# Scattering Theory for the Dirac Equation of Hartree Type in 2+1 Dimensions

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**Abstract**

Consider a scattering problem for the Dirac equation with a nonlocal term including the Hartree type in two dimensions. We show the existence of scattering operators for small data in the subcritical and critical Sobolev spaces.

## 1 Introduction

We consider a scattering problem for the Dirac equation with a nonlocal term

$$\partial_t\psi + \alpha\cdot\nabla\psi + i\beta\psi = \lambda[V * |\psi|^{p-1}]\psi, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^2. \tag{1.1}$$

Here $p>3,\ \partial_t=\partial/\partial t,\ \nabla=(\partial_1,\partial_2),\ \partial_j=\partial/\partial x_j,\ j=1,2,\ \lambda\in\mathbf{C},\ \alpha\cdot\nabla=\sum_{j=1}^{2}\alpha_j\partial_j,\ \ \alpha_j$'s, $j=1,2$ and $\beta$ are $2\times 2$ Hermitian matrices satisfying the usual anticommutation relations [‡]. The unknown function $\psi$ is a 2-spinor field defined on $\mathbf{R}\times\mathbf{R}^2$. The function $V=V(x)$ satisfies $|V(x)|\le|x|^{-\gamma}$ with $\gamma>0$, and $*$ denotes the convolution in space. Initial and final data in the Sobolev space $H^s$ are assumed to be small.

As a special case of (1.1), the Dirac equation of Hartree type, say $p=3$, with the Coulomb potential $V(x)=|x|^{-1}$ is derived from the Maxwell-Dirac equations with zero magnetic field. See Chadam and Glassey [4]. See also [5] and [6]. For related results, we refer to [1, 3, 8].

The scaling argument for the massless Dirac equation in $n$ dimensions, say $\partial_t\psi+\alpha\cdot\nabla\psi=\lambda(V*|\psi|^{p-1})\psi$ with $V(x)=|x|^{-\gamma}$ gives the value of the critical Sobolev exponent

[‡] $\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk}I,\ \alpha_j\beta+\beta\alpha_j=0,\ \beta^2=I$ for $1\le j,\ k\le 2$.

$s_c = (\gamma - 1 + n(p-3)/2)/(p-1)$. In [11] we have studied the scattering problem for (1.1) with small initial data in the Sobolev space $H^s$ in three and higher dimensions ($n \geq 3$), dividing the problem into the subcritical case $s > s_c$ and the critical one $s = s_c$. We state our results in [11]. Assume that $s < (p-1)/2$ if $p$ is not an odd integer. For the case $s > s_c$, let $p \geq 3$ and let $(\gamma, s)$ satisfy the following conditions:

$$\text{(H1)} \quad \begin{cases} s > s_c = \dfrac{\gamma - 1}{p-1} + \dfrac{n(p-3)}{2(p-1)}, \\[2ex] s > \dfrac{\gamma}{n(p-1)} + \dfrac{1}{2}, \\[2ex] \max\left\{ \dfrac{n(p-3)}{2(p-2)},\ 2 - \dfrac{n(p-3)}{2} \right\} < \gamma < n. \end{cases}$$

For the Sobolev space including the critical exponent $H^s$ with $s \geq s_c$, let $p > 3$ and let $(\gamma, s)$ satisfy

$$\text{(H2)} \quad \begin{cases} s \geq s_c, \\[2ex] \max\left\{ \dfrac{n(p-3)}{2(p-2)},\ \dfrac{2n}{n-1} - \dfrac{n(p-3)}{2} \right\} < \gamma < n. \end{cases}$$

We have proved the existence of scattering operators for (1.1) with small initial data in the subcritical Sobolev space $H^s$ with $s > s_c$ (resp. in the Sobolev space including the critical exponent $H^s$ with $s \geq s_c$) under the condition (H1) (resp. (H2)).

Our aim of this paper is to show that the similar results hold also in two dimensions. Our basic tools for the proofs are the Strichartz estimate for the Klein-Gordon equation proved by Machihara, Nakanishi and Ozawa [10], and the interpolation inequality by Escobedo and Vega [7].

# 2 Subcritical case

We first give some notation. We denote by $H^s_r$ and $B^s_r$ the usual inhomogeneous Sobolev and Besov spaces on $\mathbf{R}^2$, respectively. We write $H^s = H^s_2$. For the definitions of these spaces, see, e.g., [2]. For functions defined on space-time, we write $L^q_t B^s_r = L^q_t(\mathbf{R}; B^s_r)$.

Let $p > 3$. We assume that $(\gamma, s)$ satisfy the condition (H1) with $n = 2$, say

$$\text{(H1)'} \quad \begin{cases} s > s_c = \dfrac{\gamma + p - 4}{p - 1}, \\[2ex] s > \dfrac{\gamma}{2(p-1)} + \dfrac{1}{2}, \\[2ex] \max\left\{\dfrac{p-3}{p-2},\ 5 - p\right\} < \gamma < 2. \end{cases}$$

The set of $(\gamma, s)$ satisfying (H1)' when $p > 5$ and $7 - p > (p-3)/(p-2)$ is shown in Figure 1.

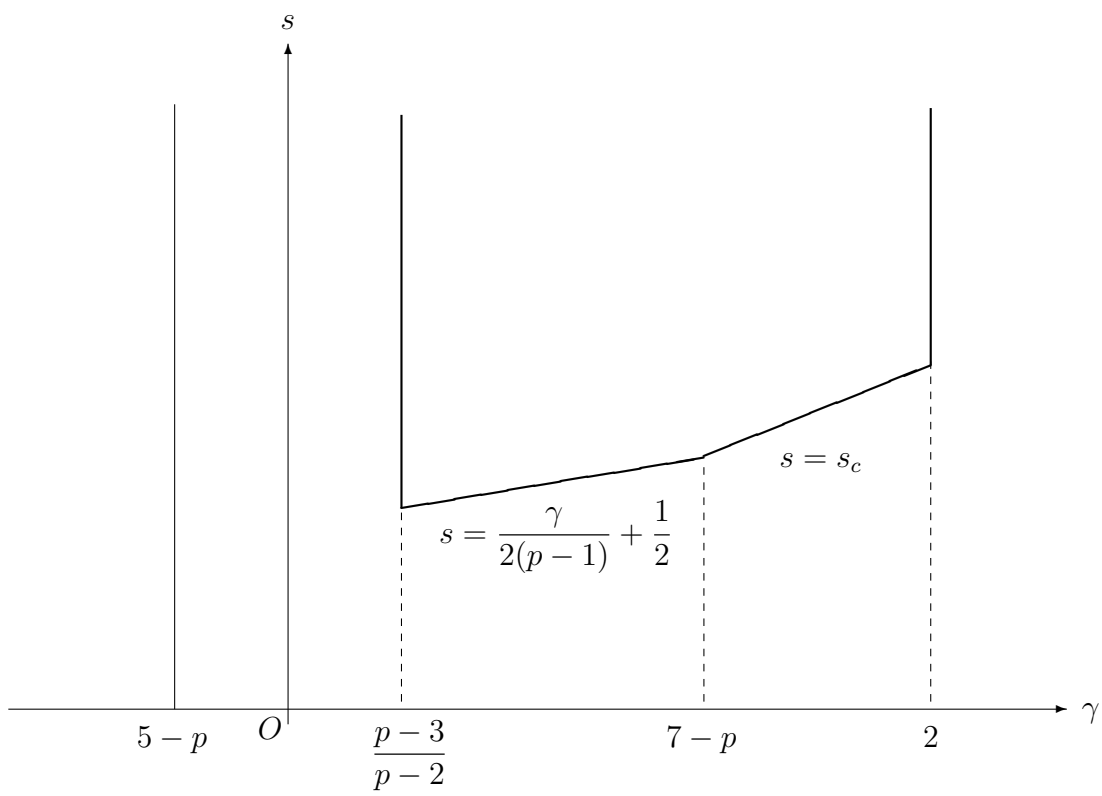


Figure 1: Domain of $(\gamma, s)$ satisfying (H1)' when $p > 5$ and $7 - p > (p-3)/(p-2)$

By choosing $0 \le \theta \le 1$ depending on $s$, $\gamma$ and $p$, we put

$$\frac{1}{r} = \frac{1}{2} - \frac{2}{(p-1)(1+\theta)}, \quad \sigma = \frac{1}{p-1} + \frac{2}{(p-1)(1+\theta)}. \tag{2.1}$$

We determine $\theta$ so that $r > 0$ and $s > \sigma$. See [11] for details.

For $p$, $r$ and $\sigma$ above and $s > 0$, we define $X_\theta^s$ by

$$X_\theta^s = (C^0 \cap L^\infty)(\mathbf{R}; H^s) \bigcap L^{p-1}(\mathbf{R}; B_r^{s-\sigma}). \tag{2.2}$$

We now state our first main result, which is about the Cauchy problem for (1.1) with the initial data $\psi_0(x)$:

$$\begin{cases} \partial_t \psi + \alpha \cdot \nabla \psi + i\beta\psi = \lambda[V * |\psi|^{p-1}]\psi, & (t,x) \in \mathbf{R} \times \mathbf{R}^2, \\ \psi(0,x) = \psi_0(x), & x \in \mathbf{R}^2. \end{cases} \tag{2.3}$$

**Theorem 2.1** *(i) Let $p > 3$. Assume (H1)' and let $s < (p-1)/2$ if $p$ is not an odd integer. Then there exists $0 \le \theta \le 1$ , depending only on $s$, $\gamma$ and $p$ such that if $\|\psi_0\|_{H^s}$ is sufficiently small, then (2.3) admits a unique global solution $\psi \in X_\theta^s$ with $s > \sigma$ defined in (2.2).*

*(ii) For the global solution $\psi$ given in (i), there exist unique $\psi_\pm \in H^s$ such that*

$$\lim_{t \to \pm\infty} \|\psi(t) - U(t)\psi_\pm\|_{H^s} = 0, \tag{2.4}$$

*where $U(t)$ denotes the Dirac group, which solves the free Dirac equation.*

**Remark.** (i) The set of $(\gamma, s)$ satisfying (H1)' and $s < (p-1)/2$ is not empty since $p > 3$.

(ii) The condition $s < (p-1)/2$ in the theorem is not necessary if $p$ is an odd integer. See the proof of Theorem 2.1 in [11].

(iii) If we treat the case $s$ close to $s_c$, it is necessary to assume that $p > 5$. See Figure 1.

We next consider the final value problem for (1.1) with data given at $t = -\infty$:

$$\psi(t,x) = U(t)\psi^-(x) + \int_{-\infty}^t U(t-t')F(\psi(t'))dt', \tag{2.5}$$

where $F(\psi) = \lambda[V * |\psi|^{p-1}]\psi$. Note that $U(t)\psi^-(x)$ is a solution of the linear Dirac equation $\partial_t \psi + \alpha \cdot \nabla \psi + i\beta\psi = 0$ with the initial data $\psi^-$ at $t = 0$.

**Theorem 2.2** *Let $s$, $\gamma$ and $p$ be as in the preceding theorem. Then there exists $0 \le \theta \le 1$ , depending only on $s$, $\gamma$ and $p$ such that if $\|\psi^-\|_{H^s}$ is sufficiently small, then the integral equation (2.5) has a unique solution $\psi \in X_\theta^s$ with $s > \sigma$ satisfying*

$$\lim_{t \to -\infty} \|\psi(t) - U(t)\psi^-\|_{H^s} = 0. \tag{2.6}$$

From Theorems 2.1 and 2.2, we can define the scattering operator for small initial data in the case $p > 3$ under the assumptions of Theorem 2.1. See, e.g., [12].

*Sketch of Proofs of Theorems*

To prove Theorem 2.1, we rewrite (2.3) as the following integral equation

$$\psi(t) = U(t)\psi_0 + \int_0^t U(t-t')F(\psi(t'))dt', \tag{2.7}$$

where $F(\psi) = \lambda[V * |\psi|^{p-1}]\psi$, and $U(t)$ is the free propagator defined on $L^2(\mathbf{R}^2; \mathbf{C}^2)$ given by

$$U(t) = I\cos t(1-\Delta)^{1/2} - (\alpha \cdot \nabla + i\beta)(1-\Delta)^{-1/2}\sin t(1-\Delta)^{1/2}.$$

We use the following lemma, which is the Strichartz estimates for $U(t)$.

**Lemma 2.3** *Let $n \geq 2$. Then one has the estimates*

$$\|U(t)u\|_{L_t^{q_1}B_{r_1}^{-s_1}} \lesssim \|u\|_{L^2}, \tag{2.8}$$

$$\left\| \int_{t'<t} U(t-t')F(t')dt' \right\|_{L_t^{q_2}B_{r_2}^{-s_2}} \lesssim \|F\|_{L_t^{q_3'}B_{r_3'}^{s_3}}, \tag{2.9}$$

*where $2/q_j = (n-1+\theta)(1/2 - 1/r_j)$, $2s_j = (n+1+\theta)(1/2-1/r_j)$ for $0 \leq \theta \leq 1$, $2 \leq q_j,\ r_j \leq \infty$, $(q_j, r_j) \neq (2, \infty)$, $j = 1, 2, 3$ when $n = 2,\ 3$, and $p'$ denotes the conjugate exponent to $p$, i.e., $1/p + 1/p' = 1$.*

See [10] for its proof.

**Remark.** The pair $(q_j, r_j) = (2, \infty)$ is called "end point" where the Strichartz estimates fail. Since we put (2.1) and $p > 3$, our admissible pair $(q, r) = (\frac{1}{p-1}, 1/2 - 2/((p-1)(1+\theta)))$ satisfies $(q, r) \neq (2, \infty)$.

The global existence is proved by using Lemma 2.3 and contraction argument. We can show the rest of Theorems 2.1 and 2.2 in the same way as in [11].

# 3 Critical case

In this section, we consider the same problem as in the previous section for the Sobolev exponent including the critical one. Let $p > 5$. We make some preparations to state the

result. We assume that $(\gamma, s)$ satisfy the condition (H2) with $n = 2$, say

$$\text{(H2)'} \quad \begin{cases} s \geq s_c = \dfrac{\gamma + p - 4}{p - 1}, \\ \\ \max\left\{\dfrac{p-3}{p-2},\ 7 - p\right\} < \gamma < 2. \end{cases}$$

We define some function spaces. We put

$$\begin{cases} \dfrac{1}{r_0} = \dfrac{1}{2} - \dfrac{2}{p-1}, \\ \\ \sigma_0 = \dfrac{3}{p-1}. \end{cases} \tag{3.1}$$

Note that $0 < 1/r_0 < 1/2$ since $p > 5$. For any $(\gamma, s)$ satisfying (H2), we have $s > \sigma_0$, hence we can choose $\sigma_1$ and $\sigma_2$ such that

$$0 < \sigma_2 < \sigma_0 < \sigma_1 < \min\{(s+1)\sigma_0/(1+\sigma_0),\ (3-\gamma)\sigma_0,\ 3/2\}. \tag{3.2}$$

For each $\sigma_j,\ j = 1, 2$, we define $q_j$ and $r_j,\ j = 1, 2$ by

$$\frac{1}{q_j} = \frac{\sigma_j}{3} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r_j}\right),\ j = 1, 2. \tag{3.3}$$

Then we see that

$$2 < q_1 < p - 1 < q_2. \tag{3.4}$$

For $p,\ r_i,\ \sigma_i,\ i = 0, 1, 2,\ q_j,\ j = 1, 2$ above and $s > 0$, we set

$$Y^s = (C^0 \cap L^\infty)(\mathbf{R}; H^s) \bigcap L^{p-1}(\mathbf{R}; B^{s-\sigma_0}_{r_0}) \bigcap L^{q_1}(\mathbf{R}; B^{s-\sigma_1}_{r_1}) \bigcap L^{q_2}(\mathbf{R}; B^{s-\sigma_2}_{r_2}). \tag{3.5}$$

**Theorem 3.1** *Let $p > 5$ and let $(\gamma, s)$ satisfy (H2)'. Assume that $s < (p-1)/2$ if $p$ is not an odd integer. If $\|\psi_0\|_{H^s}$ is sufficiently small, then there exists a unique global solution $\psi$ of (2.3) such that $\psi \in Y^s$ defined in (3.5).*

*Moreover, for the global solution $\psi$ given in (i), there exist unique $\psi_\pm \in H^s$ satisfying (2.4).*

**Theorem 3.2** *Let $s,\ \gamma$ and $p$be as in the preceding theorem. If $\|\psi^-\|_{H^s}$ is sufficiently small, then the integral equation (2.5) has a unique solution $\psi \in Y^s$ satisfying (2.6).*

From Theorems 3.1 and 3.2, we can also define the scattering operator for small initial data in the case $p > 5$ under the assumptions of Theorem 3.1.

*Sketch of Proofs of Theorems*

We can show Theorems 3.1 and 3.2 in the same way as in the proofs of Theorems 4.1 and 4.2 in [11]. We use the following lemma, which was proved by Escobedo and Vega [7].

**Lemma 3.3** *Let* $1 < a, b < \infty,\ 0 < \alpha,\ \beta < n$ *and* $0 < \delta < 1$ *satisfy*

$$\delta\left(\frac{1}{a} - \frac{\alpha}{n}\right) + (1-\delta)\left(\frac{1}{b} - \frac{\beta}{n}\right) = 0, \tag{3.6}$$
$$\frac{1}{a} - \frac{\alpha}{n} \neq 0 \quad \text{and} \quad \frac{1}{b} - \frac{\beta}{n} \neq 0.$$

*Then we have*

$$\|u\|_{L^\infty(\mathbf{R}^n)} \leq C\|u\|_{H_a^\alpha}^{\delta}\|u\|_{H_b^\beta}^{1-\delta}.$$

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