Characterization of the critical Sobolev space on the optimal singularity at the origin

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Abstract

In the present paper, we investigate the optimal singularity at the origin for the functions belonging to the critical Sobolev space $H^{\frac{n}{p},p}(\mathbb{R}^n)$, $1<p<\infty$. With this purpose, we shall show the weighted Gagliardo-Nirenberg type inequality:

$$\|u\|_{L^q\left(\mathbb{R}^n\,;\,\frac{dx}{|x|^s}\right)} \le C\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}} q^{\frac{1}{p'}} \|u\|_{L^p(\mathbb{R}^n)}^{\frac{(n-s)p}{nq}} \|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}^{1-\frac{(n-s)p}{nq}}, \qquad \text{(GN)}$$

where C depends only on n and p. Here, $0\le s<n$ and $\tilde{p}\le q<\infty$ with some $\tilde{p}\in(p,\infty)$ determined only by n and p. Additionally, in the case $n\ge 2$ and $\frac{n}{n-1}\le p<\infty$, we can prove the growth orders for s as $s\uparrow n$ and for q as $q\to\infty$ are both optimal. (GN) allows us to prove the Trudinger type estimate with the homogeneous weight. Furthermore, it is obvious that (GN) can not hold with the weight $|x|^n$ itself. However, with a help of the logarithmic weight of the type $\left(\log\frac{1}{|x|}\right)^r |x|^n$ at the origin, we cover this critical weight. Simultaneously, we shall give the minimal exponent $r=\frac{q+p'}{p'}$ so that the continuous embedding can hold.

2000 Mathematics Subject Classification. Primary 46E35; Secondary 26D10.

Key words: Sobolev embedding theorem, Gagliardo-Nirenberg type inequality, Trudinger type inequality, Caffarelli-Kohn-Nirenberg inequality

1 Introduction and main results

In the present paper, we give some characterization of the functions in $H^{\frac{n}{p},p}(\mathbb{R}^n)$ with $n\in\mathbb{N}$ and $1<p<\infty$ called the critical Sobolev space in the sense that the continuous embedding $H^{\frac{n}{p},p}(\mathbb{R}^n)\hookrightarrow L^q(\mathbb{R}^n)$ holds for all $p\le q<\infty$, but $H^{\frac{n}{p},p}(\mathbb{R}^n)\not\subset L^\infty(\mathbb{R}^n)$ which implies that $H^{\frac{n}{p},p}(\mathbb{R}^n)$ possibly has a singularity at some point. Indeed, at least in the case $n\ge 2$ and $\frac{n}{n-1}\le p<\infty$, a compactly supported function such as $\left[\log\left(\frac{1}{|x|}\right)\right]^\tau$ with $0<\tau<\frac{1}{p'}$ at the origin implies the failure of the embedding in the case $q=\infty$, see Lemma 2.6 in Section 2. More precisely, Ozawa [12] gave the Gagliardo-Nirenberg type estimate of the following type:

$$\|u\|_{L^q(\mathbb{R}^n)} \le C\,q^{\frac{1}{p'}}\|u\|_{L^p(\mathbb{R}^n)}^{\frac{p}{q}}\|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}^{1-\frac{p}{q}} \qquad (1.1)$$

holds for all $u \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ and $p \leq q < \infty$, where C depends only on n and p, and $p' := \frac{p}{p-1}$ denotes the Hölder conjugate exponent of p. The inequality (1.1) was originally obtained by Ogawa [9] and Ogawa-Ozawa [10] in the case $p = 2$, i.e., $H^{\frac{n}{2},2}(\mathbb{R}^n)$. Moreover, we refer to Kozono-Wadade [6] which treats the marginal case of (1.1) as $p \to \infty$ in $H^{\frac{n}{p},p}(\mathbb{R}^n)$. In fact, the functions having bounded mean oscillation BMO can be expressed as the limit case of $H^{\frac{n}{p},p}(\mathbb{R}^n)$ with $p = \infty$ in some sense, and [6] proved (1.1) with $\|(-\Delta)^{\frac{n}{2p}} u\|_{L^p(\mathbb{R}^n)}$ replaced by $\|u\|_{BMO}$. In addition, Wadade [18] is also a generalization of (1.1) in terms of the Besov and the Triebel-Lizorkin spaces.

Our purpose in this article is to generalize (1.1) with the weighted Lebesgue space. In general, for a measurable weight function $w(x)$, we define $L^q\left(\mathbb{R}^n\,;\,\frac{dx}{w(x)}\right)$ as the function space endowed with the norm:

$$\|u\|_{L^q\left(\mathbb{R}^n\,;\,\frac{dx}{w(x)}\right)} := \left(\int_{\mathbb{R}^n} |u(x)|^q \frac{dx}{w(x)}\right)^{\frac{1}{q}} \quad \text{for } 1 < q < \infty.$$

We shall show the following inequality with the homogeneous weight $w(x) = |x|^s$:

Theorem 1.1. *Let $n \in \mathbb{N}$ and $1 < p < \infty$. Then there exist positive constants $\tilde{p} \in (p, \infty)$ and C which both depend only on n and p such that the inequality*

$$\|u\|_{L^q\left(\mathbb{R}^n\,;\,\frac{dx}{|x|^s}\right)} \leq C \left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}} q^{\frac{1}{p'}} \|u\|_{L^p(\mathbb{R}^n)}^{\theta} \|(-\Delta)^{\frac{n}{2p}} u\|_{L^p(\mathbb{R}^n)}^{1-\theta} \tag{1.2}$$

holds for all $u \in H^{\frac{n}{p},p}(\mathbb{R}^n)$, $0 \leq s < n$ and $\tilde{p} \leq q < \infty$, where $\theta := \frac{(n-s)p}{nq} \in (0,1)$. Furthermore, if $n \geq 2$ and $\frac{n}{n-1} \leq p < \infty$, the growth orders $\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}}$ as $s \uparrow n$ and $q^{\frac{1}{p'}}$ as $q \to \infty$ are both optimal in the sense that we can not replace $\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}}$ and $q^{\frac{1}{p'}}$ by $\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}-\varepsilon}$ and $q^{\frac{1}{p'}-\varepsilon}$ for any small $\varepsilon > 0$, respectively.

Remark 1.2. (i) If we do not pay attention to the growth orders of s and q, the inequality (1.2) itself is shown by Caffarelli-Kohn-Nirenberg [1] with the first order derivative, i.e., $\frac{n}{p} = 1$. However, we aim to obtain the optimal growth orders of s and q, and in fact we can prove that those orders are optimal in the case $n \geq 2$ and $\frac{n}{n-1} \leq p < \infty$. Unfortunately, we do not know the optimality in the cases $n = 1$ and $1 < p < \infty$, or $n \geq 2$ and $1 < p < \frac{n}{n-1}$ because of some technical reason, see Lemma 2.6 in Section 2. Moreover, we shall prove a weighted Trudinger type estimate as an effect of this growth order $q^{\frac{1}{p'}}$ as $q \to \infty$, which will be stated below.
(ii) The exponent $\tilde{p}$ actually can be chosen as $\tilde{p} := \max\{p+1, p'+1, n+1\}$. This restriction for the range of q will be used to prove Lemma 2.4.

As stated in Remark 1.2 (i), we can prove a weighted Trudinger type estimate as an application of Theorem 1.1:

Corollary 1.3. *Let $n \in \mathbb{N}$, $1 < p < \infty$, and define the function $\Phi_{n,p}$ by*

$$\Phi_{n,p}(t) := \exp t - \sum_{j=0}^{j_0-1} \frac{t^j}{j!} \quad \text{for } t \in \mathbb{R} \quad \text{with} \quad j_0 := \min\{j \in \mathbb{N}\,;\, p'j \geq \tilde{p}\},$$

where $\tilde{p} \in (p, \infty)$ is the positive constant given by Theorem 1.1. Then there exist two positive constants α and β which depend only on n and p such that

$$\int_{\mathbb{R}^n} \Phi_{n,p}\left(\alpha(n-s)|u(x)|^{p'}\right) \frac{dx}{|x|^s} \leq \frac{\beta}{n-s} \|u\|_{L^p(\mathbb{R}^n)}^{\frac{(n-s)p}{n}}$$

holds for all $u \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ with $\|(-\Delta)^{\frac{n}{2p}} u\|_{L^p(\mathbb{R}^n)} \leq 1$ and $0 \leq s < n$.

Remark 1.4. The procedure to get the Trudinger type estimate from the Gagliardo-Nirenberg type estimate was originally seen in [9], [10], [11] and [12]. Especially, [11] clarified the relationship between the positive constants in the Trudinger and the Gagliardo-Nirenberg type estimates with the exact formula, which shows these two inequalities are actually equivalent each other.

Next, we shall state the result which deals with the critical weight $s = n$ in Theorem 1.1. Obviously, the inequality (1.1) can not hold with the weight $|x|^n$ itself. However, with a help of the logarithmic weight, we shall show the following inequality:

Theorem 1.5. *Let $n \in \mathbb{N}$, $1 < p < r < \infty$ and $p \leq q \leq (r-1)p'$. Then there exists a positive constant C which depends only on n, p, q and r such that*

$$\|u\|_{L^q\left(\mathbb{R}^n\,;\, \frac{dx}{w_r(x)}\right)} \leq C\|u\|_{H^{\frac{n}{p},p}(\mathbb{R}^n)} \tag{1.3}$$

holds for all $u \in H^{\frac{n}{p},p}(\mathbb{R}^n)$, where the weight function $w_r(x)$ is given by

$$w_r(x) := \left[\log\left(e + \frac{1}{|x|}\right)\right]^r |x|^n. \tag{1.4}$$

Furthermore, if $n \geq 2$ and $\frac{n}{n-1} \leq p < \infty$, the bound $(r-1)p'$ is sharp in the sense that the inequality (1.3) no longer holds provided $q > (r-1)p'$.

Remark 1.6. (i) There are more general results of such embeddings in case of Besov and Triebel-Lizorkin spaces including the Sobolev scale, cf. [5] and [7, 8], but restricted to Muckenhoupt weights or so-called admissible weights, the former allows the weight to have a local singularity, while the latter is some class of smooth functions. We emphasize that these classes of weight functions do not cover the above limiting situation. Indeed, it is well-known that the weight $\frac{1}{w_r}$ as in (1.4) no longer belongs to even the class of Muckenhoupt weights.

(ii) Since there exists an upper bound $(r-1)p'$ with respect to q so that the inequality (1.3) holds, we can not deduce the Trudinger type estimate from the inequality (1.3) unlike the case with the subcritical weight $|x|^s$ with $0 \leq s < n$. We additionally note that the critical exponent $q = (r-1)p'$ comes from the following computation:

$$\left(\int_{\{|x|<\frac{1}{2}\}} \left[\log\left(\frac{1}{|x|}\right)\right]^{\frac{q}{p'}} \frac{dx}{w_r(x)}\right)^{\frac{1}{q}} = \infty$$

provided that $q \geq (r-1)p'$. Here, note that the marginal case $q = (r-1)p'$ is included in the above observation. However, we shall overcome this difficulty to get (1.3) by using the generalized Young inequality by O'Neil [13], see Theorem B in Chapter 2.

Finally let us describe the organization of this article. Section 2 is devoted to prepare the several lemmas for the proof of main theorems, and we shall show our theorems in Section 3.

2 Preliminaries

This chapter is devoted to prepare several lemmas for the proof of main theorems. First, let us introduce the higher-dimensional Hardy inequality proved by Drábek-Heinig-Kufner [2] :

Theorem A. (i) *Let U_1 and V_1 be non-negative weight functions in $\mathbb{R}^n$, and $1 < p \leq q < \infty$. Then the inequality*

$$\left(\int_{\mathbb{R}^n} \left(\int_{\{|y|<|x|\}} f(y)dy \right)^q U_1(x)dx \right)^{\frac{1}{q}} \leq C_1 \left(\int_{\mathbb{R}^n} f(x)^p V_1(x)dx \right)^{\frac{1}{p}}$$

holds for all $f \geq 0$ a.e. in $\mathbb{R}^n$ if and only if

$$A_1 := \sup_{R>0} \left(\int_{\{|x|>R\}} U_1(x)dx \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} V_1(x)^{-(p'-1)}dx \right)^{\frac{1}{p'}} < \infty.$$

Moreover, the constant C_1 can be taken as

$$C_1 = (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_1.$$

(ii) *Let U_2 and V_2 be non-negative weight functions in $\mathbb{R}^n$, and $1 < p \leq q < \infty$. Then the inequality*

$$\left(\int_{\mathbb{R}^n} \left(\int_{\{|y|>|x|\}} f(y)dy \right)^q U_2(x)dx \right)^{\frac{1}{q}} \leq C_2 \left(\int_{\mathbb{R}^n} f(x)^p V_2(x)dx \right)^{\frac{1}{p}}$$

holds for all $f \geq 0$ a.e. in $\mathbb{R}^n$ if and only if

$$A_2 := \sup_{R>0} \left(\int_{\{|x|<R\}} U_2(x)dx \right)^{\frac{1}{q}} \left(\int_{\{|x|>R\}} V_2(x)^{-(p'-1)}dx \right)^{\frac{1}{p'}} < \infty.$$

Moreover, the constant C_2 can be taken as

$$C_2 = (p')^{\frac{1}{p'}} p^{\frac{1}{q}} A_2.$$

By scaling and changing a variable, we have the following variant of Theorem A :

Theorem A′. (i) *Let* U_1 *and* V_1 *be non-negative weight functions in* $\mathbb{R}^n$*, and* $1 < p \le q < \infty$*. Then the inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{\{2|y|<|x|\}} f(y)dy\right)^q U_1(x)dx\right)^{\frac{1}{q}} \le \tilde{C}_1\left(\int_{\mathbb{R}^n} f(x)^p V_1(x)dx\right)^{\frac{1}{p}}$$

holds for all $f \ge 0$ *a.e. in* $\mathbb{R}^n$ *if and only if*

$$\tilde{A}_1 := \sup_{R>0}\left(\int_{\{|x|>2R\}} U_1(x)dx\right)^{\frac{1}{q}}\left(\int_{\{|x|<R\}} V_1(x)^{-(p'-1)}dx\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the constant $\tilde{C}_1$ *can be taken as*

$$\tilde{C}_1 = (p')^{\frac{1}{p'}} p^{\frac{1}{q}} \tilde{A}_1.$$

(ii) *Let* U_2 *and* V_2 *be non-negative weight functions in* $\mathbb{R}^n$*, and* $1 < p \le q < \infty$*. Then the inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{\{|y|>2|x|\}} f(y)dy\right)^q U_2(x)dx\right)^{\frac{1}{q}} \le \tilde{C}_2\left(\int_{\mathbb{R}^n} f(x)^p V_2(x)dx\right)^{\frac{1}{p}}$$

holds for all $f \ge 0$ *a.e. in* $\mathbb{R}^n$ *if and only if*

$$\tilde{A}_2 := \sup_{R>0}\left(\int_{\{|x|<R\}} U_2(x)dx\right)^{\frac{1}{q}}\left(\int_{\{|x|>2R\}} V_2(x)^{-(p'-1)}dx\right)^{\frac{1}{p'}} < \infty.$$

Moreover, the constant $\tilde{C}_2$ *can be taken as*

$$\tilde{C}_2 = (p')^{\frac{1}{p'}} p^{\frac{1}{q}} \tilde{A}_2.$$

In what follows, C denotes a positive constant which may vary from line to line. We shall show key lemmas by applying Theorem A′ below. The idea of this procedure was inspired by Rakotondratsimba [14, 15], who proved the weighted Young inequalities for convolutions towards the functions behaving like the Riesz potential $|x|^{-(n-\alpha)}$ with $0 < \alpha < n$. However, we need to consider not only the Riesz potential but also φ, ψ and the Bessel potential G_α which are defined below, and for the purpose to get exact growth orders concerning s and q, we investigate these individual kernels precisely. Lemma 2.1–Lemma 2.4 will be used to prove Theorem 1.1 in Section 3.

Lemma 2.1. *Let* $n \in \mathbb{N}$, $1 < p < \infty$ *and* $\varphi(x) = e^{-\pi|x|^2}$*. Then there exists a positive constant* C *which depends only on* n *and* p *such that*

$$\|\varphi * u\|_{L^q\left(\mathbb{R}^n;\,\frac{dx}{|x|^s}\right)} \le C\left(\frac{1}{n-s}\right)^{\frac{1}{q}} \|u\|_{L^p(\mathbb{R}^n)} \tag{2.1}$$

holds for all $u \in L^p(\mathbb{R}^n)$, $0 \le s < n$ *and* $p \le q < \infty$.

Proof. Obviously, it is enough to show the inequality (2.1) for non-negative functions. First, we decompose the integral into three parts:

$$\int_{\mathbb{R}^n} (\varphi * u)(x)^q \frac{dx}{|x|^s} \le 3^q \Bigg[\int_{\mathbb{R}^n} \left(\int_{\{|y|<\frac{|x|}{2}\}} \varphi(x-y)u(y)dy \right)^q \frac{dx}{|x|^s}$$
$$+ \int_{\mathbb{R}^n} \left(\int_{\{\frac{|x|}{2}\le |y|\le 2|x|\}} \varphi(x-y)u(y)dy \right)^q \frac{dx}{|x|^s} + \int_{\mathbb{R}^n} \left(\int_{\{|y|>2|x|\}} \varphi(x-y)u(y)dy \right)^q \frac{dx}{|x|^s} \Bigg]$$
$$=: 3^q (S_1 + S_2 + S_3).$$

We first estimate S_1. Note that $|y| < \frac{|x|}{2}$ implies $\frac{|x|}{2} < |x-y|$. Hence, we see

$$\int_{\{|y|<\frac{|x|}{2}\}} \varphi(x-y)u(y)dy \le \left(\sup_{\{\frac{|x|}{2}<|z|\}} \varphi(z) \right) \int_{\{|y|<\frac{|x|}{2}\}} u(y)dy = e^{-\frac{\pi|x|^2}{4}} \int_{\{|y|<\frac{|x|}{2}\}} u(y)dy.$$

Thus we have

$$S_1 \le \int_{\mathbb{R}^n} \left(\int_{\{|y|<\frac{|x|}{2}\}} u(y)dy \right)^q e^{-\frac{\pi q|x|^2}{4}} |x|^{-s} dx.$$

To apply Theorem A′ (i), we need to check the following condition:

$$\left(\int_{\{2R<|x|\}} e^{-\frac{\pi q|x|^2}{4}} |x|^{-s} dx \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx \right)^{\frac{1}{p'}} \le \tilde{A}_1 \tag{2.2}$$

holds for all $R > 0$, where $\tilde{A}_1$ is independent of R. Indeed, once (2.2) has been established, the Hardy inequality yields that

$$S_1^{\frac{1}{q}} \le (p')^{\frac{1}{p'}} p^{\frac{1}{q}} \tilde{A}_1 \|u\|_{L^p(\mathbb{R}^n)} \le C\tilde{A}_1 \|u\|_{L^p(\mathbb{R}^n)},$$

where C is independent of p and q since $p \le q$ and $\sup_{1<p<\infty} p^{\frac{1}{p}} < \infty$.

To check the condition (2.2), we distinguish two cases:

Case 1. We assume $R \ge 1$. In this case, we have

$$\left(\int_{\{2R<|x|\}} e^{-\frac{\pi q|x|^2}{4}} |x|^{-s} dx \right)^{\frac{1}{q}} \le \left(\int_{\{2R<|x|\}} e^{-\frac{\pi qR|x|}{2}} |x|^{-s} dx \right)^{\frac{1}{q}}$$
$$= \left(\int_{\{2R<|x|\}} e^{-\frac{\pi qR|x|}{4}} e^{-\frac{\pi qR|x|}{4}} |x|^{-s} dx \right)^{\frac{1}{q}} \le e^{-\frac{\pi R^2}{2}} \left(\int_{\{2R<|x|\}} e^{-\frac{\pi qR|x|}{4}} |x|^{-s} dx \right)^{\frac{1}{q}}$$
$$\le e^{-\frac{\pi R^2}{2}} \left(\int_{\{2<|x|\}} e^{-\frac{\pi|x|}{4}} dx \right)^{\frac{1}{q}} \le Ce^{-\frac{\pi R^2}{2}}.$$

By using the above estimate, we obtain

$$\left(\int_{\{2R<|x|\}} e^{-\frac{\pi q|x|^2}{4}}|x|^{-s}dx\right)^{\frac{1}{q}}\left(\int_{\{|x|<R\}} dx\right)^{\frac{1}{p'}} \le Ce^{-\frac{\pi R^2}{2}}R^{\frac{n}{p'}} \le C \tag{2.3}$$

for all $R \ge 1$.

Case 2. We assume $0 < R < 1$. In this case, we have

$$\begin{aligned}&\left(\int_{\{2R<|x|\}} e^{-\frac{\pi q|x|^2}{4}}|x|^{-s}dx\right)^{\frac{1}{q}} \le \left(\int_{\mathbb{R}^n} e^{-\frac{\pi q|x|^2}{4}}|x|^{-s}dx\right)^{\frac{1}{q}}\\
&\le \left(\int_{\{|x|<1\}} e^{-\frac{\pi q|x|^2}{4}}|x|^{-s}dx\right)^{\frac{1}{q}} + \left(\int_{\{|x|\ge 1\}} e^{-\frac{\pi q|x|^2}{4}}|x|^{-s}dx\right)^{\frac{1}{q}}\\
&\le \left(\int_{\{|x|<1\}} |x|^{-s}dx\right)^{\frac{1}{q}} + \left(\int_{\{|x|\ge 1\}} e^{-\frac{\pi|x|^2}{4}}dx\right)^{\frac{1}{q}} \le C\left(\frac{1}{n-s}\right)^{\frac{1}{q}} + C.\end{aligned}$$

Thus we have,

$$\left(\int_{\{2R<|x|\}} e^{-\frac{\pi q|x|^2}{4}}|x|^{-s}dx\right)^{\frac{1}{q}}\left(\int_{\{|x|<R\}} dx\right)^{\frac{1}{p'}} \le C\left(\frac{1}{n-s}\right)^{\frac{1}{q}} \tag{2.4}$$

for all $0 < R < 1$. Therefore, combining (2.3) with (2.4), we can take $\tilde{A}_1 = C\left(\frac{1}{n-s}\right)^{\frac{1}{q}}$.

Next, we estimate S_3 in the similar way as S_1. Note that $|y| > 2|x|$ implies $\frac{|y|}{2} < |x-y|$. Hence, we see

$$\int_{\{|y|>2|x|\}} \varphi(x-y)u(y)dy \le \int_{\{|y|>2|x|\}} \left(\sup_{\{\frac{|y|}{2}<|z|\}} \varphi(z)\right) u(y)dy = \int_{\{|y|>2|x|\}} e^{-\frac{\pi|y|^2}{4}}u(y)dy.$$

Hence, we have

$$S_3 \le \int_{\mathbb{R}^n}\left(\int_{\{|y|>2|x|\}} h(y)dy\right)^q |x|^{-s}dx, \quad \text{where} \quad h(y) := e^{-\frac{\pi|y|^2}{4}}u(y).$$

To apply Theorem A′ (ii), we need to check the following condition:

$$\left(\int_{\{|x|<R\}} |x|^{-s}dx\right)^{\frac{1}{q}}\left(\int_{\{2R<|x|\}} \left(e^{\frac{\pi p|x|^2}{4}}\right)^{-(p'-1)} dx\right)^{\frac{1}{p'}} \le \tilde{A}_2 \tag{2.5}$$

holds for all $R > 0$, where $\tilde{A}_2$ is independent of R. Indeed, once (2.5) has been established, the Hardy inequality yields that

$$S_3^{\frac{1}{q}} \le (p')^{\frac{1}{p'}}p^{\frac{1}{q}}\tilde{A}_2\left(\int_{\mathbb{R}^n} h(x)^p e^{\frac{\pi p|x|^2}{4}}dx\right)^{\frac{1}{p}} \le C\tilde{A}_2\|u\|_{L^p(\mathbb{R}^n)}.$$

To check the condition (2.5), we distinguish two cases:

Case 1. We assume $R \geq 1$. Then we have

$$\left(\int_{\{|x|<R\}} |x|^{-s} dx \right)^{\frac{1}{q}} \leq C \left(\frac{1}{n-s} \right)^{\frac{1}{q}} R^{\frac{n-s}{q}} \leq C \left(\frac{1}{n-s} \right)^{\frac{1}{q}} R^n.$$

On the other hand, we see

$$\begin{aligned} \left(\int_{\{2R<|x|\}} e^{-\frac{\pi p'|x|^2}{4}} dx \right)^{\frac{1}{p'}} &\leq \left(\int_{\{2R<|x|\}} e^{-\frac{\pi p' R|x|}{2}} dx \right)^{\frac{1}{p'}} = \left(\int_{\{2R<|x|\}} e^{-\frac{\pi p' R|x|}{4}} e^{-\frac{\pi p' R|x|}{4}} dx \right)^{\frac{1}{p'}} \\ &\leq e^{-\frac{\pi R^2}{2}} \left(\int_{\{2R<|x|\}} e^{-\frac{\pi p' R|x|}{4}} dx \right)^{\frac{1}{p'}} \leq e^{-\frac{\pi R^2}{2}} \left(\int_{\{2<|x|\}} e^{-\frac{\pi |x|}{4}} dx \right)^{\frac{1}{p'}} \leq C e^{-\frac{\pi R^2}{2}}. \end{aligned}$$

Thus by using above estimates, we obtain

$$\left(\int_{\{|x|<R\}} |x|^{-s} dx \right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} e^{-\frac{\pi p'|x|^2}{4}} dx \right)^{\frac{1}{p'}} \leq C \left(\frac{1}{n-s} \right)^{\frac{1}{q}} R^n e^{-\frac{\pi R^2}{2}} \leq C \left(\frac{1}{n-s} \right)^{\frac{1}{q}} \tag{2.6}$$

for all $R \geq 1$.

Case 2. We assume $0 < R < 1$. In this case, we see

$$\begin{aligned} &\left(\int_{\{|x|<R\}} |x|^{-s} dx \right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} e^{-\frac{\pi p'|x|^2}{4}} dx \right)^{\frac{1}{p'}} \\ &\leq \left(\int_{\{|x|<1\}} |x|^{-s} dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} e^{-\frac{\pi p'|x|^2}{4}} dx \right)^{\frac{1}{p'}} \leq C \left(\frac{1}{n-s} \right)^{\frac{1}{q}} \end{aligned} \tag{2.7}$$

for all $0 < R < 1$. Thus combining (2.6) with (2.7), we can take $\tilde{A}_2 = C \left(\frac{1}{n-s} \right)^{\frac{1}{q}}$.

Finally, we estimate S_2. Note that $\frac{|x|}{2} \leq |y| \leq 2|x|$ and $2^k \leq |x| < 2^{k+1}$ imply that $2^{k-1} \leq |y| < 2^{k+2}$, and take $r := \frac{nq}{n-s} \in [q, \infty)$. Then by the Hölder inequality and the Young inequality, we see

$$\begin{aligned} S_2 &= \sum_{k \in \mathbb{Z}} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} \varphi(x-y) u(y) dy \right)^q |x|^{-s} dx \\ &\leq \sum_{k \in \mathbb{Z}} 2^{-ks} \int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} \varphi(x-y) u(y) dy \right)^q dx \\ &\leq \sum_{k \in \mathbb{Z}} 2^{-ks} \mu \left(\left\{ 2^k \leq |x| < 2^{k+1} \right\} \right)^{1-\frac{q}{r}} \left(\int_{\{2^k \leq |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} \varphi(x-y) u(y) dy \right)^r dx \right)^{\frac{q}{r}} \end{aligned}$$

$$\le C\sum_{k\in\mathbb{Z}}2^{-ks+kn\left(1-\frac{q}{r}\right)}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\varphi(x-y)u(y)\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}(y)dy\right)^r dx\right)^{\frac{q}{r}}$$

$$= C\sum_{k\in\mathbb{Z}}\|\varphi * u\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|_{L^r(\mathbb{R}^n)}^q \le C\|\varphi\|_{L^{\tilde r}(\mathbb{R}^n)}^q\sum_{k\in\mathbb{Z}}\|u\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|_{L^p(\mathbb{R}^n)}^q$$

$$\le C^q\sum_{k\in\mathbb{Z}}\left(\int_{\{2^{k-1}\le|x|<2^{k+2}\}}u(x)^p dx\right)^{\frac{q}{p}} \le C^q\left(\sum_{k\in\mathbb{Z}}\int_{\{2^{k-1}\le|x|<2^{k+2}\}}u(x)^p dx\right)^{\frac{q}{p}} = C^q\|u\|_{L^p(\mathbb{R}^n)}^q,$$

where $\tilde r\in[1,\infty)$ is defined by $\frac{1}{r}+1=\frac{1}{\tilde r}+\frac{1}{p}$, and μ denotes the Lebesgue measure. In the above estimates, we used the fact that $\max_{1\le\tilde r\le\infty}\|\varphi\|_{L^{\tilde r}(\mathbb{R}^n)}<\infty$ since $\varphi\in\mathcal{S}(\mathbb{R}^n)$. Thus we finish the proof. □

We proceed to prove the following lemma:

Lemma 2.2. *Let $n\in\mathbb{N}$, $0<\alpha<n$ and define the function ψ by*

$$\psi(x) := \int_{\mathbb{R}^n}\left||x|^{-(n-\alpha)}-|x-y|^{-(n-\alpha)}\right|e^{-\pi|y|^2}dy \quad \text{for } x\in\mathbb{R}^n\setminus\{0\}.$$

Then there exists a positive constant C which depends only on n and α such that ψ satisfies

$$\psi(x)\le C\min\{|x|^{-(n-\alpha)},|x|^{-(n-\alpha+1)}\} \quad \text{for all } x\in\mathbb{R}^n\setminus\{0\}.$$

Proof. We first decompose ψ into three integrals:

$$\psi(x) = \int_{\{|y|\le\frac{|x|}{2}\}}\left||x|^{-(n-\alpha)}-|x-y|^{-(n-\alpha)}\right|e^{-\pi|y|^2}dy$$
$$+\int_{\{\frac{|x|}{2}<|y|<2|x|\}}\left||x|^{-(n-\alpha)}-|x-y|^{-(n-\alpha)}\right|e^{-\pi|y|^2}dy$$
$$+\int_{\{|y|\ge2|x|\}}\left||x|^{-(n-\alpha)}-|x-y|^{-(n-\alpha)}\right|e^{-\pi|y|^2}dy =: \psi_1(x)+\psi_2(x)+\psi_3(x).$$

First, we estimate ψ_1. For $|y|\le\frac{|x|}{2}$, we see

$$\left||x|^{-(n-\alpha)}-|x-y|^{-(n-\alpha)}\right| = \left|\int_0^1\frac{d}{d\tau}\left[|x-\tau y|^{-(n-\alpha)}\right]d\tau\right| \le (n-\alpha)|y|\int_0^1|x-\tau y|^{-(n-\alpha+1)}d\tau$$
$$=(n-\alpha)|y||x|^{-(n-\alpha+1)}\int_0^1\left|\frac{x}{|x|}-\tau\frac{y}{|x|}\right|^{-(n-\alpha+1)}d\tau \le (n-\alpha)|y||x|^{-(n-\alpha+1)}\int_0^1\left|1-\tau\frac{|y|}{|x|}\right|^{-(n-\alpha+1)}d\tau$$
$$\le(n-\alpha)|y||x|^{-(n-\alpha+1)}\int_0^1\left(1-\frac{\tau}{2}\right)^{-(n-\alpha+1)}d\tau = C|y||x|^{-(n-\alpha+1)}.$$

Thus on one hand, we see

$$\psi_1(x)\le C|x|^{-(n-\alpha+1)}\int_{\{|y|\le\frac{|x|}{2}\}}|y|e^{-\pi|y|^2}dy \le C|x|^{-(n-\alpha+1)}\int_{\mathbb{R}^n}|y|e^{-\pi|y|^2}dy = C|x|^{-(n-\alpha+1)},$$

and on the other hand, we obtain

$$\psi_1(x) \le C|x|^{-(n-\alpha+1)} \int_{\{|y|\le \frac{|x|}{2}\}} |y| e^{-\pi|y|^2} dy \le C|x|^{-(n-\alpha)} \int_{\{|y|\le \frac{|x|}{2}\}} e^{-\pi|y|^2} dy$$
$$\le C|x|^{-(n-\alpha)} \int_{\mathbb{R}^n} e^{-\pi|y|^2} dy = C|x|^{-(n-\alpha)}.$$

Next, we estimate ψ_2 as follows:

$$\psi_2(x) \le |x|^{-(n-\alpha)} \int_{\{\frac{|x|}{2}<|y|<2|x|\}} e^{-\pi|y|^2} dy + \int_{\{\frac{|x|}{2}<|y|<2|x|\}} |x-y|^{-(n-\alpha)} e^{-\pi|y|^2} dy$$
$$\le |x|^{-(n-\alpha)} e^{-\frac{\pi|x|^2}{4}} \int_{\{|y|<2|x|\}} dy + e^{-\frac{\pi|x|^2}{4}} \int_{\{\frac{|x|}{2}<|y|<2|x|\}} |x-y|^{-(n-\alpha)} dy$$
$$\le C|x|^{\alpha} e^{-\frac{\pi|x|^2}{4}} + e^{-\frac{\pi|x|^2}{4}} \int_{\{|z|<3|x|\}} |z|^{-(n-\alpha)} dz = C|x|^{\alpha} e^{-\frac{\pi|x|^2}{4}}$$
$$\le C \min\{|x|^{-(n-\alpha)}, |x|^{-(n-\alpha+1)}\}.$$

Finally, we estimate ψ_3.

$$\psi_3(x) \le |x|^{-(n-\alpha)} \int_{\{|y|\ge 2|x|\}} e^{-\pi|y|^2} dy + \int_{\{|y|\ge 2|x|\}} |x-y|^{-(n-\alpha)} e^{-\pi|y|^2} dy =: \psi_{31}(x) + \psi_{32}(x).$$

On one hand, we see

$$\psi_{31}(x) \le |x|^{-(n-\alpha)} \int_{\mathbb{R}^n} e^{-\pi|y|^2} dy = C|x|^{-(n-\alpha)},$$

and on the other hand, for $|x| \ge 1$ we have

$$\psi_{31}(x) \le |x|^{-(n-\alpha)} \int_{\{|y|\ge 2|x|\}} e^{-2\pi|x||y|} dy \le |x|^{-(n-\alpha)} e^{-2\pi|x|^2} \int_{\{|y|\ge 2|x|\}} e^{-\pi|x||y|} dy$$
$$\le |x|^{-(2n-\alpha)} e^{-2\pi|x|^2} \int_{\mathbb{R}^n} e^{-\pi|z|} dz \le C|x|^{-(n-\alpha+1)}.$$

Next, note that $|y| \ge 2|x|$ implies $|x-y| \ge |x|$. Then we see

$$\psi_{32}(x) \le |x|^{-(n-\alpha)} \int_{\{|y|\ge 2|x|\}} e^{-\pi|y|^2} dy = \psi_{31}(x).$$

Thus the estimate of ψ_{32} can be reduced to the estimate of $\psi_{31}(x)$, and we finish the proof. □

We can get the following lemma by applying Lemma 2.2:

Lemma 2.3. *Let $n \in \mathbb{N}$, $1 < p < \infty$ and set the function ψ as in Lemma 2.2 with $\alpha = \frac{n}{p}$. Then there exists a positive constant C which depends only on n and p such that the estimate*

$$\|\psi\|_{L^r(\mathbb{R}^n)} \le C \left(\frac{1}{p'-r} + \frac{1}{(n+p')r - np'} \right)^{\frac{1}{r}}$$

holds for all $\frac{np'}{n+p'} < r < p'$.

Proof. By using Lemma 2.2, we see that

$$\|\psi\|_{L^r(\mathbb{R}^n)}^r = \int_{\{|x|<1\}} \psi(x)^r dx + \int_{\{|x|\geq 1\}} \psi(x)^r dx$$
$$\leq C\left(\int_{\{|x|<1\}} |x|^{-\frac{nr}{p'}} dx + \int_{\{|x|\geq 1\}} |x|^{-\left(\frac{n}{p'}+1\right)r} dx\right) \leq C\left(\frac{1}{p'-r} + \frac{1}{(n+p')r - np'}\right).$$

□

Furthermore, by applying Lemmas 2.2 and 2.3, we prove the following:

Lemma 2.4. *Let $n \in \mathbb{N}$, $1 < p < \infty$ and set the function ψ as in Lemma 2.2 with $\alpha = \frac{n}{p}$. Then there exist positive constants $\tilde{p} \in (p, \infty)$ and C which both depend only on n and p such that the inequality*

$$\|\psi * u\|_{L^q\left(\mathbb{R}^n : \frac{dx}{|x|^s}\right)} \leq C\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}} q^{\frac{1}{p'}} \|u\|_{L^p(\mathbb{R}^n)}$$

holds for all $u \in L^p(\mathbb{R}^n)$, $0 \leq s < n$ and $\tilde{p} \leq q < \infty$.

We prove Lemma 2.4 in the similar way to the proof of Lemma 2.1. However, it should be noted that the function ψ has a singularity at the origin which is a major difference between ψ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We may assume u is non-negative, and we take $\tilde{p} := \max\{p+1, p'+1, n+1\} \leq q < \infty$.

The integral is decomposed into three parts:

$$\int_{\mathbb{R}^n} (\psi * u)(x)^q |x|^{-s} dx \leq 3^q \left[\int_{\mathbb{R}^n} \left(\int_{\{|y|<\frac{|x|}{2}\}} \psi(x-y)u(y)dy\right)^q |x|^{-s} dx\right.$$
$$+ \int_{\mathbb{R}^n} \left(\int_{\{\frac{|x|}{2}\leq |y| \leq 2|x|\}} \psi(x-y)u(y)dy\right)^q |x|^{-s} dx + \left.\int_{\mathbb{R}^n} \left(\int_{\{|y|>2|x|\}} \psi(x-y)u(y)dy\right)^q |x|^{-s} dx\right]$$
$$=: 3^q(T_1 + T_2 + T_3).$$

First, we estimate T_1. Note that $|y| < \frac{|x|}{2}$ implies $\frac{|x|}{2} < |x-y|$. Hence, we see

$$T_1 \leq \int_{\mathbb{R}^n} \left(\int_{\{|y|<\frac{|x|}{2}\}} u(y)dy\right)^q \tilde{\psi}(x)^q |x|^{-s} dx, \quad \text{where} \quad \tilde{\psi}(x) := \sup_{\{|z|>\frac{|x|}{2}\}} \psi(z).$$

To apply Theorem A′ (i), we need to check the following condition:

$$\left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^q |x|^{-s} dx\right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx\right)^{\frac{1}{p'}} \leq \tilde{A}_1 \tag{2.8}$$

holds for all $R > 0$. Indeed, once (2.8) has been established, the Hardy inequality yields

$$T_1^{\frac{1}{q}} \leq C\tilde{A}_1 \|u\|_{L^p(\mathbb{R}^n)}.$$

Note that Lemma 2.2 shows

$$\tilde{\psi}(x) \leq C \min\left\{ |x|^{-\frac{n}{p'}}, |x|^{-\left(\frac{n}{p'}+1\right)} \right\} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{2.9}$$

We distinguish two cases:

Case 1. We assume $R \geq 1$. By using the latter estimate of (2.9), we see

$$\left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^q |x|^{-s} dx \right)^{\frac{1}{q}} \leq C \left(\int_{\{2R<|x|\}} |x|^{-\left(\frac{n}{p'}+1\right)q-s} dx \right)^{\frac{1}{q}}$$

$$\leq C \left(\frac{1}{\left(\frac{n}{p'}+1\right)q-(n-s)} \right)^{\frac{1}{q}} R^{-\frac{n}{p'}+\frac{n-s}{q}-1} \leq C \left(\frac{1}{\left(\frac{n}{p'}+1\right)q-n} \right)^{\frac{1}{q}} R^{-\frac{n}{p'}+\frac{n-s}{q}-1} \leq CR^{-\frac{n}{p'}+\frac{n-s}{q}-1},$$

where we used $q \geq n+1$ to get a constant C independent of s and q. Thus since $q \geq n+1$ and $R \geq 1$, the above estimate yields that

$$\left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^q |x|^{-s} dx \right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx \right)^{\frac{1}{p'}} \leq CR^{\frac{n-s}{q}-1} \leq C \quad \text{for all } R \geq 1.$$

Case 2. We assume $0 < R < 1$. In this case, we have

$$\left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^q |x|^{-s} dx \right)^{\frac{1}{q}} \leq \left(\int_{\{2R<|x|<2\}} \tilde{\psi}(x)^q |x|^{-s} dx \right)^{\frac{1}{q}} + \left(\int_{\{|x|\geq 2\}} \tilde{\psi}(x)^q |x|^{-s} dx \right)^{\frac{1}{q}} =: B_1 + B_2.$$

By using $q \geq n+1$ and the latter estimate of (2.9), we see

$$B_2 \leq C \left(\int_{\{|x|\geq 2\}} |x|^{-\left(\frac{n}{p'}+1\right)q-s} \right)^{\frac{1}{q}}$$

$$\leq C \left(\frac{1}{\left(\frac{n}{p'}+1\right)q-(n-s)} 2^{-\left[\left(\frac{n}{p'}+1\right)q-(n-s)\right]} \right)^{\frac{1}{q}} \leq C \left(\frac{1}{\left(\frac{n}{p'}+1\right)q-n} \right)^{\frac{1}{q}} \leq C,$$

and by $q \geq p'+1$ and the former estimate of (2.9), we obtain

$$B_1 \leq C \left(\int_{\{2R<|x|<2\}} |x|^{-\frac{nq}{p'}-s} dx \right)^{\frac{1}{q}} \leq C \left(\frac{R^{-\left[\frac{nq}{p'}-(n-s)\right]}-1}{\frac{nq}{p'}-(n-s)} \right)^{\frac{1}{q}}$$

$$\leq C \left(\frac{1}{\frac{nq}{p'}-n} \right)^{\frac{1}{q}} R^{-\frac{n}{p'}+\frac{n-s}{q}} \leq CR^{-\frac{n}{p'}+\frac{n-s}{q}} \quad \text{for all } 0 < R < 1.$$

Thus we get

$$\left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^q |x|^{-s} dx\right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx\right)^{\frac{1}{p'}} \leq C\left(R^{\frac{n-s}{q}} + R^{\frac{n}{p'}}\right) \leq C \quad \text{for all } 0 < R < 1.$$

As a consequence, we can take $\tilde{A}_1 = C$ which depends only on n and p.

Next, we estimate T_3. Note that $2|x| < |y|$ implies $\frac{|y|}{2} < |x-y|$. Then we see

$$T_3 \leq \int_{\mathbb{R}^n} \left(\int_{\{|y|>2|x|\}} h(y)dy\right)^q |x|^{-s} dx, \quad \text{where} \quad h(y) := \tilde{\psi}(y)u(y).$$

To apply Theorem A′ (ii), we need to check the following condition:

$$\left(\int_{\{|x|<R\}} |x|^{-s} dx\right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \left(\tilde{\psi}(x)^{-p}\right)^{-(p'-1)} dx\right)^{\frac{1}{p'}} \leq \tilde{A}_2 \tag{2.10}$$

holds for all $R > 0$ with some $\tilde{A}_2$ independent of R. Indeed, once (2.10) has been established, the Hardy inequality yields

$$T_3^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} \tilde{A}_2 \left(\int_{\mathbb{R}^n} h(x)^p \tilde{\psi}(x)^{-p} dx\right)^{\frac{1}{p}} \leq C\tilde{A}_2 \|u\|_{L^p(\mathbb{R}^n)}.$$

We distinguish two cases:

Case 1. We assume $R \geq 1$. In this case, by $q \geq n+1$ and the latter estimate of (2.9), we have

$$\left(\int_{\{|x|<R\}} |x|^{-s} dx\right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^{p'} dx\right)^{\frac{1}{p'}} \leq C\left(\frac{1}{n-s}\right)^{\frac{1}{q}} R^{\frac{n-s}{q}} \left(\int_{\{2R<|x|\}} |x|^{-\left(\frac{n}{p'}+1\right)p'} dx\right)^{\frac{1}{p'}}$$

$$= C\left(\frac{1}{n-s}\right)^{\frac{1}{q}} R^{\frac{n-s}{q}-1} \leq C\left(\frac{1}{n-s}\right)^{\frac{1}{q}} \quad \text{for all } R \geq 1.$$

Case 2. We assume $0 < R < 1$. In this case, we see

$$\left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^{p'} dx\right)^{\frac{1}{p'}} \leq \left(\int_{\{2R<|x|<2\}} \tilde{\psi}(x)^{p'} dx\right)^{\frac{1}{p'}} + \left(\int_{\{|x|\geq 2\}} \tilde{\psi}(x)^{p'} dx\right)^{\frac{1}{p'}} =: \tilde{B}_1 + \tilde{B}_2.$$

The latter estimate of (2.9) yields $\tilde{B}_2 < \infty$, and the former estimate of (2.9) shows

$$\tilde{B}_1 \leq C\left(\int_{\{2R<|x|<2\}} |x|^{-n} dx\right)^{\frac{1}{p'}} = C\left(\log\frac{1}{R}\right)^{\frac{1}{p'}} \quad \text{for all } 0 < R < 1.$$

Thus we get

$$\left(\int_{\{|x|<R\}} |x|^{-s}dx\right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^{p'}dx\right)^{\frac{1}{p'}} \leq C\left(\frac{1}{n-s}\right)^{\frac{1}{q}} R^{\frac{n-s}{q}} \left(1+\left(\log\frac{1}{R}\right)^{\frac{1}{p'}}\right) \tag{2.11}$$

for all $0 < R < 1$. Elementary calculus gives

$$\max_{\{0<R<1\}} g(R) := \max_{\{0<R<1\}} R^{\frac{n-s}{q}} \left(\log\frac{1}{R}\right)^{\frac{1}{p'}} = g\left(e^{-\frac{q}{p'(n-s)}}\right) = \left(\frac{q}{ep'(n-s)}\right)^{\frac{1}{p'}} = C\left(\frac{1}{n-s}\right)^{\frac{1}{p'}} q^{\frac{1}{p'}}. \tag{2.12}$$

Combining (2.11) with (2.12) yields

$$\left(\int_{\{|x|<R\}} |x|^{-s}dx\right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} \tilde{\psi}(x)^{p'}dx\right)^{\frac{1}{p'}} \leq C\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}} q^{\frac{1}{p'}},$$

and then we can take $\tilde{A}_2 = C\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}} q^{\frac{1}{p'}}$.

Finally, we estimate T_2. Note that $\frac{|x|}{2} \leq |y| \leq 2|x|$ and $2^k \leq |x| < 2^{k+1}$ imply that $2^{k-1} \leq |y| < 2^{k+2}$, and take $r := \frac{nq}{n-s} \in [q,\infty)$. Then by the Hölder inequality and the Young inequality, we see

$$\begin{aligned}
T_2 &= \sum_{k\in\mathbb{Z}} \int_{\{2^k\leq|x|<2^{k+1}\}} \left(\int_{\{\frac{|x|}{2}\leq|y|\leq2|x|\}} \psi(x-y)u(y)dy\right)^q |x|^{-s}dx \\
&\leq \sum_{k\in\mathbb{Z}} 2^{-ks} \int_{\{2^k\leq|x|<2^{k+1}\}} \left(\int_{\{\frac{|x|}{2}\leq|y|\leq2|x|\}} \psi(x-y)u(y)dy\right)^q dx \\
&\leq C\sum_{k\in\mathbb{Z}} 2^{-ks+kn\left(1-\frac{q}{r}\right)} \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \psi(x-y)u(y)\chi_{\{2^{k-1}\leq|\cdot|<2^{k+2}\}}(y)dy\right)^r dx\right)^{\frac{q}{r}} \\
&= C\sum_{k\in\mathbb{Z}} \|\psi * u\chi_{\{2^{k-1}\leq|\cdot|<2^{k+2}\}}\|^q_{L^r(\mathbb{R}^n)} \leq C\|\psi\|^q_{L^{\tilde{r}}(\mathbb{R}^n)} \sum_{k\in\mathbb{Z}} \|u\chi_{\{2^{k-1}\leq|\cdot|<2^{k+2}\}}\|^q_{L^p(\mathbb{R}^n)} \\
&\leq \left(C\|\psi\|_{L^{\tilde{r}}(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R}^n)}\right)^q,
\end{aligned}$$

where $\tilde{r}$ is defined by $\frac{1}{r}+1 = \frac{1}{\tilde{r}}+\frac{1}{p}$, i.e., $\frac{1}{\tilde{r}} = \frac{1}{p'}+\frac{n-s}{nq}$. Since $q \geq \max\{p+1, n+1\}$ and $0 \leq s < n$, we see $\max\{1, \frac{np'}{n+p'}\} < \tilde{r} < p'$. By Lemma 2.3, we easily see that

$$\|\psi\|_{L^{\tilde{r}}(\mathbb{R}^n)} \leq C\left(\frac{q}{n-s}\right)^{\frac{1}{p'}}.$$

Therefore, we get

$$T_2^{\frac{1}{q}} \leq C\left(\frac{q}{n-s}\right)^{\frac{1}{p'}} \|u\|_{L^p(\mathbb{R}^n)},$$

which finishes the proof. □

Next, for the proof of Theorem 1.5, we prepare several tools. First, we recall the Bessel potential $G_\alpha(x)$ with $0 < \alpha < n$ defined by

$$G_\alpha(x) := \frac{1}{(4\pi)^{\frac{\alpha}{2}}\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty t^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi|x|^2}{t}-\frac{t}{4\pi}} dt \quad \text{for } x \in \mathbb{R}^n,$$

where Γ denotes the Gamma function. By virtue of the identity $(I-\Delta)^{-\frac{\alpha}{2}} f = G_\alpha * f$ for $f \in \mathcal{S}'(\mathbb{R}^n)$, Theorem 1.5 can be changed into the following equivalent form:

Theorem 1.5′. *Let $n \in \mathbb{N}$, $1 < p < r < \infty$, $p \le q \le (r-1)p'$, and let $w_r(x)$ be the weight function as in Theorem 1.5. Then there exists a positive constant C which depends only on n, p, q and r such that*

$$\left\| G_{\frac{n}{p}} * f \right\|_{L^q\left(\mathbb{R}^n;\, \frac{dx}{w_r(x)}\right)} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

holds for all $f \in L^p(\mathbb{R}^n)$.

For the treatment of the marginal case $q = (r-1)p'$ in Theorem 1.5′, we need to recall the weak Lebesgue space $L^p_w(\mathbb{R}^n)$ for $1 < p < \infty$ and the generalized Young inequality: We say $f \in L^p_w(\mathbb{R}^n)$ if the following norm is finite, i.e.,

$$\|f\|_{L^p_w(\mathbb{R}^n)} := \sup_{\lambda > 0} \mu\left(\{x \in \mathbb{R}^n\,;\, |f(x)| > \lambda\}\right)^{\frac{1}{p}} \lambda,$$

where μ denotes the Lebesgue measure. Then O'Neil [13] proved the following inequality which generalizes the usual Young inequality:

Theorem B. *Let $n \in \mathbb{N}$ and $1 < p < q < \infty$. Then there exists a positive constant C which depends only on n, p and q such that*

$$\|f * g\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^r_w(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}, \tag{2.13}$$

where the exponent $r \in (1,\infty)$ is determined by $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$.

Actually, O'Neil [13] proved more general inequality than (2.13) in terms of Lorentz spaces $L^p_q(\mathbb{R}^n)$ with $1 < p < \infty$ and $1 \le q \le \infty$. However, since it is well-known that $L^p_\infty(\mathbb{R}^n) = L^p_w(\mathbb{R}^n)$, we obtain Theorem B as a particular case of the result in [13].

Furthermore, we establish the decay estimate for $G_\alpha(x)$, which is essentially shown in Stein [16]. However, we shall include the verification for the sake of completeness.

Lemma 2.5. *Let $n \in \mathbb{N}$ and $0 < \alpha < n$. Then there exists a positive constant C which depends only on n and α such that*

$$G_\alpha(x) \le \begin{cases} C\,|x|^{-(n-\alpha)} & \text{for } x \in \mathbb{R}^n \setminus \{0\}, \\ C\,e^{-|x|} & \text{for } x \in \mathbb{R}^n \quad \text{with } |x| \ge 1. \end{cases}$$

Proof. For $x \in \mathbb{R}^n \setminus \{0\}$, changing a variable yields

$$G_\alpha(x) \le C \int_0^\infty t^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi|x|^2}{t}} dt = C|x|^{-(n-\alpha)} \int_0^\infty \tau^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi}{\tau}} d\tau = C|x|^{-(n-\alpha)},$$

which proves the former decay estimate in Lemma 2.5.

Next assume $|x| \ge 1$. First, we obtain

$$\begin{aligned} &\int_0^1 t^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi|x|^2}{t}-\frac{t}{4\pi}} dt \le \int_0^1 t^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi|x|^2}{2t}-\frac{\pi|x|^2}{2t}} dt \le e^{-\frac{\pi|x|^2}{2}} \int_0^1 t^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi|x|^2}{2t}} dt \\ &\le e^{-\frac{\pi|x|^2}{2}} \int_0^1 t^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi}{2t}} dt = Ce^{-\frac{\pi|x|^2}{2}} \le Ce^{-|x|}. \end{aligned} \tag{2.14}$$

Moreover, elementary calculus shows that the function $e^{-\frac{\pi|x|^2}{t}-\frac{t}{4\pi}}$ for $t > 0$ has a maximum at $t = 2\pi|x|$, and then we have

$$\int_1^\infty t^{-\frac{n-\alpha}{2}-1} e^{-\frac{\pi|x|^2}{t}-\frac{t}{4\pi}} dt \le e^{-|x|} \int_1^\infty t^{-\frac{n-\alpha}{2}-1} dt = Ce^{-|x|}. \tag{2.15}$$

Combining (2.14) with (2.15), we get the latter estimate in Lemma 2.5. □

In the end of this chapter, we prove a lemma which is necessary for the proof of the optimality of both Theorem 1.1 and Theorem 1.5. We first define

$$v_\tau(x) := \left(\log \frac{1}{|x|}\right)^\tau \eta(|x|) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, \tag{2.16}$$

where we take $\eta \in C^\infty([0,\infty))$ satisfying the followings:

$$\text{(i) } 0 \le \eta(t) \le 1 \text{ for } t \in [0,\infty)\,; \text{ (ii) } \eta(t) \equiv 1 \text{ for } 0 \le t \le \frac{1}{6}\,; \text{ (iii) } \eta(t) \equiv 0 \text{ for } t \ge \frac{1}{5}.$$

We remark that the following lemma can be understood as the explicit version of the extremal function studied in [3, Theorem 2.7.1] and [4, Theorem 2.1].

Lemma 2.6. *Let $n \ge 2$ and $\frac{n}{n-1} \le p < \infty$. Then $v_\tau \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ holds for any $\tau \in (0, \frac{1}{p'})$ with the estimate:*

$$\|v_\tau\|_{H^{\frac{n}{p},p}(\mathbb{R}^n)} \le C \left(\frac{1}{\frac{1}{p'} - \tau}\right)^{\frac{1}{p}},$$

where a positive constant C depends only on n and p.

Proof. We first remark that the direct computation yields the derivative estimates of v_τ: for any $l \in \mathbb{N}$, there exists c_l depending only on l such that

$$|\partial_x^\beta v_\tau(x)| \le c_l \tilde{v}_{\tau,l}(|x|) \quad \text{holds for } 1 \le |\beta| \le l, \tau \in (0,1) \text{ and } x \in \mathbb{R}^n \setminus \{0\},$$

where $\tilde{v}_{\tau,l}(t) := t^{-l}(\log \frac{1}{t})^{\tau-1}\chi_{[0,\frac{1}{4}]}(t)$, $t \in (0,\infty)$. We note that the function $\tilde{v}_{\tau,l}$ is non-increasing on $(0,\infty)$ for $l \geq 1$ and $\tau \in (0,1)$. In this proof, C denotes a positive constant which depends only on n and p. It is easy to show $\|v_\tau\|_{L^p(\mathbb{R}^n)} \leq C$ for all $\tau \in (0,1)$. Now let $\frac{n}{p} = m + \alpha$, where m is a non-negative integer and $\alpha \in [0,1)$. In the case of $\alpha = 0$, we can prove

$$\|\partial_x^\beta v_\tau\|_{L^p(\mathbb{R}^n)} \leq C \left(\frac{1}{\frac{1}{p'} - \tau}\right)^{\frac{1}{p}} \quad \text{for } 1 \leq |\beta| \leq m \text{ and } 0 < \tau < \frac{1}{p'}$$

by directly estimating the $L^p(\mathbb{R}^n)$-norm of the derivatives of v_τ. Then hereafter we assume that $\alpha \in (0,1)$. We have $0 \leq m \leq n-2$ by the assumption $p \geq \frac{n}{n-1}$ and $\alpha \neq 0$. We prove this lemma by applying the characterization of $H^{\frac{n}{p},p}(\mathbb{R}^n)$ in [17, §1.7, §2.1]. Thus it is enough to show that

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{|(\partial_x^\beta v_\tau)(x+y) - (\partial_x^\beta v_\tau)(x)|}{|y|^{n+\alpha}} dy \right)^p dx \leq \frac{C}{\frac{1}{p'} - \tau} \quad \text{for } |\beta| \leq m \text{ and } 0 < \tau < \frac{1}{p'} \tag{2.17}$$

because we already obtained $\|v_\tau\|_{L^p(\mathbb{R}^n)} \leq C$ for all $0 < \tau < 1$. Then we focus on the estimate of the integrand of (2.17). We first divide the integrand into three parts as follows:

$$\begin{aligned}
J(x) &:= \int_{\mathbb{R}^n} \frac{|(\partial_x^\beta v_\tau)(x+y) - (\partial_x^\beta v_\tau)(x)|}{|y|^{n+\alpha}} dy \leq \int_{\mathbb{R}^n} \int_0^1 \left|(\nabla \partial_x^\beta v_\tau)(x+ty)\right| dt\, |y|^{-n-\alpha+1} dy \\
&\leq C \int_{\mathbb{R}^n} \int_0^1 \tilde{v}_{\tau,m+1}\big(|x+ty|\big) dt\, |y|^{-n-\alpha+1} dy \leq C \Bigg(\int_{\{|y|<\frac{|x|}{2}\}} \int_0^1 \tilde{v}_{\tau,m+1}\big(|x+ty|\big) dt\, |y|^{-n-\alpha+1} dy \\
&+ \int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} \int_0^1 \tilde{v}_{\tau,m+1}\big(|x+ty|\big) dt\, |y|^{-n-\alpha+1} dy \\
&+ \int_{\{|y|>2|x|\}} \int_0^1 \tilde{v}_{\tau,m+1}\big(|x+ty|\big) dt\, |y|^{-n-\alpha+1} dy \Bigg) =: C(J_1(x) + J_2(x) + J_3(x)).
\end{aligned}$$

Since we have $|x+ty| \geq \frac{|x|}{2}$ for any $|y| < \frac{|x|}{2}$ and $0 \leq t \leq 1$, we estimate J_1 as follows:

$$J_1(x) \leq \int_{\{|y|<\frac{|x|}{2}\}} \int_0^1 \tilde{v}_{\tau,m+1}\left(\frac{|x|}{2}\right) dt\, |y|^{-n-\alpha+1} dy = C|x|^{-\frac{n}{p}} \left(\log \frac{2}{|x|}\right)^{\tau-1} \chi_{[0,\frac{1}{2}]}\big(|x|\big).$$

Next, we estimate J_2. By changing a variable $z = x + ty$, we have

$$\begin{aligned}
J_2(x) &\leq C|x|^{-n-\alpha+1} \int_0^1 \int_{\{\frac{|x|}{2} \leq |y| \leq 2|x|\}} \tilde{v}_{\tau,m+1}\big(|x+ty|\big) dy\, dt \\
&= C|x|^{-n-\alpha+1} \int_0^1 \int_{\{\frac{t|x|}{2} \leq |z-x| \leq 2t|x|\}} \tilde{v}_{\tau,m+1}\big(|z|\big) dz\, t^{-n} dt \\
&= C|x|^{-n-\alpha+1} \Bigg[\int_{\{\frac{|x|}{2} \leq |z-x| \leq 2|x|\}} \int_{\frac{|z-x|}{2|x|}}^1 t^{-n} dt\, \tilde{v}_{\tau,m+1}\big(|z|\big) dz
\end{aligned}$$

$$+\int_{\{|z-x|<\frac{|x|}{2}\}}\int_{\frac{|z-x|}{2|x|}}^{\frac{2|z-x|}{|x|}} t^{-n}dt\,\tilde{v}_{\tau,m+1}(|z|)dz\Bigg] =: C|x|^{-n-\alpha+1}(J_{21}(x)+J_{22}(x)).$$

Note that $\frac{|x|}{2}\le|z-x|\le 2|x|$ implies $\frac{|z-x|}{2|x|}\ge\frac{1}{4}$ and $|z|\le 3|x|$. Then by using the condition $m\le n-2$, we can estimate J_{21} as

$$\begin{aligned}
J_{21}(x) &\le C\int_{\{\frac{|x|}{2}\le|z-x|\le 2|x|\}}\tilde{v}_{\tau,m+1}(|z|)dz \le C\int_{\{|z|\le 3|x|\}}\tilde{v}_{\tau,m+1}(|z|)dz\\
&= \begin{cases} \dfrac{C}{(n-m-1)^\tau}\displaystyle\int_{(n-m-1)\log 4}^{\infty}\sigma^{\tau-1}e^{-\sigma}d\sigma\le C & \text{if } |x|>\dfrac{1}{12},\\ \dfrac{C}{(n-m-1)^\tau}\displaystyle\int_{(n-m-1)\log\frac{1}{3|x|}}^{\infty}\sigma^{\tau-1}e^{-\sigma}d\sigma\le C|x|^{n-m-1}\left(\log\dfrac{1}{3|x|}\right)^{\tau-1} & \text{if } |x|\le\dfrac{1}{12}, \end{cases}
\end{aligned}$$

where we used the following claim:

Claim. The estimate $\int_t^\infty \sigma^{\tau-1}e^{-\sigma}d\sigma\le t^{\tau-1}e^{-t}$ holds for any $t>0$ and $0<\tau<1$.

Indeed, this claim is shown as

$$\int_t^\infty \sigma^{\tau-1}e^{-\sigma}d\sigma = t^{\tau-1}e^{-t}-(1-\tau)\int_t^\infty\sigma^{\tau-2}e^{-\sigma}d\sigma\le t^{\tau-1}e^{-t}.$$

On the other hand, we can estimate J_{22} as

$$\begin{aligned}
J_{22}(x) &\le C|x|^{n-1}\int_{\{|z-x|\le\frac{|x|}{2}\}}\frac{1}{|z-x|^{n-1}}\tilde{v}_{\tau,m+1}(|z|)dz\\
&\le C|x|^{n-1}\tilde{v}_{\tau,m+1}\left(\frac{|x|}{2}\right)\int_{\{|z-x|\le\frac{|x|}{2}\}}\frac{1}{|z-x|^{n-1}}dz = C|x|^{n-m-1}\left(\log\frac{2}{|x|}\right)^{\tau-1}\chi_{[0,\frac{1}{2}]}(|x|)
\end{aligned}$$

since $|z|\ge\frac{|x|}{2}$ holds for $|z-x|\le\frac{|x|}{2}$. Lastly, we estimate J_3. By changing a variable $z=ty$, we divide the integral into two parts as follows:

$$\begin{aligned}
J_3(x) &= \int_0^1\int_{\{|z|>2t|x|\}}\tilde{v}_{\tau,m+1}(|x+z|)|z|^{-n-\alpha+1}t^{\alpha-1}dz\,dt\\
&= \int_{\{|z|>2|x|\}}\int_0^1 t^{\alpha-1}dt\,\tilde{v}_{\tau,m+1}(|x+z|)|z|^{-n-\alpha+1}dz\\
&\quad+\int_{\{|z|\le 2|x|\}}\int_0^{\frac{|z|}{2|x|}} t^{\alpha-1}dt\,\tilde{v}_{\tau,m+1}(|x+z|)|z|^{-n-\alpha+1}dz =: J_{31}(x)+J_{32}(x).
\end{aligned}$$

We now estimate J_{31}. We first remark that we have $J_{31}(x)\equiv 0$ for $|x|>\frac{1}{4}$ since $|x+z|>\frac{1}{4}$ holds for $|z|>2|x|$ and $|x|>\frac{1}{4}$. Then we consider $|x|\le\frac{1}{4}$. Since we have $|x+z|>\frac{|z|}{2}$ for any $|z|>2|x|$, we have

$$J_{31}(x)=\frac{1}{\alpha}\int_{\{|z|>2|x|\}}\tilde{v}_{\tau,m+1}(|x+z|)|z|^{-n-\alpha+1}dz\le\frac{1}{\alpha}\int_{\{|z|>2|x|\}}\tilde{v}_{\tau,m+1}\left(\frac{|z|}{2}\right)|z|^{-n-\alpha+1}dz$$

$$= \frac{C}{(\frac{n}{p})^\tau} \int_{\frac{n}{p}\log 4}^{\frac{n}{p}\log\frac{1}{|x|}} \sigma^{\tau-1} e^\sigma d\sigma \le C|x|^{-\frac{n}{p}} \left(\log \frac{1}{|x|}\right)^{\tau-1}$$

for $|x| \le \frac{1}{4}$, where we used the following claim:

Claim. Fix $a > 0$. Then there exists a positive constant C_a depending only on a such that

$$\int_a^t \sigma^{\tau-1} e^\sigma d\sigma \le C_a t^{\tau-1} e^t$$

holds for any $t > a$ and $0 < \tau < 1$.

Indeed, this claim is shown as follows. First, it is easy to show $\int_2^t \sigma^{\tau-1} e^\sigma d\sigma \le 2t^{\tau-1}e^t$ for $t \ge 2$. Hence we may assume $a < 2$. Then we have

$$\int_a^2 \sigma^{\tau-1} e^\sigma d\sigma \le C'_a \qquad \text{and} \qquad t^{\tau-1}e^t \ge (1-\tau)^{\tau-1} e^{1-\tau} \ge 1 \qquad \text{for } t > 0 \text{ and } 0 < \tau < 1,$$

where a positive constant C'_a depends only on a. Therefore, this claim is true. Now we estimate J_{32}. We divide it into two parts as follows:

$$\begin{aligned} J_{32}(x) &= C|x|^{-\alpha} \int_{\{|z| \le 2|x|\}} \tilde{v}_{\tau,m+1}\big(|x+z|\big)|z|^{-n+1} dz \\ &= C|x|^{-\alpha} \left(\int_{\{|z|<\frac{|x|}{2}\}} \tilde{v}_{\tau,m+1}\big(|x+z|\big)|z|^{-n+1} dz + \int_{\{\frac{|x|}{2} \le |z| \le 2|x|\}} \tilde{v}_{\tau,m+1}\big(|x+z|\big)|z|^{-n+1} dz \right) \\ &=: C|x|^{-\alpha}(J_{321}(x) + J_{322}(x)). \end{aligned}$$

Since $|z| < \frac{|x|}{2}$ yields $|x+z| > \frac{|x|}{2}$, we have

$$J_{321}(x) \le \int_{\{|z|<\frac{|x|}{2}\}} \tilde{v}_{\tau,m+1}\left(\frac{|x|}{2}\right) |z|^{-n+1} dz = C|x|^{-m} \left(\log \frac{2}{|x|}\right)^{\tau-1} \chi_{[0,\frac{1}{2}]}\big(|x|\big).$$

On the other hand, we remark that $|x+z| \le 3|x|$ holds for $|z| \le 2|x|$. Hence, we have

$$\begin{aligned} J_{322}(x) &\le C|x|^{-n+1} \int_{\{\frac{|x|}{2} \le |z| \le 2|x|\}} \tilde{v}_{\tau,m+1}\big(|x+z|\big) dz \le C|x|^{-n+1} \int_{\{|y| \le 3|x|\}} \tilde{v}_{\tau,m+1}\big(|y|\big) dy \\ &\le \begin{cases} C|x|^{-n+1} & \text{if } |x| > \dfrac{1}{12}, \\ C|x|^{-m} \left(\log \dfrac{1}{3|x|}\right)^{\tau-1} & \text{if } |x| \le \dfrac{1}{12} \end{cases} \end{aligned}$$

since $m \le n-2$ in the same way as the estimate of J_{21}.

Summing up, we obtain

$$J(x) \le C \left[|x|^{-\frac{n}{p}} \sum_{l=\frac{1}{2},1,3} \left(\log \frac{1}{l|x|}\right)^{\tau-1} \chi_{[0,\frac{1}{4l}]}\big(|x|\big) + |x|^{-n-\alpha+1} \chi_{(\frac{1}{12},\infty)}\big(|x|\big) \right].$$

Therefore, we have

$$\|J\|_{L^p(\mathbb{R}^n)} \le C \sum_{l=\frac{1}{2},1,3} \left\| |\cdot|^{-\frac{n}{p}} \left(\log \frac{1}{l|\cdot|}\right)^{\tau-1} \chi_{[0,\frac{1}{4l}]}(|\cdot|) \right\|_{L^p(\mathbb{R}^n)} + C \left\| |\cdot|^{-n-\alpha+1} \chi_{(\frac{1}{12},\infty)}(|\cdot|) \right\|_{L^p(\mathbb{R}^n)}$$
$$= C\left(\int_0^{\frac{1}{4}} \frac{1}{t}\left(\log\frac{1}{t}\right)^{p(\tau-1)} dt\right)^{\frac{1}{p}} + C\left(\int_{\frac{1}{12}}^{\infty} t^{-p(n-m-1)-1} dt\right)^{\frac{1}{p}} \le C\left(\frac{1}{\frac{1}{p'}-\tau}\right)^{\frac{1}{p}}$$

since $m \le n-2$ and $0 < \tau < \frac{1}{p'}$, which is the desired estimate. □

3 Proof of main theorems

In this chapter, we prove Theorem 1.1 and Theorem 1.5′ by using lemmas in Section 2.

Proof of Theorem 1.1. We first prove the optimality of the growth orders with respect to s and q, which is easily seen by applying Lemma 2.6. Indeed, let v_τ be the function defined in (2.16) for $0 < \tau < \frac{1}{p'}$. Then the direct computation yields

$$\|v_\tau\|_{L^q\left(\mathbb{R}^n;\frac{dx}{|x|^s}\right)} \ge \left(\int_{\{|x|\le\frac{1}{6}\}} \left[\log\left(\frac{1}{|x|}\right)\right]^{\tau q} \frac{dx}{|x|^s}\right)^{\frac{1}{q}} = O\left(\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\tau} q^\tau\right) \tag{3.1}$$

as $s \uparrow n$ and $q \to \infty$ for all $\tau \in (0, \frac{1}{p'})$. Since $v_\tau \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ for all $\tau \in (0, \frac{1}{p'})$ if $n \ge 2$ and $\frac{n}{n-1} \le p < \infty$ by Lemma 2.6, (3.1) clearly implies the growth orders of s and q are both optimal.

Thus we proceed to the proof of the affirmative part of Theorem 1.1. We may assume $u \in \mathcal{S}(\mathbb{R}^n)$ since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{\frac{n}{p},p}(\mathbb{R}^n)$. In what follows, $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and the Fourier inverse transforms, respectively. Then for any $K > 0$, the function u can be decomposed into two parts such as

$$u(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} (\mathcal{F}u)(\xi) d\xi$$
$$= \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} (\mathcal{F}u)(\xi)\, \varphi\left(\frac{\xi}{K}\right) d\xi + \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} (\mathcal{F}u)(\xi) \left(1 - \varphi\left(\frac{\xi}{K}\right)\right) d\xi =: u_1(x) + u_2(x),$$

where φ is the function as in Lemma 2.1. We first estimate the integral of u_1. Since $\mathcal{F}\varphi = \varphi$, we have

$$\mathcal{F}^{-1}\left[\varphi\left(\frac{\cdot}{K}\right)\right](x) = K^n \varphi(Kx) =: \varphi_K(x).$$

Then note that $u_1(x) = \varphi_K * u(x)$. Since we have the scaling $(\varphi_K * u)\left(\frac{x}{K}\right) = \varphi * \left(u\left(\frac{\cdot}{K}\right)\right)(x)$, Lemma 2.1 yields

$$\left(\int_{\mathbb{R}^n} |u_1(x)|^q |x|^{-s} dx\right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^n} |\varphi_K * u(x)|^q |x|^{-s} dx\right)^{\frac{1}{q}}$$

$$= K^{-\frac{n-s}{q}} \left(\int_{\mathbb{R}^n} \left| (\varphi_K * u) \left(\frac{y}{K} \right) \right|^q |y|^{-s} dy \right)^{\frac{1}{q}} = K^{-\frac{n-s}{q}} \left(\int_{\mathbb{R}^n} \left| \left[\varphi * \left(u \left(\frac{\cdot}{K} \right) \right) \right] (y) \right|^q |y|^{-s} dy \right)^{\frac{1}{q}}$$

$$\leq C \left(\frac{1}{n-s} \right)^{\frac{1}{q}} K^{-\frac{n-s}{q}} \left\| u \left(\frac{\cdot}{K} \right) \right\|_{L^p(\mathbb{R}^n)} = C \left(\frac{1}{n-s} \right)^{\frac{1}{q}} K^{\frac{n}{p} - \frac{n-s}{q}} \|u\|_{L^p(\mathbb{R}^n)} \tag{3.2}$$

for all $K > 0$.

Next, we estimate the integral of u_2. For any $K > 0$, the function u_2 can be rewritten as

$$u_2(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \frac{1 - \varphi\left(\frac{\xi}{K}\right)}{(2\pi|\xi|)^{\frac{n}{p}}} (2\pi|\xi|)^{\frac{n}{p}} (\mathcal{F}u)(\xi) d\xi = \tilde{\psi}_K * (-\Delta)^{\frac{n}{2p}} u(x),$$

where

$$\tilde{\psi}_K(x) := \mathcal{F}^{-1} \left[(2\pi| \cdot |)^{-\frac{n}{p}} \left(1 - \varphi \left(\frac{\cdot}{K} \right) \right) \right] (x) = C \left(|x|^{-\frac{n}{p'}} - | \cdot |^{-\frac{n}{p'}} * \mathcal{F}^{-1} \left[\varphi \left(\frac{\cdot}{K} \right) \right] (x) \right)$$

$$= C \left(|x|^{-\frac{n}{p'}} - | \cdot |^{-\frac{n}{p'}} * \varphi_K(x) \right) = C K^n \int_{\mathbb{R}^n} \left(|x|^{-\frac{n}{p'}} - |x-y|^{-\frac{n}{p'}} \right) e^{-\pi |Ky|^2} dy, \tag{3.3}$$

where the last equality follows from $K^n \int_{\mathbb{R}^n} e^{-\pi|Ky|^2} dy = \int_{\mathbb{R}^n} e^{-\pi|y|^2} dy = 1$ for all $K > 0$. Moreover, we have the scaling such as

$$\left(\tilde{\psi}_K * (-\Delta)^{\frac{n}{2p}} u \right) \left(\frac{x}{K} \right) = K^{-\frac{n}{p}} \tilde{\psi}_1 * \left[\left((-\Delta)^{\frac{n}{2p}} u \right) \left(\frac{\cdot}{K} \right) \right] (x) \quad \text{for all } K > 0. \tag{3.4}$$

Thus by (3.3), (3.4) and Lemma 2.4, we have

$$\left(\int_{\mathbb{R}^n} |u_2(x)|^q |x|^{-s} dx \right)^{\frac{1}{q}} = \left(\int_{\mathbb{R}^n} |\tilde{\psi}_k * (-\Delta)^{\frac{n}{2p}} u(x)|^q |x|^{-s} dx \right)^{\frac{1}{q}}$$

$$= K^{-\frac{n-s}{q}} \left(\int_{\mathbb{R}^n} \left| \left(\tilde{\psi}_K * (-\Delta)^{\frac{n}{2p}} u \right) \left(\frac{x}{K} \right) \right|^q |x|^{-s} dx \right)^{\frac{1}{q}}$$

$$= K^{-\frac{n}{p} - \frac{n-s}{q}} \left(\int_{\mathbb{R}^n} \left| \tilde{\psi}_1 * \left[\left((-\Delta)^{\frac{n}{2p}} u \right) \left(\frac{\cdot}{K} \right) \right] (x) \right|^q |x|^{-s} dx \right)^{\frac{1}{q}}$$

$$\leq C K^{-\frac{n}{p} - \frac{n-s}{q}} \left(\int_{\mathbb{R}^n} \left(\psi * \left| \left[\left((-\Delta)^{\frac{n}{2p}} u \right) \left(\frac{\cdot}{K} \right) \right] \right| (x) \right)^q |x|^{-s} dx \right)^{\frac{1}{q}}$$

$$\leq C \left(\frac{1}{n-s} \right)^{\frac{1}{q} + \frac{1}{p'}} q^{\frac{1}{p'}} K^{-\frac{n}{p} - \frac{n-s}{q}} \left\| \left((-\Delta)^{\frac{n}{2p}} u \right) \left(\frac{\cdot}{K} \right) \right\|_{L^p(\mathbb{R}^n)}$$

$$= C \left(\frac{1}{n-s} \right)^{\frac{1}{q} + \frac{1}{p'}} q^{\frac{1}{p'}} K^{-\frac{n-s}{q}} \|(-\Delta)^{\frac{n}{2p}} u\|_{L^p(\mathbb{R}^n)} \tag{3.5}$$

for all $K > 0$, where ψ is the function as in Lemma 2.4, and we used $|\tilde{\psi}_1| \leq C \psi$.

By combining (3.2) with (3.5), we have

$$\|u\|_{L^q\left(\mathbb{R}^n\,;\,\frac{dx}{|x|^s}\right)} \le C\left(\frac{1}{n-s}\right)^{\frac{1}{q}+\frac{1}{p'}} q^{\frac{1}{p'}}\left(K^{\frac{n}{p}-\frac{n-s}{q}}\|u\|_{L^p(\mathbb{R}^n)} + K^{-\frac{n-s}{q}}\|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}\right)$$

for all $K > 0$. In the end, in order to optimize the right-hand side with respect to K, we especially take K as

$$K := \left(\frac{\|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^p(\mathbb{R}^n)}}\right)^{\frac{p}{n}},$$

which provides the desired interpolation inequality, and we finish the proof. $\square$

Next, we shall prove Theorem 1.5.

Proof of Theorem 1.5. Firstly, we prove the optimality of the bound $(r-1)p'$, which is shown by Lemma 2.6 again. Indeed, it is easy to see that if $q > (r-1)p'$, taking τ close enough to $\frac{1}{p'}$ shows that

$$\|v_\tau\|_{L^q\left(\mathbb{R}^n\,;\,\frac{dx}{w_r(x)}\right)} = \infty. \tag{3.6}$$

On the other hand, $v_\tau \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ for all $\tau \in (0, \frac{1}{p'})$ if $n \ge 2$ and $\frac{n}{n-1} \le p < \infty$ by Lemma 2.6. Thus (3.6) implies the optimality of the bound $(r-1)p'$ in that case.

Thus we proceed to the proof of the affirmative part of Theorem 1.5. As we discussed in Section 2, it suffices to prove Theorem 1.5$'$. We may assume the function f is non-negative, and we first decompose the integral into three parts:

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(G_{\frac{n}{p}} * f\right)(x)^q \frac{dx}{w_r(x)} \le 3^q\Bigg[\int_{\mathbb{R}^n}\left(\int_{\{|y|<\frac{|x|}{2}\}} G_{\frac{n}{p}}(x-y)f(y)dy\right)^q \frac{dx}{w_r(x)}\\ &+\int_{\mathbb{R}^n}\left(\int_{\{\frac{|x|}{2}\le|y|\le 2|x|\}} G_{\frac{n}{p}}(x-y)f(y)dy\right)^q \frac{dx}{w_r(x)} + \int_{\mathbb{R}^n}\left(\int_{\{|y|>2|x|\}} G_{\frac{n}{p}}(x-y)f(y)dy\right)^q \frac{dx}{w_r(x)}\Bigg]\\ &=: 3^q(U_1+U_2+U_3).\end{aligned}$$

We first investigate U_1. Note that $G_{\frac{n}{p}}(x)$ is radial function and non-increasing with respect to $|x|$. Moreover, $|y| < \frac{|x|}{2}$ implies that $\frac{|x|}{2} < |x-y|$. Thus we see

$$\begin{aligned}U_1 &\le \int_{\mathbb{R}^n}\left(\int_{\{|y|<\frac{|x|}{2}\}} f(y)dy\right)^q\left(\sup_{\{\frac{|x|}{2}<|z|\}} G_{\frac{n}{p}}(z)\right)^q \frac{dx}{w_r(x)}\\ &= \int_{\mathbb{R}^n}\left(\int_{\{|y|<\frac{|x|}{2}\}} f(y)dy\right)^q G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)}.\end{aligned}$$

To apply Theorem A$'$ (i), we need to check the following condition:

$$\left(\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)}\right)^{\frac{1}{q}}\left(\int_{\{|x|<R\}} dx\right)^{\frac{1}{p'}} \le \tilde{A}_1$$

holds for all $R>0$. Indeed, once the above estimate has been established, the Hardy inequality yields

$$U_1^{\frac{1}{q}} \leq (p')^{\frac{1}{p'}} p^{\frac{1}{q}} \tilde{A}_1 \|f\|_{L^p(\mathbb{R}^n)}.$$

We distinguish two cases:

Case 1. We assume $R \geq 1$. In this case, by the latter estimate in Lemma 2.5, we have

$$\begin{aligned}
\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)} &\leq C \int_{\{2R<|x|\}} e^{-\frac{q|x|}{2}} \frac{dx}{\left[\log\left(e+\frac{1}{|x|}\right)\right]^r |x|^n} \\
&\leq C \int_{\{2R<|x|\}} e^{-\frac{q|x|}{4}-\frac{q|x|}{4}} dx \leq Ce^{-\frac{qR}{2}} \int_{\{2<|x|\}} e^{-\frac{q|x|}{4}} dx = Ce^{-\frac{qR}{2}}.
\end{aligned}$$

Thus we have for any $R \geq 1$,

$$\left(\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)}\right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx\right)^{\frac{1}{p'}} \leq Ce^{-\frac{R}{2}} R^{\frac{n}{p'}} \leq C.$$

Case 2. We assume $0<R<1$. In this case, we see

$$\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)} = \int_{\{2R<|x|<2\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)} + \int_{\{|x|\geq 2\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)}.$$

By the latter estimate in Lemma 2.5, the second term is integrable, and by the former estimate in Lemma 2.5, the first term can be estimated as follows:

$$\int_{\{2R<|x|<2\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)} \leq \int_{\{2R<|x|<2\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{|x|^n} \leq C \int_{\{2R<|x|<2\}} |x|^{-\frac{nq}{p'}-n} dx \leq CR^{-\frac{nq}{p'}}.$$

Thus we obtain for any $0<R<1$,

$$\left(\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^q \frac{dx}{w_r(x)}\right)^{\frac{1}{q}} \left(\int_{\{|x|<R\}} dx\right)^{\frac{1}{p'}} \leq C\left(R^{-\frac{n}{p'}}+1\right) R^{\frac{n}{p'}} \leq C.$$

Next, we estimate U_3. Note that $2|x|<|y|$ implies $\frac{|y|}{2}<|x-y|$. Thus we see

$$U_3 \leq \int_{\mathbb{R}^n} \left(\int_{\{|y|>2|x|\}} G_{\frac{n}{p}}\left(\frac{y}{2}\right) f(y) dy\right)^q \frac{dx}{w_r(x)}.$$

To apply Theorem A′ (ii), we need to check the following condition:

$$\left(\int_{\{|x|<R\}} \frac{dx}{w_r(x)}\right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^{p'} dx\right)^{\frac{1}{p'}} \leq \tilde{A}_2.$$

We distinguish two cases :

Case 1. We assume $R \geq 1$. By Lemma 2.5, we have

$$\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^{p'} dx \leq C \int_{\{2R<|x|\}} e^{-\frac{p'|x|}{2}} dx$$
$$= C \int_{\{2R<|x|\}} e^{-\frac{p'|x|}{4}-\frac{p'|x|}{4}} dx \leq C e^{-\frac{p'R}{2}} \int_{\{2<|x|\}} e^{-\frac{p'|x|}{4}} dx = C e^{-\frac{p'R}{2}}.$$

Furthermore, we see

$$\int_{\{|x|<R\}} \frac{dx}{w_r(x)} = \int_{\{|x|<\frac{1}{2}\}} \frac{dx}{w_r(x)} + \int_{\{\frac{1}{2}\leq|x|<R\}} \frac{dx}{w_r(x)},$$

and it is easy to see that the first term is integrable since $r > 1$. The second term will be estimated as

$$\int_{\{\frac{1}{2}\leq|x|<R\}} \frac{dx}{w_r(x)} \leq \int_{\{\frac{1}{2}\leq|x|<R\}} \frac{dx}{|x|^n} \leq C(1+\log R).$$

Combining the above estimates, we have for any $R \geq 1$,

$$\left(\int_{\{|x|<R\}} \frac{dx}{w_r(x)}\right)^{\frac{1}{q}} \left(\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^{p'} dx\right)^{\frac{1}{p'}} \leq C\left[1+(\log R)^{\frac{1}{q}}\right] e^{-\frac{R}{2}} \leq C.$$

Case 2. We assume $0 < R < 1$, which is a crucial case where we use the condition $q \leq (r-1)p'$. First, Lemma 2.5 yields

$$\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^{p'} dx = \int_{\{2R<|x|<2\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^{p'} dx + \int_{\{|x|\geq 2\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^{p'} dx \leq C\left[1+\log\left(\frac{1}{R}\right)\right].$$

Moreover, it is easy to see that

$$\int_{\{|x|<R\}} \frac{dx}{w_r(x)} \leq C\left[\log\left(e+\frac{1}{R}\right)\right]^{-(r-1)}.$$

Thus combining above two estimates shows that

$$\left(\int_{\{|x|<R\}} \frac{dx}{w_r(x)}\right)^{\frac{1}{q}}\left(\int_{\{2R<|x|\}} G_{\frac{n}{p}}\left(\frac{x}{2}\right)^{p'} dx\right)^{\frac{1}{p'}} \leq C\left[\log\left(e+\frac{1}{R}\right)\right]^{-\frac{r-1}{q}} \left[1+\left[\log\left(\frac{1}{R}\right)\right]^{\frac{1}{p'}}\right] \leq C$$

since $r > 1$ and $-\frac{r-1}{q}+\frac{1}{p'} \leq 0$, i.e., $q \leq (r-1)p'$.

Finally, we estimate U_2. We first write U_2 as

$$U_2 = \sum_{k\in\mathbb{Z}} \int_{\{2^k\leq|x|<2^{k+1}\}} \left(\int_{\{\frac{|x|}{2}\leq|y|\leq 2|x|\}} G_{\frac{n}{p}}(x-y)f(y)dy\right)^q \frac{dx}{w_r(x)}.$$

Since $\tilde{w}_r(x) := \left[\log\left(\frac{1}{|x|}\right)\right]^r |x|^n$ is non-decreasing with respect to $|x|$ near the origin, there exists $k_0 \in \mathbb{Z}$ with $k_0 \le -3$ such that $\tilde{w}_r(x)$ is non-decreasing in $|x| \in (0, 2^{k_0+1})$. We decompose U_2 with k_0:

$$U_2 = \sum_{k=-\infty}^{k_0} \int_{\{2^k \le |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} G_{\frac{n}{p}}(x-y) f(y) dy \right)^q \frac{dx}{w_r(x)}$$
$$+ \sum_{k=k_0+1}^{\infty} \int_{\{2^k \le |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} G_{\frac{n}{p}}(x-y) f(y) dy \right)^q \frac{dx}{w_r(x)} =: U_{21} + U_{22}.$$

We first investigate U_{22} which is easier to estimate compared to U_{21}. Note that $\frac{|x|}{2} \le |y| \le 2|x|$ and $2^k \le |x| < 2^{k+1}$ imply $2^{k-1} \le |y| < 2^{k+2}$. Then by the Young inequality, we see

$$U_{22} \le C \sum_{k=k_0+1}^{\infty} \int_{\{2^k \le |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} G_{\frac{n}{p}}(x-y) f(y) dy \right)^q dx$$
$$\le C \left\| G_{\frac{n}{p}} * f\chi_{\{2^{k-1} \le |\cdot| < 2^{k+2}\}} \right\|_{L^q(\mathbb{R}^n)}^q \le C \|G_{\frac{n}{p}}\|_{L^{\tilde{r}}(\mathbb{R}^n)}^q \sum_{k=k_0+1}^{\infty} \|f\chi_{\{2^{k-1} \le |\cdot| < 2^{k+2}\}}\|_{L^p(\mathbb{R}^n)}^q$$
$$= C \sum_{k=k_0+1}^{\infty} \left(\int_{\{2^{k-1} \le |x| < 2^{k+2}\}} f(x)^p dx \right)^{\frac{q}{p}} \le C \left(\sum_{k\in\mathbb{Z}} \int_{\{2^{k-1} \le |x| < 2^{k+2}\}} f(x)^p dx \right)^{\frac{q}{p}} = C\|f\|_{L^p(\mathbb{R}^n)}^q,$$

where the exponent $\tilde{r} \in [1, \infty)$ is determined by $1 + \frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$, and in the above estimate, we used $G_{\frac{n}{p}} \in L^{\tilde{r}}(\mathbb{R}^n)$ which is easily seen by using Lemma 2.5.

Next, we estimate U_{21}. Recall that $\tilde{w}_r(x)$ is non-decreasing in $|x| \in (0, 2^{k_0+1})$, and note that $|y| \le 2|x|$ implies $|x| \ge \frac{|x-y|}{3}$. Thus by Lemma 2.5, we have

$$U_{21} \le C \sum_{k=-\infty}^{k_0} \int_{\{2^k \le |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} |x-y|^{-\frac{n}{p'}} f(y) dy \right)^q \frac{dx}{\tilde{w}_r(x)}$$
$$= C \sum_{k=-\infty}^{k_0} \int_{\{2^k \le |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} \frac{|x-y|^{-\frac{n}{p'}} f(y)}{\tilde{w}_r(x)^{\frac{1}{q}}} dy \right)^q dx$$
$$\le C \sum_{k=-\infty}^{k_0} \int_{\{2^k \le |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} \frac{|x-y|^{-\frac{n}{p'}} f(y)}{\tilde{w}_r\left(\frac{x-y}{3}\right)^{\frac{1}{q}}} dy \right)^q dx.$$

Here, note that $\frac{|x|}{2} \le |y| \le 2|x|$ and $2^k \le |x| < 2^{k+1}$ with $k \le k_0$ yield

$$2^{k-1} \le |y| < 2^{k+2} \quad \text{and} \quad |x-y| \le 3|x| < 3 \cdot 2^{k_0+1} \le \frac{3}{4} \quad \text{since} \quad k_0 \le -3.$$

Then we further keep evaluating U_{21}:

$$U_{21} \le C \sum_{k=-\infty}^{k_0} \int_{\{2^k \le |x| < 2^{k+1}\}} \left(\int_{\{\frac{|x|}{2} \le |y| \le 2|x|\}} \frac{f(y)}{\left[\log\left(\frac{1}{|x-y|}\right)\right]^{\frac{r}{q}} |x-y|^{\frac{n}{q}+\frac{n}{p'}}} dy \right)^q dx$$

$$\leq C \sum_{k=-\infty}^{k_0} \left\| W * f\chi_{\{2^{k-1}\leq|\cdot|<2^{k+2}\}} \right\|_{L^q(\mathbb{R}^n)}^q, \quad \text{where} \quad W(x) := \frac{\chi_{B_{\frac{3}{4}}(0)}(x)}{\left[\log\left(\frac{1}{|x|}\right)\right]^{\frac{r}{q}} |x|^{\frac{n}{q}+\frac{n}{p'}}}.$$

Now we distinguish two cases:

Case 1. We assume $p \leq q < (r-1)p'$. In this case, the Young inequality yields

$$U_{21} \leq C\|W\|_{L^{\tilde{r}}(\mathbb{R}^n)}^q \sum_{k=-\infty}^{k_0} \|f\chi_{\{2^{k-1}\leq|\cdot|<2^{k+2}\}}\|_{L^p(\mathbb{R}^n)}^q \leq C\|W\|_{L^{\tilde{r}}(\mathbb{R}^n)}^q \|f\|_{L^p(\mathbb{R}^n)}^q,$$

where the exponent $\tilde{r} \in [1,\infty)$ is determined by $1+\frac{1}{q} = \frac{1}{\tilde{r}} + \frac{1}{p}$. To complete the above estimate, we check $W \in L^{\tilde{r}}(\mathbb{R}^n)$ below. Note that $\left(\frac{n}{q}+\frac{n}{p'}\right)\tilde{r} = n$ and $\frac{r\tilde{r}}{q} = \frac{rp'}{p'+q}$. Thus by changing a variable, we see

$$\|W\|_{L^{\tilde{r}}(\mathbb{R}^n)}^{\tilde{r}} = \int_{B_{\frac{3}{4}}(0)} \frac{dx}{\left[\log\left(\frac{1}{|x|}\right)\right]^{\frac{rp'}{p'+q}} |x|^n} = C\int_{\log\left(\frac{4}{3}\right)}^{\infty} \frac{dt}{t^{\frac{rp'}{p'+q}}} < \infty$$

since $\frac{rp'}{p'+q} > 1$, i.e., $q < (r-1)p'$.

Case 2. We assume $q = (r-1)p'$. In this case, one sees that $W \notin L^{\tilde{r}}(\mathbb{R}^n)$ but $W \in L^{\tilde{r}}_w(\mathbb{R}^n)$ with $\tilde{r} = \frac{p'}{r'} \in (1,\infty)$. Indeed, we have

$$W(x) = \frac{\chi_{B_{\frac{3}{4}}(0)}(x)}{\left[\log\left(\frac{1}{|x|}\right)\right]^{\frac{r'}{p'}} |x|^{\frac{nr'}{p'}}} \leq \frac{C}{|x|^{\frac{nr'}{p'}}} =: \tilde{W}(x) \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\},$$

and we can easily observe that $\tilde{W} \in L^{\tilde{r}}_w(\mathbb{R}^n)$ from the definition of the weak Lebesgue space. Then by the above observation and Theorem B, we have

$$U_{21} \leq C\|W\|_{L^{\tilde{r}}_w(\mathbb{R}^n)}^q \sum_{k=-\infty}^{k_0} \|f\chi_{\{2^{k-1}\leq|\cdot|<2^{k+2}\}}\|_{L^p(\mathbb{R}^n)}^q \leq C\|\tilde{W}\|_{L^{\tilde{r}}_w(\mathbb{R}^n)}^q \|f\|_{L^p(\mathbb{R}^n)}^q = C\|f\|_{L^p(\mathbb{R}^n)}^q.$$

Thus we finish the proof. □

We end this chapter with the proof of Corollary 1.3 which is an immediate consequence of Theorem 1.1:

Proof of Corollary 1.3. Let $\tilde{p}$ and C be positive constants depending only n and p given by Theorem 1.1, and let $\alpha > 0$ which will be chosen small enough later. Then for any $u \in H^{\frac{n}{p},p}(\mathbb{R}^n)$ with $\|(-\Delta)^{\frac{n}{2p}}u\|_{L^p(\mathbb{R}^n)} \leq 1$, the Taylor expansion yields

$$\int_{\mathbb{R}^n} \Phi_{n,p}\left(\alpha(n-s)|u(x)|^{p'}\right)\frac{dx}{|x|^s} = \sum_{j=j_0}^{\infty} \frac{[\alpha(n-s)]^j}{j!} \|u\|_{L^{p'j}\left(\mathbb{R}^n;\frac{dx}{|x|^s}\right)}^{p'j}.$$

Furthermore, we apply Theorem 1.1 for each norm $\|u\|_{L^{p'j}\left(\mathbb{R}^n\,;\,\frac{dx}{|x|^s}\right)}$ with $p'j \geq \tilde{p}$, and we get

$$\int_{\mathbb{R}^n} \Phi_{n,p}\left(\alpha(n-s)|u(x)|^{p'}\right)\frac{dx}{|x|^s} \leq \frac{1}{n-s}\left[\sum_{j=1}^{\infty}\frac{\left(\alpha\, C^{p'}p'j\right)^j}{j!}\right]\|u\|_{L^p(\mathbb{R}^n)}^{\frac{(n-s)p}{n}}.$$

In the end, we take $\alpha = \frac{1}{2C^{p'}p'e}$ so that $\beta = \sum_{j=1}^{\infty}\frac{\left(\frac{j}{2e}\right)^j}{j!} < \infty$. □

4 Acknowledgments

The authors would like to express their gratitude to the referee for his/her valuable comments. In addition, the first author is supported by Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists, and the second author is supported by the Fujukai Foundation.

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