)–(6) are changed for

\[ P_k F^{(N)}_k = 0 \begin{cases} h_{k-1} < x < h_k & (1 \leq k \leq N - 1) \\ h_{N-1} < x & (k = N) \end{cases}, \]  

(7)

\[ D_x F^{(N)}_1 |_{x=0} = 0, \]  

(8)

\[ (F^{(N)}_k - F^{(N)}_{k+1}) |_{x=h_k} = 0 \quad (1 \leq k \leq N - 2), \]  

(9)

\[ (a_k b_k D_x F^{(N)}_k - a_{k+1} b_{k+1} D_x F^{(N)}_{k+1}) |_{x=h_k} = 0 \quad (1 \leq k \leq N - 2), \]  

(10)

\[ (F^{(N)}_{N-1} + F^{(N)}_N) |_{x=h_{N-1}} = u_{N-1} |_{x=h_{N-1}}, \]  

(11)

\[ (a_{N-1} b_{N-1} D_x F^{(N)}_{N-1} + a_N b_N D_x F^{(N)}_N) |_{x=h_{N-1}} = a_{N-1} b_{N-1} D_x u_{N-1} |_{x=h_{N-1}}, \]  

(12)

where $P_k = P_k(D_t, D_x)$, $F^{(N)}_k = F^{(N)}_k(t, x)$ and

\[ u_{N-1} = u_{N-1}(t, x; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}; y) \]

for short notation. We solve these equations. We apply the Fourier-Laplace transformation along $\rho = \tau - i m \log(2 + |\tau|)$ with respect to $t$ to these equations as in Matsumura [3], where $m$ is a positive real large enough. Then by (7) we can write

\[ \hat{F}^{(N)}_k(\rho, x) = \Phi^{(N)}_k(\rho) e \left( -\frac{x}{a_k} \right) + \Psi^{(N)}_k(\rho) e \left( \frac{x}{a_k} \right) \quad (1 \leq k \leq N - 1), \]  

(13)

\[ \hat{F}^{(N)}_N(\rho, x) = \Phi^{(N)}_N(\rho) e \left( -\frac{x}{a_N} \right), \]  

(14)

where $e(s) := e(s; \rho) := \exp(i \rho s)$. In the same way as $F^{(N)}_k(t, x)$, we write

\[ \Phi^{(N)}_k(\rho) = \Phi^{(N)}_k(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y), \]  

\[ \Psi^{(N)}_k(\rho) = \Psi^{(N)}_k(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) \]

in order to make the dependence on the coefficients clearly. We define $K_M[\text{resp. } L_M](\rho; a_1, \ldots, a_M; b_1, \ldots, b_M; h_1, \ldots, h_{M-1}, h_M; y)$ by

\[ K_M(\rho; a_1, \ldots, a_M; b_1, \ldots, b_M; h_1, \ldots, h_{M-1}, h_M; y) := \hat{u}_M(\rho, x; a_1, \ldots, a_M; b_1, \ldots, b_M; h_1, \ldots, h_{M-1}; y) |_{x=h_M}, \]  

6
\[ L_M(\rho; a_1, \ldots, a_M; b_1, \ldots, b_M; h_1, \ldots, h_{M-1}, h_M; y) := -\frac{a_M b_M}{\rho} D_x \hat{u}_M(\rho, x; a_1, \ldots, a_M; b_1, \ldots, b_M; h_1, \ldots, h_{M-1}; y)|_{x=h_M} \]

for \( M = 1, 2, \ldots \). Now, we substitute (13) and (14) into the Fourier-Laplace transform of the equations (8)–(12) and simplify them. Then we have

\[ \mathcal{Z}_N \begin{bmatrix} \Phi_1^{(N)} \\ \Psi_1^{(N)} \\ \Phi_2^{(N)} \\ \Psi_2^{(N)} \\ \vdots \\ \Phi_{N-1}^{(N)} \\ \Psi_{N-1}^{(N)} \\ \Phi_N^{(N)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ K_{N-1} \\ L_{N-1} \end{bmatrix}, \tag{15} \]

where we define the \((j, l)\)-components

\[ \mathcal{Z}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1})_{jl} \]

of the \((2N - 1) \times (2N - 1)\) matrix

\[ \mathcal{Z}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}) \]
by

\[
Z_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1})_{jl} :=
\begin{cases}
1, & j = 1, l = 1, \\
-1, & j = 1, l = 2, \\
e\left(-\frac{h_k}{a_k}\right), & j = 2k, l = 2k - 1 \ (k = 1, \ldots, N - 1), \\
e\left(\frac{h_k}{a_k}\right), & j = 2k, l = 2k \ (k = 1, \ldots, N - 1), \\
b_k e\left(-\frac{h_k}{a_k}\right), & j = 2k + 1, l = 2k - 1 \ (k = 1, \ldots, N - 1), \\
-b_k e\left(\frac{h_k}{a_k}\right), & j = 2k + 1, l = 2k \ (k = 1, \ldots, N - 1), \\
-e\left(-\frac{h_k}{a_{k+1}}\right), & j = 2k, l = 2k + 1 \ (k = 1, \ldots, N - 2), \\
-e\left(\frac{h_k}{a_{k+1}}\right), & j = 2k, l = 2k + 2 \ (k = 1, \ldots, N - 2), \\
b_{k+1} e\left(-\frac{h_k}{a_{k+1}}\right), & j = 2k + 1, l = 2k + 1 \ (k = 1, \ldots, N - 2), \\
b_{k+1} e\left(\frac{h_k}{a_{k+1}}\right), & j = 2k + 1, l = 2k + 2 \ (k = 1, \ldots, N - 2), \\
e\left(-\frac{h_{N-1}}{a_N}\right), & j = 2N - 2, l = 2N - 1, \\
b_N e\left(-\frac{h_{N-1}}{a_N}\right), & j = 2N - 1, l = 2N - 1, \\
0, & \text{otherwise}
\end{cases}
\]

and we write

\[
Z_N = Z_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}),
\]

\[
\Phi_k^{(N)}[\Psi_k^{(N)}] = \Phi_k^{(N)}[\Psi_k^{(N)}](\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y),
\]

\[
K_{N-1}[L_{N-1}] = K_{N-1}[L_{N-1}](\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-1}; y)
\]

for short notation in the equation (15).
We need to express the explicit formula of \( u(t, 0) \) in order to discuss our inverse problem. Then we construct \( \Phi^{(N)}_1(\rho) \) and \( \Psi^{(N)}_1(\rho) \). Now, for short notation we write

\[
K_N[L_N] = K_N[L_N](\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}, h_N; y),
\]

\[
\Phi^{(N)}_N = \Phi^{(N)}_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y),
\]

\[
Z_N = Z_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1})
\]

in Lemmas 1, 2, 3 and Corollary 4. We first express \( \det Z_N \) explicitly.

**Lemma 1.** Let \( N \geq 2 \). Then

\[
\det Z_N = (-1)^N e \left( -\frac{h_{N-1}}{a_N} \right) \times Z_N \left( \rho; b_1, \ldots, b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right)
\]

(16)

holds, where we denote

\[
Z_N(\rho; b_1, b_2, \ldots, b_N; \Theta_1, \Theta_2, \ldots, \Theta_{N-1})
\]

\[
:= \sum_{\alpha_k = \pm 1} \alpha_1 \left\{ \prod_{j=1}^{N-2} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) \right\} (b_{N-1} + \alpha_{N-1} b_N) e \left( \sum_{j=1}^{N-1} \alpha_j \Theta_j \right)
\]

for \( N \geq 2 \), and we define \( \prod_{j=1}^{N-2} (b_j + \alpha_j \alpha_{j+1} b_{j+1}) = 1 \) when \( N = 2 \).

Proof. We prove this lemma by induction on \( N \). It is easy to obtain the equation (16) for the case of \( N = 2 \). Then we assume that the equation (16) for \( N \geq 2 \) holds, and we show the equation (16) for \( N + 1 \). We first expand \( \det Z_{N+1} \) along the \((2N + 1)\)-st column, and expand them along the \((2N)\)-th row. Then we have

\[
\det Z_{N+1}(\rho; a_1, \ldots, a_N, a_{N+1}; b_1, \ldots, b_N, b_{N+1}; h_1, \ldots, h_{N-1}, h_N)
\]

\[
= -e \left( -\frac{h_N}{a_{N+1}} \right) \left\{ b_N e \left( -\frac{h_N}{a_N} \right) e \left( \frac{2h_{N-1}}{a_N} \right) \det Z_{N} + b_N e \left( \frac{h_N}{a_N} \right) \det Z_{N}^+ \right\}
\]

\[
+ b_{N+1} e \left( -\frac{h_N}{a_{N+1}} \right) \left\{ e \left( -\frac{h_N}{a_N} \right) e \left( \frac{2h_{N-1}}{a_N} \right) \det Z_{N} - e \left( \frac{h_N}{a_N} \right) \det Z_{N}^+ \right\}
\]

for \( N \geq 2 \).
\[
\begin{align*}
= -e \left( -\frac{h_N}{a_{N+1}} \right) \left\{ (b_N + b_{N+1}) e \left( \frac{h_N}{a_N} \right) \det Z_N^+ + (b_N - b_{N+1}) e \left( \frac{2h_{N-1} - h_N}{a_N} \right) \det Z_N^- \right\} \\
\overset{(*)} = -e \left( -\frac{h_N}{a_{N+1}} \right) \left\{ (b_N + b_{N+1}) e \left( \frac{h_N}{a_N} \right) (-1)^N e \left( -\frac{h_{N-1}}{a_N} \right) Z_N^+ + (b_N - b_{N+1}) e \left( \frac{2h_{N-1} - h_N}{a_N} \right) (-1)^N e \left( -\frac{h_{N-1}}{a_N} \right) Z_N^- \right\} \\
= (-1)^{N+1} e \left( -\frac{h_N}{a_{N+1}} \right) \\
\times \left\{ \sum_{\alpha_1 = \pm 1} \prod_{1 \leq k \leq N-1} \left( b_k + \alpha_j a_{j+1} \right) \right\} \left\{ \sum_{j=1}^{N-1} \alpha_j \frac{h_j - h_{j-1}}{a_j} \right\} \\
\times \left\{ \sum_{\alpha_N = \pm 1} (b_{N-1} + \alpha_N a_N b_N) (b_N + \alpha_N b_{N+1}) e \left( \frac{h_N - h_{N-1}}{a_N} \right) \right\} \\
= (-1)^{N+1} e \left( -\frac{h_N}{a_{N+1}} \right) Z_{N+1} \left( b_1, \ldots, b_N, b_{N+1}; \right. \\
\left. \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}}, \frac{h_N - h_{N-1}}{a_N} \right),
\end{align*}
\]

where we write

\[
Z_N^\pm = Z_N(\rho; a_1, \ldots, a_{N-1}, a_N; b_1, \ldots, b_{N-1}, \pm b_N; h_1, \ldots, h_{N-1}),
\]

\[
Z_N^\pm = Z_N \left( \rho; b_1, \ldots, b_{N-1}, \pm b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right)
\]

for short notation and we use the inductive hypothesis at \((*)\). Hence we obtain the equation \((16)\) for \(N + 1\).

Next, we express \(K_N\) and \(L_N\) explicitly.

**Lemma 2.** For \(N \geq 2\)

\[
K_N = \Phi_N^{(N)} e \left( -\frac{h_N}{a_N} \right), \quad L_N = b_N K_N
\]

hold.
Proof. Because of $h_{N-1} < h_N$, we have
\[ D^j_x \tilde{a}_N(x)|_{x=h_N} = D^j_x \tilde{F}^{(N)}_N(x)|_{x=h_N} \]
for $j = 0, 1$. From this equation we can obtain this lemma easily.

Lemma 3. For $N \geq 2$ we have
\[
\Phi^{(N)}_N = -\frac{(-2)^{N-1}}{2a_1i\rho} \frac{1}{\det Z_N} \left( \prod_{j=1}^{N-1} b_j \right) \sum_{\nu=\pm 1} e \left( \frac{\nu y}{a_1} \right) .
\] (17)

Proof. We prove the equation (17) by induction on $N$. First we consider the case of $N = 2$. We remark that we obtain
\[
K_1(\rho; a_1; b_1; h_1; y) = \frac{1}{2a_1i\rho} e \left( -\frac{h_1}{a_1} \right) \sum_{\nu=\pm 1} e \left( \frac{\nu y}{a_1} \right) ,
\]
\[
L_1(\rho; a_1; b_1; h_1; y) = b_1 K_1(\rho; a_1; b_1; h_1; y)
\]
from the definition of $K_1$ and $L_1$, and Section 2.1. By these equations, we have this lemma for $N = 2$. Then we assume that the equation (17) for $N(\geq 2)$ holds, and we show the equation (17) for $N+1$. We have
\[
\Phi^{(N+1)}_N(\rho; a_1, \ldots, a_N, a_{N+1}; b_1, \ldots, b_N, b_{N+1}; h_1, \ldots, h_{N-1}, h_N; y)
\]
\[= \left( \text{the } (2N+1)\text{-st component of } (Z_{N+1}^{-1}) \cdot \begin{bmatrix} 0 \\ K_N \\ L_N \end{bmatrix} \right) \]
\[= \frac{1}{\det Z_{N+1}} \{ \text{(the } (2N, 2N+1)\text{-cofactor of } Z_{N+1})K_N \]
\[+ \text{(the } (2N + 1, 2N + 1)\text{-cofactor of } Z_{N+1})L_N \} \]
\[\overset{\overset{\text{eq.}}{\text{V)}}}{=} \frac{1}{\det Z_{N+1}} \left\{ -b_N e \left( -\frac{h_N}{a_N} \right) e \left( 2\frac{h_{N-1}}{a_N} \right) \det Z_N - b_N e \left( \frac{h_N}{a_N} \right) \det Z_N \right. \]
\[\left. + b_N e \left( -\frac{h_N}{a_N} \right) e \left( 2\frac{h_{N-1}}{a_N} \right) \det Z_N - b_N e \left( \frac{h_N}{a_N} \right) \det Z_N \} \}
\[\times \Phi^{(N)}_N \left( -\frac{h_N}{a_N} \right) \]
\[= -\frac{2}{\det Z_{N+1}} b_N \Phi^{(N)}_N \]
\[\overset{\overset{\text{eq.}}{\text{V)}}}{=} -2 \frac{\det Z_N}{\det Z_{N+1}} b_N \Phi^{(N)}_N \]
\[= \frac{\det Z_N}{\det Z_{N+1}} b_N \frac{(-2)^{N-1}}{2a_1i\rho} \frac{1}{\det Z_N} \left( \prod_{j=1}^{N-1} b_j \right) \sum_{\nu=\pm 1} e \left( \frac{\nu y}{a_1} \right) .
\]
\[- \frac{(-2)^N}{2a_1i\rho} \frac{1}{\det Z_{N+1}} \left( \prod_{j=1}^{N} b_j \right) \sum_{\nu = \pm 1} e \left( \nu \frac{y}{a_1} \right), \]

where we write

\begin{align*}
Z_N &= Z_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_{N-1}, -b_N; h_1, \ldots, h_{N-1}), \\
Z_{N+1} &= Z_{N+1}(\rho; a_1, \ldots, a_N, a_{N+1}; b_1, \ldots, b_{N+1}; h_1, \ldots, h_{N-1}, h_N)
\end{align*}

for short notation, and we expand the determinant along the \((2N)\)-th row and use Lemma 2 at (♯), and we use the inductive hypothesis at (*). \(\square\)

**Corollary 4.** For \(N \geq 2\),

\[K_N = \frac{(-2)^{N-1}}{2a_1i\rho} \frac{1}{\det Z_N} \left( \prod_{j=1}^{N-1} b_j \right) \sum_{\nu = \pm 1} e \left( \nu \frac{y}{a_1} - \frac{h_N}{a_N} \right), \quad L_N = b_N K_N\]

hold.

**Proof.** By Lemmas 2 and 3, we obtain this corollary easily. \(\square\)

**Remark 5.** We define \(Z_1(\rho; a_1; b_1; \cdot) = (-1), \ Z_1(\rho; b_1; \cdot) = 1\). Then Lemma 1 and Corollary 4 hold also for \(N = 1\), where we define \(\prod_{j=1}^{N-1} b_j = 1\) for \(N = 1\).

Now, we express \(\Phi_1^{(N)}\) and \(\Psi_1^{(N)}\) explicitly.

**Lemma 6.** For \(N \geq 2\),

\[\Phi_1^{(N)}(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = \Psi_1^{(N)}(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)\]

\[= \frac{2^{2N-4}}{2a_1i\rho \det Z_N \det Z_{N-1}} \left( \prod_{j=1}^{N-2} (b_j b_{j+1}) \right) \sum_{\nu = \pm 1} e \left( \nu \frac{y}{a_1} - \frac{h_{N-1}}{a_{N-1}} - \frac{h_{N-1}}{a_N} \right)\]

holds, where we write

\begin{align*}
Z_N &= Z_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}), \\
Z_{N-1} &= Z_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2})
\end{align*}

for short notation, and we define \(\prod_{j=1}^{N-2} (b_j b_{j+1}) = 1\) for \(N = 2\).
Proof. It is easy to obtain $\Phi_1^{(N)}(\rho) = \Psi_1^{(N)}(\rho)$ from the equation (15). Then we find the explicit formula of $\Phi_1^{(N)}(\rho)$. We have

$$\Phi_1^{(N)}(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)$$

$$= \left(\text{the first component of } (Z^{-1}) \cdot \begin{pmatrix} 0 \\ K_{N-1} \\ L_{N-1} \end{pmatrix} \right)$$

$$= \frac{1}{\det Z_N} \{ (\text{the } (2N-2, 1)\text{-cofactor of } Z_N)K_{N-1}$$

$$+ (\text{the } (2N-1, 1)\text{-cofactor of } Z_N)L_{N-1} \}$$

$$\equiv \frac{b_N - b_{N-1}}{\det Z_N} e \left( -\frac{h_{N-1}}{a_N} \right) K_{N-1}$$

$$\times \prod_{j=1}^{N-2} \det \left[ \begin{array}{cc} -e \left( -\frac{h_j}{a_{j+1}} \right) & -e \left( \frac{h_{j+1}}{a_{j+1}} \right) \\ -b_{j+1} e \left( -\frac{h_j}{a_{j+1}} \right) & b_{j+1} e \left( \frac{h_j}{a_{j+1}} \right) \end{array} \right]$$

$$= \frac{b_N - b_{N-1}}{\det Z_N} e \left( -\frac{h_{N-1}}{a_N} \right) K_{N-1} (-2)^{N-2} \left( \prod_{j=1}^{N-2} b_{j+1} \right)$$

$$\equiv \frac{b_N - b_{N-1}}{\det Z_N} e \left( -\frac{h_{N-1}}{a_N} \right) (-2)^{N-2} \frac{1}{2a_1 \rho} \frac{1}{\det Z_{N-1}} \left( \prod_{j=1}^{N-2} b_j \right)$$

$$\times \sum_{\nu = \pm 1} e \left( \nu \frac{y}{a_1} - \frac{h_{N-1}}{a_{N-1}} \right) (-2)^{N-2} \left( \prod_{j=1}^{N-2} b_{j+1} \right)$$

$$= \frac{2^{2N-4}}{2a_1 \rho} \frac{b_{N-1} - b_N}{\det Z_N \det Z_{N-1}} \left\{ \prod_{j=1}^{N-2} (b_j b_{j+1}) \right\} \sum_{\nu = \pm 1} e \left( \nu \frac{y}{a_1} - \frac{h_{N-1}}{a_{N-1}} - \frac{h_{N-1}}{a_N} \right),$$

where we use Corollary 4 at (12) and we write

$$Z_N = Z_N(\rho; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}),$$

$$Z_{N-1} = Z_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}),$$

$$K_{N-1} = K_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}, h_{N-1}; y),$$

$$L_{N-1} = L_{N-1}(\rho; a_1, \ldots, a_{N-1}; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-2}, h_{N-1}; y)$$

for short notation. \qed
Proposition 7. For $N \geq 2$,
\[
F_1^{(N)}(t, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)
= f^{(N)} \left( t, x; b_1, \ldots, b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}}; y \right)
\]
holds, where we put
\[
f^{(N)}(t, x; b_1, \ldots, b_N; \Theta_1, \ldots, \Theta_{N-1}; y)
:= -\frac{1}{2a_1} \sum_{0 \leq m_k < \infty \atop (k=1, \ldots, N-1)} \psi_N(m_1, \ldots, m_{N-1}; b_1, \ldots, b_N)
\times \sum_{\nu, \bar{\nu} = \pm 1} H \left( t - \left( \nu \frac{y}{a_1} + \bar{\nu} \frac{x}{a_1} + 2 \sum_{J=1}^{N-1} (m_J + 1) \Theta_J \right) \right),
\]
and define $\psi_N$ by
\[
\psi_2(m_1; b_1, b_2) = \left( \frac{b_1 - b_2}{b_1 + b_2} \right)^{m_1+1}
\]
Here we define

\[
\psi_N(m_1, \ldots, m_{N-1}; b_1, \ldots, b_N) = \sum_{(j_0)_{a \in C_N}, (i_0)_{b \in A_{N-1}}} 2^{2N-4} \times \prod_{k=1}^{N-1} \left( \frac{m_k}{\prod_{a \in C_N} (j_0)!} \prod_{b \in A_{N-1}} (i_0)! \right) \left( \prod_{j=1}^{N-3} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right) \frac{b_{N-2} b_{N-1}}{(b_{N-2} + b_{N-1})^2} \frac{b_{N-1} - b_N}{b_{N-1} + b_N} \right)
\]

for \( N = 2 \) and as following for \( N \geq 3 \):

\[
\psi_N(m_1, \ldots, m_{N-1}; b_1, \ldots, b_N) = \sum_{(j_0)_{a \in C_N}, (i_0)_{b \in A_{N-1}}} 2^{2N-4} \times (-1)^{k-2} \frac{\prod_{k=1}^{N-1} \left( \sum_{a \in C_N} \sum_{\beta \in A_{N-1}} \right) \left( m_k - \sum_{a \in C_{N-1}} j_0 - \sum_{\beta \in A_{N-1}} i_0 \right) i_0}{\prod_{a \in C_N} (j_0)!} \prod_{b \in A_{N-1}} (i_0)! \left( \prod_{j=1}^{N-3} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right) \frac{b_{N-2} b_{N-1}}{(b_{N-2} + b_{N-1})^2} \frac{b_{N-1} - b_N}{b_{N-1} + b_N} \right)
\]

Here we define \( \prod_{j=1}^{N-3} (\ast) = 1 \) for \( N = 3 \) and we put

\[
A_N = \{ \alpha = (\alpha_1, \ldots, \alpha_{N-1}) : \alpha_k = \pm 1, \ \alpha \neq (1, 1, \ldots, 1) \}, \quad B_N = \{ \alpha \in A_N : \# \{ k : \alpha_k = -1 \} = 1 \}, \quad C_N = A_N \setminus B_N, \quad A_N^{(k_1, \ldots, k_v)} = \{ \alpha \in A_N : \alpha_{k_1} = \cdots = \alpha_{k_v} = \pm 1 \}, \quad C_N^{(k_1, \ldots, k_v)} = \{ \alpha \in C_N : \alpha_{k_1} = \cdots = \alpha_{k_v} = \pm 1 \},
\]

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\[ G_N = G_N(m_1, \ldots, m_{N-1}) \]
\[ = \left\{ (j_\alpha)_{\alpha \in C_N}, (i_\beta)_{\beta \in A_{N-1}} : 
  \begin{array}{l}
  j_\alpha \geq 0, \quad i_\beta \geq 0, \\
  \sum_{\alpha \in C_N^{(m_1)}} j_\alpha + \sum_{\beta \in A_{N-1}^{(m_2)}} i_\beta \leq m_k \quad (1 \leq k \leq N-2), \\
  \sum_{\alpha \in C_N^{(m_{N-1})}} j_\alpha \leq m_{N-1}.
  \end{array} \right\} \]

Remark 8. For example, \( \psi_3 \) and \( \psi_4 \) are as following:

\[ \psi_3(m_1, m_2; b_1, b_2, b_3) = \sum_{j_1 \geq 0, \atop j_1 + j_2 \leq m_2} (-1)^{m_2-j_2} \frac{(m_1 + m_2 - j - \iota)!}{(m_1 - j - \iota)! (m_2 - j)! \iota!} \frac{b_1 b_2}{(b_1 + b_2)^2} \left( \frac{b_1 - b_2}{b_1 + b_2} \right)^{m_1 + m_2 - 2j} \left( \frac{b_2 - b_3}{b_2 + b_3} \right)^{m_2 + 1}, \]

\[ \psi_4(m_1, m_2, m_3; b_1, b_2, b_3, b_4) = \sum_{j_1, j_2, j_3, j_4, j_1 + j_2 \leq m_1, \atop j_1 + j_2 + j_3 \leq m_2, \atop j_1 + j_2 + j_3 + j_4 \leq m_3} (-1)^{m_2 + m_3 - j_1 - j_2 - j_3 - 2j_4} \frac{(m_1 + m_2 + m_3 - j_1 - j_2 - j_3 - 2j_4 - \iota_1 - \iota_2 - 2\iota_3)!}{(m_1 - j_2 - j_3 - j_4 - \iota_1 - \iota_2 - \iota_3)! \iota_1! \iota_2! \iota_3!} \frac{b_1 b_2}{(b_1 + b_2)^2} \left( \frac{b_1 - b_2}{b_1 + b_2} \right)^{m_1 + m_2 - 2(j_3 + j_4 + \iota_3)} \left( \frac{b_3 - b_4}{b_3 + b_4} \right)^{m_3 + 1}. \]

In the case of \( \psi_3 \), the indices \( j \) and \( \iota \) correspond to the indices \( j_\alpha \) and \( i_\beta \) in Proposition 7, respectively. We remark that \( A_2 = \{(-1)\} \) and \( C_3 = \{(-1, -1)\} \). In the same way, the indices \( j_\kappa \) (\( \kappa = 1, 2, 3, 4 \)) and \( i_\kappa \) (\( \kappa = 1, 2, 3 \)) correspond to the indices \( j_\alpha \) and \( i_\beta \) in Proposition 7 as following, respectively:
Proof of Proposition 7. By the equation (13) for \( k = 1 \) and Lemmas 6 and 1, we have

\[
\hat{F}_1^{(N)}(\rho, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = -\frac{1}{2a_1} \frac{1}{ip} 2^{2N-4}(b_{N-1} - b_N) \left\{ \prod_{j=1}^{N-2} (b_j b_{j+1}) \right\} \\
\times \frac{1}{Z_N Z_{N-1}} \sum_{\nu, \bar{\nu} = \pm 1} e \left( \frac{y}{\nu a_1} + \frac{x}{\bar{\nu} a_1} - \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right),
\]  

where we write

\[
Z_N = Z_N \left( \rho; b_1, \ldots, b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right),
\]

\[
Z_{N-1} = Z_{N-1} \left( \rho; b_1, \ldots, b_{N-1}; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-2} - h_{N-3}}{a_{N-2}} \right)
\]

for short notation. Then we discuss \( 1/Z_M(\rho; b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1}) \). We first have

\[
Z_M(\rho; b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1}) = \left\{ \prod_{j=1}^{M-1} (b_j + b_{j+1}) \right\} e \left( \sum_{j=1}^{M-1} \Theta_j \right) \\
\times \left[ 1 - \left\{ - \sum_{\alpha \in A_M} \alpha_1 \left( \prod_{j=1}^{M-2} \frac{b_j + \alpha_j \alpha_{j+1} b_{j+1}}{b_j + b_{j+1}} \right) \right. \right. \\
\left. \left. \times \frac{b_{M-1} + \alpha_{M-1} b_M}{b_{M-1} + b_M} e \left( \sum_{j=1}^{M-1} (\alpha_j - 1) \Theta_j \right) \right\} \right]
\]

for \( M \geq 2 \). Here, we remark that the absolute value of

\[
\sum_{\alpha \in A_M} \alpha_1 \left( \prod_{j=1}^{M-2} \frac{b_j + \alpha_j \alpha_{j+1} b_{j+1}}{b_j + b_{j+1}} \right) \frac{b_{M-1} + \alpha_{M-1} b_M}{b_{M-1} + b_M} e \left( \sum_{j=1}^{M-1} (\alpha_j - 1) \Theta_j \right)
\]
can be small enough when the positive number $m$ is large enough. Then we obtain

\[
Z_M(b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1}) = \frac{1}{\prod_{J=1}^{M-1} (b_J + b_{J+1})} \exp \left( - \sum_{J=1}^{M-1} \Theta_J \right) \\
\times \sum_{K=0}^\infty \left\{ \sum_{\alpha \in \Delta_M} (-\alpha_1) \left( \prod_{J=1}^{M-2} \frac{b_J + \alpha_J b_{J+1}}{b_J + b_{J+1}} \right) \times \frac{b_{M-1} + \alpha_{M-1} b_M}{b_{M-1} + b_M} \exp \left( \sum_{J=1}^{M-1} (\alpha_J - 1) \Theta_J \right) \right\}^K \\
= \frac{1}{\prod_{J=1}^{M-1} (b_J + b_{J+1})} \sum_{0 \leq j_0 < \infty} \left( \sum_{\alpha \in \Delta_M} j_\alpha \right)! \prod_{\alpha \in \Delta_M} (j_\alpha)! \\
\times (-1)^{\alpha \in \Delta_M} \left\{ \prod_{J=1}^{M-2} \frac{b_J + b_{J+1}}{b_J + b_{J+1}} \sum_{\alpha \in \Delta_M} \alpha_{J=J+1} \right\}^{j_\alpha} \\
\times \left( \frac{b_{M-1} - b_M}{b_{M-1} + b_M} \right)^{\sum_{\alpha \in \Delta_M^{(M-1)}} j_\alpha} \exp \left( - \sum_{J=1}^{M-1} \left\{ 2 \sum_{\alpha \in \Delta_M^{(J-1)}} j_\alpha + 1 \right\} \Theta_J \right).
We substitute this equation into the equation (18). Then we have

\[
\hat{f}_1^{(N)}(\rho, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = -\frac{1}{2a_1i\rho} e^{2\rho N-4} \left\{ \prod_{j=1}^{N-2} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right\} \\
\times \sum_{0 \leq j_0 < \infty} \sum_{0 \leq i_\beta < \infty} \frac{\left( \sum_{\alpha \in A_N} j_\alpha \right) ! \left( \sum_{\beta \in A_{N-1}} i_\beta \right) !}{\prod_{\alpha \in A_N} (j_\alpha !) \prod_{\beta \in A_{N-1}} (i_\beta !)} \\
\times (-1)^{j_\alpha} \sum_{\beta \in A_{N-1}^{(1)+}} \sum_{\alpha \in A_N^{(1)+}} i_\beta \left\{ \prod_{j=1}^{N-3} \left( \frac{b_j - b_{j-1}}{b_j + b_{j+1}} \right) \right\} \\
\times \left( \frac{b_{N-2} - b_{N-1}}{b_{N-2} + b_{N-1}} \right)^{a_{N-2} \alpha_{N-1}^{(1)+}} \left( \frac{b_{N-1} - b_N}{b_{N-1} + b_N} \right)^{-j_\alpha + 1} \\
\times \sum_{\nu, \bar{\nu} = \pm 1} e \left( \nu \frac{y}{a_1} + \bar{\nu} \frac{x}{a_1} - 2 \sum_{j=1}^{N-2} \left( \sum_{\alpha \in A_N^{(j)-}} j_\alpha + \sum_{\beta \in A_{N-1}^{(j)-}} i_\beta + 1 \right) \right) \frac{h_j - h_{j-1}}{a_j} \\
\times \left( \sum_{\alpha \in A_N^{(N-1)-}} j_\alpha + 1 \right) \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right)
\]

for \( N \geq 3 \). Now, we apply the inverse Fourier-Laplace transformation with respect to \( \tau \) to this equation, and we change the indices from \( j_\alpha (\alpha \in B_N) \) to \( m_k \) by the relations

\[
\hat{j}_{(1,\ldots,1,-1,1,\ldots,1)}^{(k)} = m_k - \sum_{\alpha \in C_N^{(k)-}} j_\alpha - \sum_{\beta \in A_{N-1}^{(k)-}} i_\beta \quad (1 \leq k \leq N-2),
\]

\[
\hat{j}_{(1,\ldots,1-1)} = m_{N-1} - \sum_{\alpha \in C_{N-1}^{(N-1)-}} j_\alpha.
\]

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Here we remark that

\[ F^{-1}_{\tau} \left[ \frac{e^{ips}}{i \rho} \right] (t) = H(t + s). \]

Then we obtain this proposition for \( N \geq 3 \). We can also prove the case of \( N = 2 \) in the same way.

3 The proof of the main result

In this section, we prove our main result. We first discuss the behavior of the function \( f^{(p)}(t, 0) \) near \( t = 0 \) in Lemmas 9 and 10.

**Lemma 9.** For \( p \geq 2 \) and \( \Theta_1 > y/a_1, \Theta_j > 0 \) \( (j = 2, \ldots, p - 1) \),

\[
\begin{aligned}
f^{(p)}(t, 0; b_1, \ldots, b_p; \Theta_1, \ldots, \Theta_{p-1}; y)
&= \begin{cases} 
0, & t \in \left[ 0, -\frac{y}{a_1} + 2 \sum_{j=1}^{p-1} \Theta_j \right), \\
-\frac{1}{a_1} q_p(b_1, \ldots, b_p), & t \in \left( -\frac{y}{a_1} + 2 \sum_{j=1}^{p-1} \Theta_j, -\frac{y}{a_1} + 2 \sum_{j=1}^{p-1} \Theta_j + \epsilon_p \right)
\end{cases}
\end{aligned}
\]

holds, where we define

\[ q_p(b_1, \ldots, b_p) = 2^{2p-4} \left\{ \prod_{j=1}^{p-2} \frac{b_j b_{j+1}}{(b_j + b_{j+1})^2} \right\} \frac{b_{p-1} - b_p}{b_{p-1} + b_p}, \]

\[ \epsilon_p = \epsilon_p(a_1; \Theta_1, \ldots, \Theta_{p-1}; y) = 2 \min \left\{ \frac{y}{a_1}, \Theta_1, \ldots, \Theta_{p-1} \right\}. \]

Proof. From

\[ \psi_p(0, \ldots, 0; b_1, \ldots, b_p) = q_p(b_1, \ldots, b_p) \]

and

\[ \nu \frac{y}{a_1} + 2 \sum_{j=1}^{p-1} (m_j + 1) \Theta_j \geq -\frac{y}{a_1} + 2 \sum_{j=1}^{p-1} \Theta_j + \epsilon_p \quad (\nu = \pm 1) \]

except for \((m_1, \ldots, m_{p-1}; \nu) = (0, \ldots, 0; -1)\) we obtain this lemma.
Lemma 10. For $N > k + 1 \geq 2$ and $\Theta_1 > y/a_1$, $\Theta_j > 0$ ($j = 2, \ldots, N - 1$),

\[
\sum_{p=k+1}^{N} f^{(p)}(t, 0; b_1, \ldots, b_p; \Theta_1, \ldots, \Theta_{p-1}; y)
\]

\[
= \begin{cases}
0, & t \in \left[0, -\frac{y}{a_1} + 2 \sum_{j=1}^{k} \Theta_j \right), \\
- \frac{1}{a_1} q_{k+1}(b_1, \ldots, b_{k+1}), & t \in \left(-\frac{y}{a_1} + 2 \sum_{j=1}^{k} \Theta_j, -\frac{y}{a_1} + 2 \sum_{j=1}^{k} \Theta_j + \tilde{\varepsilon}_k \right)
\end{cases}
\]

holds, where we define

\[
\tilde{\varepsilon}_k = \tilde{\varepsilon}_k(a_1; \Theta_1, \ldots, \Theta_k, \Theta_{k+1}; y) = 2 \min \left\{ \frac{y}{a_1}, \Theta_1, \ldots, \Theta_k, \Theta_{k+1} \right\}.
\]

Proof. By Lemma 9, we obtain this lemma easily. \(\square\)

Here, we state the proposition which is the key of the proof of our main result.

Proposition 11. Let $N \geq k + 1 \geq 2$ and $\Theta_1 > y/a_1$, $\Theta_j > 0$ ($j = 2, \ldots, N - 1$). Suppose $b_k \neq b_{k+1}$. Let $T > 0$. Put

\[
\tilde{v}(t) := \sum_{p=k+1}^{N} f^{(p)}(t, 0; b_1, \ldots, b_p; \Theta_1, \ldots, \Theta_{p-1}; y).
\]

Then the following holds:

- If $\tilde{v}(t) \equiv 0$ on $[0, T)$ then
  \[
  \Theta_k \geq \frac{1}{2} \left( T + \frac{y}{a_1} \right) - \sum_{j=1}^{k-1} \Theta_j. \quad (19)
  \]

- Assume $\tilde{v}(t) \not\equiv 0$ on $[0, T)$. Put $t_k := \inf \{ t \in [0, T) : \tilde{v}(t) \neq 0 \}$. Then there exist a constant $c_k$ and a positive constant $\varepsilon'_k > 0$ such that
  \[
  \tilde{v}(t) \equiv c_k \text{ on } (t_k, t_k + \varepsilon'_k).
  \]
Furthermore

\[ \Theta_k = \frac{1}{2} \left( t_k + \frac{y}{a_1} \right) - \sum_{j=1}^{k-1} \Theta_j, \]

(20)

\[ b_{k+1} = \frac{2^{2k-2} \prod_{j=1}^{k-1} (b_j b_{j+1}) + c_k a_1 \prod_{j=1}^{k-1} (b_j + b_{j+1})^2}{2^{2k-2} \prod_{j=1}^{k-1} (b_j b_{j+1}) - c_k a_1 \prod_{j=1}^{k-1} (b_j + b_{j+1})^2} b_k, \]

(21)

hold.

Proof. By Lemmas 9 and 10, there exists \( \varepsilon > 0 \) such that

\[ e_v(t) = \begin{cases} 0, & t \in \left[ 0, -\frac{y}{a_1} + 2 \sum_{j=1}^{k-1} \Theta_j \right), \\ -\frac{1}{a_1} q_{k+1}(b_1, \ldots, b_{k+1}), & t \in \left( -\frac{y}{a_1} + 2 \sum_{j=1}^{k-1} \Theta_j, -\frac{y}{a_1} + 2 \sum_{j=1}^{k-1} \Theta_j + \varepsilon \right) \end{cases} \]

holds. We remark that \( q_{k+1}(b_1, \ldots, b_{k+1}) \neq 0 \) since we assume that \( b_k \neq b_{k+1} \).

If \( \bar{v}(t) \equiv 0 \) on \([0, T)\) then we have

\[ T \leq -\frac{y}{a_1} + 2 \sum_{j=1}^{k-1} \Theta_j, \]

namely the equation (19). Hereafter we assume that \( \bar{v}(t) \not\equiv 0 \) on \([0, T)\). Then the constant \( t_k \) in this proposition satisfies

\[ t_k = -\frac{y}{a_1} + 2 \sum_{j=1}^{k-1} \Theta_j. \]

We obtain the equation (20) from this equation. On the other hand, we can take the constant \( \varepsilon'_k \) in this proposition as \( \varepsilon \), and the constant \( c_k \) in this proposition satisfies

\[ c_k = -\frac{1}{a_1} q_{k+1}(b_1, \ldots, b_{k+1}). \]

By this equation, we have the equation (21). \( \square \)
Next, we remark that there is a possibility that the same observation data can be obtained even if the unknown constants are different.

**Lemma 12.** Let \( a_j, b_j \ (j = 1, \ldots, N) \), \( h_j \ (j = 1, \ldots, N - 1) \), \( \tilde{a}_j, \tilde{b}_j \ (j = 1, \ldots, \tilde{N}) \), \( \tilde{h}_j \ (j = 1, \ldots, \tilde{N} - 1) \) be positive constants. Assume that \( h_j > h_{j-1} \) \( (j = 1, \ldots, N - 1) \) and \( \tilde{h}_j > \tilde{h}_{j-1} \) \( (j = 1, \ldots, \tilde{N} - 1) \), where we put \( h_0 := 0 \) and \( \tilde{h}_0 := 0 \). Let \( T > 0 \). Assume \( a_1 = \tilde{a}_1 \). Suppose

\[
\frac{h_j - h_{j-1}}{a_j} = \frac{\tilde{h}_j - \tilde{h}_{j-1}}{\tilde{a}_j} \quad (1 \leq j \leq N_T - 1), \quad b_j = \tilde{b}_j \quad (1 \leq j \leq N_T),
\]

where the natural number \( N_T \) satisfies

\[
T \leq -\frac{y}{a_1} + 2 \min \left\{ \sum_{j=1}^{N_T} \frac{h_j - h_{j-1}}{a_j}, \sum_{j=1}^{\tilde{N}_T} \frac{\tilde{h}_j - \tilde{h}_{j-1}}{\tilde{a}_j} \right\}.
\]

Then for \( t \in [0, T) \)

\[
u_N(t, 0; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = \nu_{\tilde{N}}(t, 0; \tilde{a}_1, \ldots, \tilde{a}_{\tilde{N}}; \tilde{b}_1, \ldots, \tilde{b}_{\tilde{N}}; \tilde{h}_1, \ldots, \tilde{h}_{\tilde{N}-1}; y)
\]

holds.

**Proof.** We remark that we have

\[
u_N(t, 0; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = \nu_1(t, 0; a_1; b_1; \ldots; y)
\]

by the definition of \( F^{(p)}_1(t, x) \) and Proposition 7. In particular, we have

\[
u_N(t, 0; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = \nu_1(t, 0; a_1; b_1; \ldots; y)
\]

by the definition of \( F^{(p)}_1(t, x) \) and Proposition 7. In particular, we have

\[
u_N(t, 0; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = \nu_1(t, 0; a_1; b_1; \ldots; y)
\]
and the last term vanishes for $t \in [0, T)$ by Lemma 10. Then we obtain this lemma.

By Lemma 12, we cannot identify $a_k$ and $h_k$ themselves even if the observation data on $[0, \infty)$ are given. We can identify only $b_k$ and $(h_k - h_{k-1})/a_k$. Then we try to reconstruct them.

Now, we state the process in order to reconstruct them.

**Theorem 13.** Suppose the constants $a_1, b_1, y$ are known. Assume $b_j \neq b_{j+1}$ for $j = 1, \ldots, N - 1$. Assume that the observation data $v(t) := u_N(t, 0)$ are given on $[0, T)$, where $u_N(t, x)$ is the solution of (1)–(6). Then $b_{k+1}$ and $(h_k - h_{k-1})/a_k (k = 1, \ldots, N_0 - 1)$ are reconstructed by the following process:

- The first step: Put $v_1(t) := (1/a_1)H(t - y/a_1) - v(t)$.

- The $(k+1)$-st step ($k = 1, 2, \ldots$): If $v_k(t) \equiv 0$ on $[0, T)$ then the process is finished. If $v_k(t) \not\equiv 0$ on $[0, T)$ then we carry out the following process: Put $t_k := \inf \{t \in [0, T) : v_k(t) \neq 0\}$. Then there exist a constant $c_k$ and a positive constant $\varepsilon'_k$ such that
  $$v_k(t) \equiv c_k \text{ on } (t_k, t_k + \varepsilon'_k).$$

The constants $(h_k - h_{k-1})/a_k$ and $b_{k+1}$ are reconstructed by

$$\frac{h_k - h_{k-1}}{a_k} := \frac{1}{2} \left( t_k + \frac{y}{a_1} \right) - \sum_{j=1}^{k-1} \frac{h_j - h_{j-1}}{a_j},$$

$$b_{k+1} := \frac{2^{2k-2} \prod_{j=1}^{k-1} (b_j b_{j+1}) + c_k a_1 \prod_{j=1}^{k-1} (b_j + b_{j+1})^2}{2^{2k-2} \prod_{j=1}^{k-1} (b_j b_{j+1}) - c_k a_1 \prod_{j=1}^{k-1} (b_j + b_{j+1})^2} b_k.$$
We define
\[ v_{k+1}(t) := v_k(t) + \frac{1}{a_1} \sum_{m_j} \psi_{k+1}(m_1, \ldots, m_k; b_1, \ldots, b_{k+1}) \]
\[ \sum_{j=1}^k (m_j + 1) \frac{h_j - h_{j-1}}{a_j} \leq \frac{1}{2} \left(T + \frac{a}{a_1}\right) \]
\[ \times \sum_{\nu = \pm 1} H \left( t - \left( \nu \frac{y}{a_1} + 2 \sum_{j=1}^k (m_j + 1) \frac{h_j - h_{j-1}}{a_j} \right) \right) \]
and go the next step, where \( \psi_{k+1} \) is defined in Proposition 7.

Furthermore, when the process is finished at the \((N_0 + 1)\)-st step, that is to say, \( v_{N_0}(t) \equiv 0 \) on \([0, T]\), we have either \( N = N_0 \) or the following:

\[ N > N_0 \text{ and } \frac{h_{N_0} - h_{N_0-1}}{a_{N_0}} \geq \frac{1}{2} \left(T + \frac{y}{a_1}\right) - \sum_{j=1}^{N_0-1} \frac{h_j - h_{j-1}}{a_j}. \]

Remark 14. For \( k = 2, 3, \ldots \), we have
\[ \frac{1}{2} \left( t_k + \frac{y}{a_1} \right) - \sum_{j=1}^{k-1} \frac{h_j - h_{j-1}}{a_j} = \frac{1}{2} (t_k - t_{k-1}), \]
that is to say, we can also reconstruct \((h_k - h_{k-1})/a_k\) by
\[ \frac{h_k - h_{k-1}}{a_k} = \frac{1}{2} (t_k - t_{k-1}). \]

Proof of Theorem 13. We first remark that
\[ u_N(t, 0; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) = u_1(t, 0; a_1; b_1; \ldots; y) \]
\[ - \sum_{k=2}^N f^{(k)}(t, 0; b_1, \ldots, b_k; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{k-1} - h_{k-2}}{a_{k-1}}; y) \]
holds as the same way in the proof of Lemma 12. Now, we put \( v_1(t) := (1/a_1)H(t - y/a_1) - v(t) \). Then we obtain
\[ v_1(t) = \sum_{k=2}^N f^{(k)}(t, 0; b_1, \ldots, b_k; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{k-1} - h_{k-2}}{a_{k-1}}; y) \]
since \( u_1(t, 0; a_1; b_1; \cdot; y) = (1/a_1)H(t - y/a_1) \) holds. From this equation and Propositions 7 and 11, we obtain this theorem. \( \square \)

4 Appendix

In this section, we discuss the case that the impedances of the adjacent media may be equal. In this case, the following lemma is a key lemma.

Lemma 15. Let \( N \geq \kappa + 1 \). If \( b_\kappa = b_{\kappa+1} \) then

\[
F_1^{(N)}(t, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y) =
\begin{cases}
0 & (N = \kappa + 1),
F_1^{(N-1)}(t, x; a_1, \ldots, a_{\kappa-1}, \overset{(\kappa)}{a}, a_{\kappa+2}, \ldots, a_N; b_1, \ldots, b_\kappa-1, b_\kappa, b_{\kappa+2}, \ldots, b_N; h_1, \ldots, h_\kappa-1, h_{\kappa+1}, h_{\kappa+2}, \ldots, h_{N-1}; y) & (N \geq \kappa + 2)
\end{cases}
\]

holds, where the constant \( \tilde{a} \) satisfies

\[
\frac{h_{\kappa+1} - h_{\kappa-1}}{\tilde{a}} = \frac{h_\kappa - h_{\kappa-1}}{a_\kappa} + \frac{h_{\kappa+1} - h_\kappa}{a_{\kappa+1}}.
\]

Proof. We remark that

\[
\hat{F}_1^{(N)}(\rho, x; a_1, \ldots, a_N; b_1, \ldots, b_N; h_1, \ldots, h_{N-1}; y)
= -\frac{1}{2a_1} \frac{1}{i\rho} 2^{2N-4}(b_{N-1} - b_N) \left\{ \prod_{J=1}^{N-2} (b_J b_{J+1}) \right\} \\
\times \frac{1}{Z_N} \frac{1}{Z_{N-1}} \sum_{\nu, \overline{\nu} = \pm 1} e \left( \nu \frac{y}{a_1} + \overline{\nu} \frac{x}{a_1} - \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right)
\]

which appears in the proof of Proposition 7, where we write

\[
Z_N = Z_N \left( \rho; b_1, \ldots, b_N; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-1} - h_{N-2}}{a_{N-1}} \right),
\]

\[
Z_{N-1} = Z_{N-1} \left( \rho; b_1, \ldots, b_{N-1}; \frac{h_1}{a_1}, \frac{h_2 - h_1}{a_2}, \ldots, \frac{h_{N-2} - h_{N-3}}{a_{N-2}} \right)
\]
for short notation. Hence
\[ F_1^{(\kappa+1)}(\rho, x; a_1, \ldots, a_{\kappa+1}; b_1, \ldots, b_{\kappa}; h_1, \ldots, h_{\kappa}; y) \equiv 0 \] (22)
holds since \( b_\kappa - b_{\kappa+1} = 0 \). Let \( N \geq \kappa + 2 \). We remark that we obtain
\[ Z_{\kappa+1}(\rho; b_1, \ldots, b_{\kappa+1}; \Theta_1, \ldots, \Theta_\kappa) = 2b_\kappa e(\Theta_\kappa)Z_\kappa(\rho; b_1, \ldots, b_\kappa; \Theta_1, \ldots, \Theta_{\kappa-1}) \]
and
\[ Z_M(\rho; b_1, \ldots, b_M; \Theta_1, \ldots, \Theta_{M-1})^{(\kappa)} = 2b_\kappa Z_{M-1}(\rho; b_1, \ldots, b_\kappa, b_{\kappa+2}, \ldots, b_M; \Theta_1, \ldots, \Theta_{\kappa+1}, \ldots, \Theta_{M-1}) \]
for \( M \geq \kappa + 2 \). Then we have
\[ b_\kappa^{(\kappa)}F_1^{(\kappa)}(N)_{1}(\rho, x; a_1, \ldots, a_N; b_1, \ldots, b_{N-1}; h_1, \ldots, h_{N-1}; y) \]
\[ = b_\kappa^{(\kappa)}F_1^{(\kappa-1)}(N)(\rho, x; a_1, \ldots, a_{N-1}, \tilde{a}, a_{\kappa+2}, \ldots, a_N; b_1, \ldots, b_{\kappa-1}, b_\kappa, b_{\kappa+2}, \ldots, b_N; h_1, \ldots, h_{\kappa-1}, h_\kappa+1, h_{\kappa+2}, \ldots, h_{N-1}; y). \] (23)
Hence we have this lemma by applying the inverse Fourier-Laplace transformations with respect to \( \rho = \tau - im\log(2 + |\tau|) \) to the equations (22) and (23).

**Lemma 16.** Let \( b_k \neq b_{k+1} \) for \( k = 1, \ldots, N - 1 \). Then
\[ f^{(\kappa)}(t, x; a_1, \ldots, a_{N-1}, \lambda_1, \ldots, a_{N}, \lambda_N; b_1, \ldots, b_N; \lambda_1, \ldots, b_\kappa, \lambda_\kappa, \ldots, b_{N-1}, \lambda_{N-1}, \ldots, b_N) \]
\[ = \begin{cases} 0, & \lambda_N \geq 2, \\ f^{(N)}(t, x; b_1, \ldots, b_N; \sum_{\mu=1}^{\lambda_1} h_{1,\mu} - h_{1,\mu-1}, \ldots, \sum_{\mu=1}^{\lambda_{N-1}} h_{N-1,\mu} - h_{N-1,\mu-1}; y), & \lambda_N = 1, \end{cases} \]

\[ M := \sum_{k=1}^{N} \lambda_k, \quad h_{1,0} := 0, \quad \text{and} \quad h_{k,0} := h_{k-1, \lambda_{k-1}} - 1, \quad \text{for} \quad k = 2, \ldots, N - 1. \]

Proof. We obtain this lemma from repeating Lemma 15. \( \square \)

By Lemma 16, we can only find out that the situation is as Figure 3 when the impedances of the adjacent media may be equal, where we reconstruct \( b_{k+1} \) and \( (h_k - h_{k-1})/a_k \) as Theorem 13 and the constants \( \tilde{a}_{k,\mu} \) and \( \tilde{h}_{k,\mu} \) satisfy

\[
\sum_{\mu=1}^{\lambda_k} \frac{\tilde{h}_{k,\mu} - \tilde{h}_{k,\mu-1}}{\tilde{a}_{k,\mu}} = \frac{h_k - h_{k-1}}{a_k}.
\]
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