Measurement-Theoretic Foundation of Threshold Utility Maximiser's Preference Logic (Extended Abstract)

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Abstract. The first problem of this paper is as follows: what kind of preference logic can formalise inferences in which indifference is not transitive? (Nontransitivity Problem) The aim of this paper is to propose a new version of complete and decidable extrinsic preference logic–threshold utility maximiser's preference logic (TUMPL) that can solve the Nontransitivity Problem. Generally, preference logics are in danger of inviting the following problem: almost every principle which has been proposed as fundamental to one preference logic has been rejected by another one. (Fundamental Problem of Intrinsic Preference) A corollary of the Scott-Suppes theorem in measurement theory enables TUMPL to avoid the Fundamental Problem of Intrinsic Preference.

Key Words: preference logic, semiorder, Sorites Paradox, threshold utility maximisation, bounded rationality, measurement theory, representation theorem.

1 Introduction

The economist Armstrong ([1]) was one of the first to argue that *indifference* is not always *transitive*. Luce ([[5]: 179]) gave the following counterexample to the transitivity of indifference:

Example 1 (Avoidance of Sorites Paradox in Preference). If indifference were transitive, then he would be unable to detect any weight differences, however great, which is patently false.... Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes.... Now prepare 401 cups of coffee with $(1 + \frac{i}{100})x$ grams of sugar, $i = 0, 1, \ldots, 400$, where x is the weight of one cube of sugar. It is evident that he will be indifferent between cup i and cup $i + 1$, for any i, but by choice he is not indifferent between $i = 0$ and $i = 400$. ■

This example shows a situation where we would face the *Sorites Paradox*[1] in preference if indifference were transitive. The first problem now arises:

[1] [4] gives a comprehensive survey of topics of vagueness.

Problem 1 (Nontransitivity Problem). What kind of preference logic can formalise inferences in which indifference is not transitive? ■

We call it the *Nontransitivity Problem*. The aim of this paper is to propose a new version of complete and decidable *extrinsic* preference logic–*threshold utility maximiser's preference logic* (TUMPL) that can solve the Nontransitivity Problem.

Generally, preference-based logics are in danger of inviting the following problem. Von Wright ([14]) divided preferences into two categories: *extrinsic* and *intrinsic* preference. An agent is said to prefer φ_1 extrinsically to φ_2 if φ_1 is better than φ_2 in some explicit respect. So we can explain extrinsic preference from some explicit point of view. If we cannot explain preference from any explicit point of view, we call it intrinsic. Most preference logics that have been proposed are intrinsic but little attention has been paid to extrinsic preference. Von Wright ([15]) posed the following fundamental problem intrinsic preference logics faced.

Problem 2 (Fundamental Problem of Intrinsic Preference). The development of a satisfactory logic of preference has turned out to be unexpectedly problematic. The evidence for this lies in the fact that almost every principle which has been proposed as fundamental to one preference logic has been rejected by another one. ■

We call it the *Fundamental Problem of Intrinsic Preference*. For example, the status of such logical properties as Transitivity, Contraposition, Conjunctive expansion, Disjunctive Distribution and Conjunctive Distribution is as follows:

Example 2 (Variety of Preferences).

	von Wright ([14])	Martin ([6])	Chisholm and Sosa ([3])
Transitivity	+	+	+
Contraposition	−	+	−
Conjunctive Expansion	+	−	−
Disjunctive Distribution	−	−	−
Conjunctive Distribution	+	−	−

'+' denotes the property in question being provable in the logic in question. '−' denotes the property in question not being provable in the logic in question. Conjunctive Expansion says that an agent does not prefer φ_1 to φ_2 iff he does not prefer $\varphi_1 \& \neg\varphi_2$ to $\varphi_2 \& \neg\varphi_1$. Disjunctive Distribution says that if he does not prefer $\varphi_1 \vee \varphi_2$ to φ_3, then he does not prefer φ_1 to φ_3 or does not prefer φ_2 to φ_3. Conjunctive Distribution says that if he does not prefer φ_1 to φ_2 and does not prefer φ_3 to φ_2, then he does not prefer $\varphi_1 \vee \varphi_3$ to φ_2. ■

According to Mullen ([7]), we can analyse its cause as follows. The adequacy criteria for intrinsic preference principles considered by preference logicians have been whether the principles are consistent with our *intuitions* of reasonableness. But each intuition often disagrees even on the fundamental properties. Different theories, such as ethics, welfare economics, consumer demand theory, game theory and decision theory make different demands upon the fundamental properties of preference. So if we would like to propose preference logic that can avoid the Fundamental Problem of Intrinsic Preference, it should be constructed not from intuition but from a *theory* or a *rule in a theory*, that is, it should be *extrinsic*. In order to avoid the Fundamental Problem of Intrinsic Preference, we resort to *measurement theory*.[2] There are two main problems with measurement theory:

1. the representation problem–justifying the assignment of numbers to objects or propositions,
2. the uniqueness problem–specifying the transformation up to which this assignment is unique.

A solution to the former can be furnished by a *representation theorem*, which establishes that the chosen numerical system preserves the relations of the relational system. When we provide TUMPL with a model based on semiorders, by virtue of a *corollary* of the *Scott-Suppes representation theorem*, we adopt *threshold utility maximisation* as a rule in decision theory that makes demands upon the fundamental properties of preference, which enables TUMPL to avoid the Fundamental Problem of Intrinsic Preference.

The structure of this paper is as follows. In Section 2, we state weak orders as global rationality and semiorders as bounded rationality. In Section 3, we define the language $\mathcal{L}_{\mathsf{TUMPL}}$ of TUMPL, and define a structured Kripke model $\mathcal{M}$ for preference, and provide TUMPL with a truth definition, and provide TUMPL with a proof system, and sketch the proof of the soundness, completeness and decidability of TUMPL.

2 Weak Orders as Global Rationality and Semiorders as Bounded Rationality

The standard model of economics is based on *global rationality* that requires an *optimising behavior*. *Utility maximisation* is a typical example of an optimising behavior. Cantor ([2]) proved the representation theorem for utility maximisation.

Theorem 1 (Representation for Utility Maximisation, Cantor ([2])). *Suppose* $\mathbf{A}$ *is a countable set and* $\succeq$ *is a binary relation on* $\mathbf{A}$*. Then* $\succeq$ *is a weak order (transitive and connected) iff there is a function* $u : \mathbf{A} \to \mathbb{R}$ *such that for any* $x, y \in \mathbf{A}$,

$$x \succeq y \text{ iff } u(x) \geq u(y).$$

■

But according to Simon ([12]), cognitive and information-processing constrains on the capabilities of agents, together with the complexity of their environment, render an optimising behavior an unattainable ideal. He dismissed the idea that

[2] [8] gives a comprehensive survey of measurement theory.

agents should exhibit global rationality and suggested that they in fact exhibit *bounded rationality* that allows a *satisficing behavior*.[3] One explanation for Example 1 is that the nontransitivity of indifference results from the fact that we cannot generally discriminate very close quantities. The concept of a *semiorder* was introduced by Luce ([5]) to construct a model to interpret situations like Example 1 of nontransitive indifference with a *threshold* of discrimination. Scott and Suppes defined ([[9]: 117]) a semiorder as follows:

Definition 1 (Semiorder). *$\succ$ on $\mathbf{A}$ is called a semiorder if, for any $w, x, y, z \in \mathbf{A}$, the following conditions are satisfied:*

1. *$x \not\succ x$. (Irreflexivity),*
2. *If $w \succ x$ and $y \succ z$, then $w \succ z$ or $y \succ x$. (Intervality),*
3. *If $w \succ x$ and $x \succ y$, then $w \succ z$ or $z \succ y$. (Semitransitivity).*

■

Threshold utility maximisation is a typical example of a satisficing behavior. Scott and Suppes ([9]) proved a representation theorem for threshold utility maximisation when $\mathbf{A}$ is *finite*.

Theorem 2 (Representation for Threshold Utility Maximisation, Scott and Suppes ([9])). *Suppose that $\succ$ is a binary relation on a finite set $\mathbf{A}$ and δ is a positive number. Then $\succ$ is a semiorder iff there is a function $u : \mathbf{A} \to \mathbb{R}$ such that for any $x, y \in \mathbf{A}$,*

$$x \succ y \text{ iff } u(x) > u(y) + \delta.$$

■

Remark 1. Scott ([10]) simplified the Scott-Suppes theorem in terms of the solvability of finite system of linear inequalities. ■

3 Threshold Utility Maximiser's Preference Logic TUMPL

3.1 Language

We define the language $\mathcal{L}_{\mathsf{TUMPL}}$ of TUMPL.

Definition 2 (Language). *Let $\mathbf{S}$ denote a set of sentential variables, $\Box$ a necessity operator, $\mathbf{SPR}$ a strict preference relation symbol. The language $\mathcal{L}_{\mathsf{TUMPL}}$ of TUMPL is given by the following rule:*

$$\varphi ::= s \mid \top \mid \neg\varphi \mid \varphi_1 \& \varphi_2 \mid \Box\varphi \mid \mathbf{SPR}(\varphi_1, \varphi_2),$$

[3] We learned from van Rooij ([13]) the relation between the Sorites Paradox and bounded rationality.

where $s \in \mathbf{S}$, and nestings of **SPR** *do not occur.* $\bot, \vee, \rightarrow, \leftrightarrow$ *and* $\Diamond$ *are introduced by the standard definitions. We define an indifference relation symbol* **IND** *and a weak preference relation symbol* **WPR** *as follows:*

$$\mathbf{IND}(\varphi_1, \varphi_2) := \neg\mathbf{SPR}(\varphi_1, \varphi_2)\&\neg\mathbf{SPR}(\varphi_2, \varphi_1),$$
$$\mathbf{WPR}(\varphi_1, \varphi_2) := \mathbf{SPR}(\varphi_1, \varphi_2) \vee \mathbf{IND}(\varphi_1, \varphi_2).$$

The set of all well-formed formulae of $\mathcal{L}_{\mathsf{TUMPL}}$ *will be denoted by* $\Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$. ■

Remark 2. We introduce $\Box$ to construct a Boolean algebra of subsets of $\mathbf{W}$ that is accessible from $w \in \mathbf{W}$. ■

3.2 Semantics

We define a structured Kripke model $\mathcal{M}$ for TUMPL.

Definition 3 (Model). *$\mathcal{M}$ is a quadruple $(\mathbf{W}, R, V, \rho)$, where:*

- *$\mathbf{W}$ is a nonempty set of possible worlds,*
- *R is a binary relation on $\mathbf{W}$,*
- *V is a truth assignment to each $s \in \mathbf{S}$ for each $w \in \mathbf{W}$,*
- *ρ is a preference space assignment that assigns to each $w \in \mathbf{W}$ a preference space $(\mathcal{F}_w, \succ_w)$ such that $\mathcal{F}_w$ is a Boolean algebra of subsets of $\{w' \in \mathbf{W} : R(w, w')\}$ and $\succ_w$ on $\mathcal{F}_w$ is a semiorder.*

■

Since A is an *arbitrary* finite set, the next corollary follows directly from Theorem 2.

Corollary 1 (Representation on Finite Boolean Algebra). *Suppose that $\mathbf{W}$ is a finite set of possible worlds and $\mathcal{F}$ is a finite Boolean algebra of subsets of $\mathbf{W}$ and $\succ$ is a binary relation on $\mathcal{F}$, and δ is a positive number. Then $\succ$ is a semiorder iff there is a function $u : \mathcal{F} \rightarrow \mathbb{R}$ such that for any $\alpha, \beta \in \mathcal{F}$,*

$$\alpha \succ \beta \text{ iff } u(\alpha) > u(\beta) + \delta.$$

■

Remark 3. Corollary 1, together with Corollary 2 described later, can guarantee that $\succ_w$ on $\mathcal{F}_w$ is a threshold utility maximiser's preference relation. ■

We provide TUMPL with the following truth definition relative to $\mathcal{M}$:

Definition 4 (Truth). *The notion of $\varphi \in \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$ being true at $w \in W$ in $\mathcal{M}$, in symbols $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \varphi$ is inductively defined as follows:*

- $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} s$ *iff* $V(w)(s) = \mathbf{true}$,
- $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \top$,
- $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \varphi_1\&\varphi_2$ *iff* $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \varphi_1$ *and* $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \varphi_2$,
- $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \neg\varphi$ *iff* $(\mathcal{M}, w) \not\models_{\mathsf{TUMPL}} \varphi$,
- $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \Box\varphi$ *iff, for any w' such that $R(w, w')$,* $(\mathcal{M}, w') \models_{\mathsf{TUMPL}} \varphi$,
- $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \mathbf{SPR}(\varphi_1, \varphi_2)$ *iff* $[\![\varphi_1]\!]^{\mathcal{M}}_w \succ_w [\![\varphi_2]\!]^{\mathcal{M}}_w$,

where $[\![\varphi]\!]_w^{\mathcal{M}} := \{w' \in \mathbf{W} : R(w, w')$ *and* $(\mathcal{M}, w') \models_{\mathsf{TUMPL}} \varphi\}$. *If* $(\mathcal{M}, w) \models_{\mathsf{TUMPL}} \varphi$ *for all* $w \in \mathbf{W}$, *we write* $\mathcal{M} \models_{\mathsf{TUMPL}} \varphi$ *and say that* φ *is valid in* $\mathcal{M}$. *If* φ *is valid in all structured Kripke models for* TUMPL, *we write* $\models_{\mathsf{TUMPL}} \varphi$ *and say that* φ *is valid.* ■

We would like to provide a *counter-model* of the transitivity of indifference. We now return to Example 1. Assume that $\mathcal{U} := (\mathbf{W}, R, V, \rho)$ is given, where:

- $\mathbf{W} := \{w_0, \ldots, w_{400}\}$, where w_i is a possible world in which you try a cup of coffee with $(1 + \frac{i}{100})x$ grams of sugar, for any i $(0 \leq i \leq 400)$,
- R is a binary relation on $\mathbf{W}$,
- V is a truth assignment to each $s \in \mathbf{S}$ for each $w_i \in \mathbf{W}$,
- ρ is a preference space assignment that assigns to each $w_i \in \mathbf{W}$ a preference space $(\mathcal{F}_{w_i}, \succ_{w_i})$, where:
  - $\mathcal{F}_{w_i}$ is a Boolean algebra of subsets of $\{w'_i \in \mathbf{W} : R(w_i, w'_i)\}$,
  - $\succ_{w_i}$ on $\mathcal{F}_{w_i}$ is a semiorder,
  - $\{w_j\} \sim_{w_i} \{w_{j+1}\}$, for any j $(0 \leq j \leq 400)$,
  - $\{w_0\} \not\sim_{w_i} \{w_{400}\}$.

Let φ_i denote the sentence "You try a cup of coffee with $(1 + \frac{i}{100})x$ grams of sugar", for any i $(0 \leq i \leq 400)$. Then we have, for any i $(0 \leq i \leq 400)$,

$$[\![\varphi_j]\!]_{w_i}^{\mathcal{U}} \sim_{w_i} [\![\varphi_{j+1}]\!]_{w_i}^{\mathcal{U}}, \text{ for any } j \ (0 \leq j \leq 400),$$
$$[\![\varphi_0]\!]_{w_i}^{\mathcal{U}} \not\sim_{w_i} [\![\varphi_{400}]\!]_{w_i}^{\mathcal{U}},$$

for $[\![\varphi_j]\!]_{w_i}^{\mathcal{U}} = \{w_j\}$ holds for any j $(0 \leq j \leq 400)$. It must be noted that, for any i $(0 \leq i \leq 400)$, because $\succ_{w_i}$ on $\mathcal{F}_{w_i}$ is a semiorder, $[\![\varphi_j]\!]_{w_i}^{\mathcal{U}} \sim_{w_i} [\![\varphi_{j+1}]\!]_{w_i}^{\mathcal{U}}$ for any j $(0 \leq j \leq 400)$ does not imply $[\![\varphi_0]\!]_{w_i}^{\mathcal{U}} \sim_{w_i} [\![\varphi_{400}]\!]_{w_i}^{\mathcal{U}}$. So we have, for any i $(0 \leq i \leq 400)$,

$$(\mathcal{U}, w_i) \not\models_{\mathsf{TUMPL}} (\mathbf{IND}(\varphi_0, \varphi_1) \& \cdots \& \mathbf{IND}(\varphi_{399}, \varphi_{400})) \rightarrow \mathbf{IND}(\varphi_0, \varphi_{400}).$$

Therefore we obtain the following proposition.

Proposition 1 (Nontransitivity of Indifference).

$$\not\models_{\mathsf{TUMPL}} (\mathbf{IND}(\varphi_0, \varphi_1) \& \cdots \& \mathbf{IND}(\varphi_{399}, \varphi_{400})) \rightarrow \mathbf{IND}(\varphi_0, \varphi_{400}).$$

■

3.3 Syntax

We provide TUMPL with a proof system.

Definition 5 (Proof System). *The proof system of* TUMPL *consists of the following:*

1. *all tautologies of classical sentential logic,*
2. $\Box(\varphi_1 \rightarrow \varphi_2) \rightarrow (\Box\varphi_1 \rightarrow \Box\varphi_2)$ $\quad$ *(K)*,

3. $\Box(\varphi_1 \leftrightarrow \varphi_2) \& \Box(\psi_1 \leftrightarrow \psi_2) \to (\mathbf{SPR}(\varphi_1, \psi_1) \leftrightarrow \mathbf{SPR}(\varphi_2, \psi_2))$
(Replacement of Necessary Equivalents),
4. $\neg\mathbf{SPR}(\varphi, \varphi)$
(Syntactic Counterpart of Irreflexivity),
5. $(\mathbf{SPR}(\varphi_1, \varphi_2) \wedge \mathbf{SPR}(\varphi_3, \varphi_4)) \to (\mathbf{SPR}(\varphi_1, \varphi_4) \vee \mathbf{SPR}(\varphi_3, \varphi_2))$
(Syntactic Counterpart of Intervality),
6. $(\mathbf{SPR}(\varphi_1, \varphi_2) \wedge \mathbf{SPR}(\varphi_2, \varphi_3)) \to (\mathbf{SPR}(\varphi_1, \varphi_4) \vee \mathbf{SPR}(\varphi_4, \varphi_3))$
(Syntactic Counterpart of Semitransitivity),
7. Modus Ponens,
8. Necessitation.

A proof of $\varphi \in \Phi_{\mathsf{TUMPL}}$ *is a finite sequence of* $\mathcal{L}_{\mathsf{TUMPL}}$*-formulae having* φ *as the last formula such that either each formula is an instance of an axiom, or it can be obtained from formulae that appear earlier in the sequence by applying an inference rule. If there is a proof of* φ*, we write* $\vdash_{\mathsf{TUMPL}} \varphi$. ■

3.4 Metalogic

We prove the metatheorems of TUMPL. It is easy to prove the soundness of TUMPL.

Theorem 3 (Soundness). *For any* $\varphi \in \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$*, if* $\vdash_{\mathsf{TUMPL}} \varphi$*, then* $\models_{\mathsf{TUMPL}} \varphi$*.* ■

We now turn to the task of proving the completeness of TUMPL. We prove it by developing the idea of Segerberg ([11]) that we modify *filtration theory* in such a way that completeness can be established by *Corollary 1 (Representation on Finite Boolean Algebra).* We cannot go into details because of limited space. The outline of the proof is as follows. We begin by defining some new concepts.

Definition 6 (Stuffedness). *Suppose that* Θ *is a set of formulae such that* Θ *is closed under subformulae and* $\bot \in \Theta$*. Let*

$$\Delta := \{\varphi : \textit{for some } \psi,\ \mathbf{SPR}(\varphi, \psi) \textit{ or } \mathbf{SPR}(\psi, \varphi)\},$$

and let Δ' *be the closure of* Δ *under Boolean compounds. If* Θ *satisfies also the condition that* $\mathbf{SPR}(\varphi, \psi) \in \Theta$*, for any* $\varphi, \psi \in \Delta'$*, we say that* Θ *is stuffed.* ■

Definition 7 (Value Formula). *The formulae in* Δ' *are called the value formulae of* Θ*.* ■

Remark 4. There is no occurrence of **SPR** in value formulae. ■

Definition 8 (Base). *We say that* $\Psi_0 \subseteq \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$ *is a base (with respect to* TUMPL*) for* $\Psi \subseteq \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$ *if for any* $\varphi \in \Psi$ *there is some* $\varphi_0 \in \Psi_0$ *such that* $\vdash_{\mathsf{TUMPL}} \varphi \leftrightarrow \varphi_0$*.* ■

Definition 9 (Logical Finiteness). *We say that* Ψ *is logically finite (with respect to* TUMPL*) if there is a finite base for* Ψ*.* ■

Remark 5. A set has finite base if it has a logically finite one. ■

Lemma 1 (Logical Finiteness). *If $\Psi \subseteq \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$ is a finite set closed under subformulae, and if Θ is the smallest stuffed superset of Ψ, then Θ is logically finite.* ■

Definition 10 (Maximal Consistency). *A finite set $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$ is* TUMPL*-consistent iff $\nvdash_{\mathsf{TUMPL}} \neg(\varphi_1 \& \ldots \& \varphi_n)$. An infinite set of formulae is* TUMPL*-consistent iff all of its finite subsets are* TUMPL*-consistent. $\Gamma \subseteq \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$ is a* TUMPL*-maximal consistent set iff it is* TUMPL*-consistent and for any $\varphi \notin \Gamma$, $\Gamma \cup \{\varphi\}$ is* TUMPL*-inconsistent.* ■

Definition 11 (Canonical Model for Modal-Logical Part of TUMPL). *We define $\mathcal{U}^C := (\mathbf{W}^C, R^C, V^C)$ as the canonical model for the modal-logical part of* TUMPL *where*

- $\mathbf{W}^C := \{\Gamma \subseteq \Phi_{\mathcal{L}_{\mathsf{TUMPL}}} : \Gamma$ *is* TUMPL*-maximal consistent*$\}$,
- *for any $\Gamma, \Delta \in \mathbf{W}^C$, $R^C(\Gamma, \Delta)$ iff for any $\varphi \in \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$, if $\Box\varphi \in \Gamma$, then $\varphi \in \Delta$.*
- *for any $\Gamma \in \mathbf{W}^C$,*

$$V^C(\Gamma)(s) := \begin{cases} \textit{\textbf{true}} & \text{if } s \in \Gamma, \\ \textit{\textbf{false}} & \text{otherwise.} \end{cases}$$

■

Definition 12 ($\mathcal{U}$). *We define $\mathcal{U} := (\mathbf{W}, R, V)$ by means of $\mathcal{U}^C$ as follows:*

- *$\mathbf{W}$ is a fixed subset of $\mathbf{W}^C$, either $\mathbf{W}^C$ itself or else $\{\Delta : \Gamma = \Delta$ or $R^{C*}(\Gamma, \Delta)\}$ for some $\Gamma \in \mathbf{W}^C$, where R^{C*} is the ancestral of R^C.*
- *R and V are the restrictions of R^C and V^C to $\mathbf{W}$.*

■

Definition 13 (Equivalence Class). *Let Θ be a stuffed set of formulae that are logically finite with respect to* TUMPL*. We define, for $\Gamma, \Delta \in \mathbf{W}$,*

$$\Gamma \equiv \Delta \text{ iff } \Gamma \cap \Theta = \Delta \cap \Theta.$$

Then $\equiv$ is an equivalence relation on $\mathbf{W}$. We write $[\Gamma]$ for the equivalence class of Γ. ■

Definition 14 (Filtration). *We define $\mathcal{U}^{\equiv} := (\mathbf{W}^{\equiv}, R^{\equiv}, V^{\equiv})$ as a filtration of $\mathcal{U}$ where*

- $\mathbf{W}^{\equiv} := \{[\Gamma] : \Gamma \in \mathbf{W}\}$,
- *$R^{\equiv}$ is a binary relation on $\mathbf{W}^{\equiv}$ such that*
  1. *if $R(\Gamma, \Delta)$, then $R^{\equiv}([\Gamma], [\Delta])$.*
  2. *if $R^{\equiv}([\Gamma], [\Delta])$ and $\Box\varphi \in \Gamma$, then $\varphi \in \Delta$.*

3. $V^{\equiv}$ *is a function such that for any* $s \in \Theta$,

$$V^{\equiv}([\Gamma])(s) = V(\Gamma)(s).$$

■

Thus, for any $\xi \in \mathbf{W}^{\equiv}$,

$$[\![\varphi]\!]_{\xi}^{\mathcal{U}^{\equiv}} := \{\eta : R^{\equiv}(\xi, \eta) \text{ and } (\mathcal{U}^{\equiv}, \eta) \models_{\mathsf{TUMPL}} \varphi\}$$

is well-defined for any φ which does not contain **SPR**.

Lemma 2 (Lindenbaum). *Every* TUMPL*-consistent set of formulae is a subset of a* TUMPL*-maximal consistent set of formulae.* ■

Lemma 3 (Partial Truth). *If* $\varphi \in \Theta$ *and* φ *does not contain* **SPR***, then for any* $\Gamma \in \mathbf{W}$,

$$(\mathcal{U}^{\equiv}, [\Gamma]) \models_{\mathsf{TUMPL}} \varphi \textit{ iff } \varphi \in \Gamma.$$

■

We wish to supplement $\mathcal{U}^{\equiv}$ with preference space assignment $\rho^{\equiv}$ so as to obtain a structured Kripke model $\mathcal{U}_{\sharp}^{\equiv}$ for which Truth Lemma holds for all formulae in Θ. Doing this contributes to solving the completeness problem of TUMPL.

Definition 15 ($\mathcal{F}_{\xi}^{\equiv}$). *For any* $\xi \in \mathbf{W}^{\equiv}$*, we define* $\mathcal{F}^{\equiv}$ *as the set of all* $\alpha \subseteq \{\eta : R^{\equiv}(\xi, \eta)\}$ *such that for some value formula* $\varphi \in \Theta$, $\alpha = [\![\varphi]\!]_{\xi}^{\mathcal{U}^{\equiv}}$. ■

Lemma 4 (Boolean Algebra). *For any* $\xi \in \mathbf{W}^{\equiv}$, $\mathcal{F}_{\xi}^{\equiv}$ *is a Boolean algebra with* $\{\eta : R^{\equiv}(\xi, \eta)\}$ *as unit element.* ■

Definition 16 ($\succ_{\xi}$). *For any* $\xi \in \mathbf{W}^{\equiv}$ *we define* $\alpha \succ_{\xi} \beta$ *to hold between elements* $\alpha, \beta \in \mathcal{F}_{\xi}^{\equiv}$ *iff there are value formulae* $\varphi, \psi \in \Theta$ *such that* $\alpha = [\![\varphi]\!]_{\xi}^{\mathcal{U}^{\equiv}}, \beta = [\![\psi]\!]_{\xi}^{\mathcal{U}^{\equiv}}$ *and* $\mathbf{SPR}(\varphi, \psi) \in \Gamma$ *for any* $\Gamma \in \xi$. ■

Lemma 5 ($\succ_{\xi}$ and SPR). *For any value formula* $\varphi, \psi \in \Theta$ *and any* $\xi \in \mathbf{W}^{\equiv}$, $[\![\varphi]\!]_{\xi}^{\mathcal{U}^{\equiv}} \succ_{\xi} [\![\psi]\!]_{\xi}^{\mathcal{U}^{\equiv}}$ *iff, for any* $\Gamma \in \xi$, $\mathbf{SPR}(\varphi, \psi) \in \Gamma$. ■

Lemma 6 (Irreflexivity, Intervality and Semitransitivity). *For any* $\xi \in \mathbf{W}^{\equiv}$, $\succ_{\xi}$ *on* $\mathcal{F}_{\xi}^{\equiv}$ *satisfies Irreflexivity, Intervality and Semitransitivity.* ■

Since we assumed that Θ is logically finite, $\mathbf{W}^{\equiv}$ is *finite*. Hence for any $\xi \in \mathbf{W}^{\equiv}$, $\mathcal{F}_{\xi}^{\equiv}$ is *finite*, so the next corollary follows from Lemma 4 and Lemma 6.

Corollary 2. *For any* $\xi \in \mathbf{W}^{\equiv}$*, Corollary 1 is provable in* $(\mathcal{F}_{\xi}^{\equiv}, \succ_{\xi})$. ■

Definition 17. *We define* $\mathcal{U}_{\sharp}^{\equiv} := (\mathbf{W}^{\equiv}, R^{\equiv}, V^{\equiv}, \rho^{\equiv})$ *as a structured Kripke model for preference where* $\rho^{\equiv}$ *is a preference space assignment that assigns to each* $\xi \in \mathbf{W}^{\equiv}$ $(\mathcal{F}_{\xi}^{\equiv}, \succ_{\xi})$. ■

Lemma 7 (Full Truth). *For any $\varphi \in \Theta$ and any $\Gamma \in \mathbf{W}$,*

$$(\mathcal{U}_{\sharp}^{\equiv}, [\Gamma]) \models_{\mathsf{TUMPL}} \varphi \textit{ iff } \varphi \in \Gamma.$$

■

Remark 6. This lemma is the announced improvement of Lemma 3. ■

Theorem 4 (Completeness). *For any $\varphi \in \Phi_{\mathcal{L}_{\mathsf{TUMPL}}}$, if $\models_{\mathsf{TUMPL}} \varphi$, then $\vdash_{\mathsf{TUMPL}} \varphi$.* ■

Proof. Suppose that $\nvdash_{\mathsf{TUMPL}} \varphi_0$. Then $\{\neg\varphi_0\}$ is a TUMPL-consistent set. By Lemma 2 $\{\neg\varphi_0\}$ is a subset of a TUMPL-maximal consistent set Γ. Evidently $\varphi_0 \notin \Gamma$. Let Ψ be the set of subformulae of TUMPL that is finite and let Θ be the smallest stuffed extension of Ψ. By Lemma 1 Θ is logically finite with respect to TUMPL. Let $\mathbf{W}$ be as mentioned above, either $\mathbf{W}^C$ itself or else $\{\Delta : \Gamma = \Delta$ or $R^{C*}(\Gamma, \Delta)\}$ (where R^{C*} is the ancestral of R^C). By Lemma 7 $(\mathcal{U}_{\sharp}^{\equiv}, [\Gamma]) \not\models_{\mathsf{TUMPL}} \varphi$. Therefore $\not\models_{\mathsf{TUMPL}} \varphi$. ■

We can prove the decidability of TUMPL as follows.

Lemma 8 (Finite Model Property). TUMPL *has the finite model property that every non-theorem of* TUMPL *fails in a structured Kripke model for preference with only finitely many elements.* ■

Theorem 5 (Decidability). TUMPL *is decidable.* ■

Proof. Suppose that φ is not provable in TUMPL. By Lemma 8 φ fails in a structured Kripke model $\mathcal{U}_{\sharp}^{\equiv}$ for preference with only finitely many elements. If we take a domain $\mathbf{W}^{\equiv}$ with at most that number of elements, there are only finitely many ways in which accessibility relations of and truth assignments can be defined, and there are also only finite many ways to define the preference space assignment $\rho^{\equiv}$. Finally for any $\xi \in \mathbf{W}^{\equiv}$, there are only finitely many ways in which $\succ_{\xi}$ can be defined. To decide, of a given relation $\succ_{\xi}$, whether it satisfies Irreflexivity, Intervality and Semitransitivity can be done in finitely many steps. So we can find, in at most a finite number of steps, a counter-model of the unprovable formula. In fact, we can compute an upper bound to the number of steps needed. Thus TUMPL is decidable. ■

4 Concluding Remarks

In this paper we have proposed a new version of complete and decidable extrinsic preference logic–threshold utility maximiser's preference logic (TUMPL) that can solve the Nontransitivity Problem and avoid the Fundamental Problem of Intrinsic Preference.

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