A Uniqueness Theorem and the Myrberg Phenomenon for a Zalcman Domain

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Abstract

Let $R = \Delta_0 \setminus \cup_n \Delta_n$ be a Zalcman domain (or L-domain), where $\Delta_0 : 0 < |z| < 1$, $\Delta_n : |z - c_n| \leq r_n$, $c_n \searrow 0$, $\Delta_n \subset \Delta_0$ and $\Delta_n \cap \Delta_m = \phi$ $(n \neq m)$. For an unlimited two-sheeted covering $\varphi : \widetilde{\Delta}_0 \to \Delta_0$ with the branch points $\{\varphi^{-1}(c_n)\}$, set $\widetilde{R} = \varphi^{-1}(R)$. In the case $c_n = 2^{-n}$, it was proved that if a uniqueness theorem is valid for $H^\infty(R)$ at $z = 0$, then the Myrberg phenomenon $H^\infty(R) \circ \varphi = H^\infty(\widetilde{R})$ occurs. One might suspect that the converse also holds. In this paper, contrary to this intuition, we show that the converse of this previous result is not true. In addition, we generalize the previous result for more general sequences $\{c_n\}$. By this generalization we can even partly simplify the previous proof.

1 Introduction

Let $\Delta(c, r)$ denote the open disc in the complex plane $\mathbf{C}$ with radius $r > 0$ centered at c, and set $\Delta = \Delta(0, 1)$ and $\Delta_0 = \Delta \setminus \{0\}$. For a strictly decreasing sequence $\{c_n\}_{n=1}^\infty$ with $0 < c_n < 1$ converging to 0 and a sequence $\{r_n\}_{n=1}^\infty$ of positive numbers satisfying

$$c_{n+1} + r_{n+1} < c_n - r_n \quad (n \in \mathbf{N}), \qquad c_1 + r_1 < 1, \tag{1.1}$$

where $\mathbf{N}$ is the set of positive integers, we consider the domain

$$R := R(c_n, r_n) := \Delta_0 \setminus \bigcup_{n=1}^{\infty} \overline{\Delta(c_n, r_n)} \ . \tag{1.2}$$

A domain of this form is called a *Zalcman domain* (or *L-domain* according to [7]). The condition (1.1) says that the closed discs $\overline{\Delta(c_n, r_n)}$ are contained in Δ_0 and mutually disjoint.

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We denote by $H^\infty(R)$ the Banach space of bounded holomorphic functions f on R equipped with the supremum norm $\|f\|_\infty$. Following [3], we say that the *uniqueness theorem* is valid for $H^\infty(R(c_n, r_n))$ at $z = 0$ when the following implication holds; if $f \in H^\infty(R(c_n, r_n))$ satisfies the condition

$$\lim_{z\to 0, z<0} f^{(m)}(z) = 0 \ (m = 0, 1, \cdots) \ ,$$

then $f \equiv 0$.

We consider an unlimited two-sheeted covering $\varphi : \widetilde{\Delta}_0 \to \Delta_0$ with the branch points $\{\varphi^{-1}(c_n)\}$, and write $\widetilde{W} = \varphi^{-1}(W)$ for $W \subset \Delta_0$. We say that the *Myrberg phenomenon* occurs for the covering surface $(\widetilde{W}, W, \varphi)$ if we have

$$H^\infty(\widetilde{W}) = H^\infty(W) \circ \varphi \ . \tag{1.3}$$

In his celebrated paper [6], Myrberg showed that (1.3) holds for $(\widetilde{\Delta}_0, \Delta_0, \varphi)$. His proof goes as follows. For each $f \in H^\infty(\widetilde{\Delta}_0)$, define bounded analytic functions g and h by $g(z) = (f(z^+) - f(z^-))^2$ and $h(z) = (f(z^+) + f(z^-))/2$, where $\varphi^{-1}(z) = \{z^+, z^-\}$. Since $z^+ = z^-$ at a branch point of φ, $g(c_n) = 0$ for all n. Then, $g \equiv 0$ by the classical uniqueness theorem, and hence, $h \circ \varphi = f$.

We are particularly interested in the case $R = R(c_n, r_n)$, which gives the simplest example of plane domains of infinite connectivity. Although the covering surface $(\widetilde{R}, R, \varphi)$ has no branch points, the uniqueness theorem is valid for $H^\infty(R)$ at $z = 0$ and the Myrberg phenomenon occurs for $(\widetilde{R}, R, \varphi)$ for a kind of Zalcman domains R (cf. [2], [3]; also, [4], [5]).

In this paper we are concerned with the following result [3](Proposition 3.1, Theorem 4.1).

Theorem *Let $R = R(2^{-n}, 2^{-nN(n)})$. Suppose that the uniqueness theorem is valid for $H^\infty(R)$ at $z = 0$. Then,*

(A) $\lim_{n\to\infty} N(n) = \infty$*; and*

(B) *the Myrberg phenomenon $H^\infty(R) \circ \varphi = H^\infty(\widetilde{R})$ occurs.*

We shall generalize this theorem for Zalcman domains $R = R(c_n, c_n^{N(n)})$ under the condition

$$\limsup_{n\to\infty} \frac{c_{n+1}}{c_n} < 1 \ . \tag{1.4}$$

With this generalization, we can simplify an argument in the previous proof of the part (A). There naturally occured a guess when the above theorem was obtained that the uniqueness theorem and the Myrberg phenomenon are in fact equivalent. Contrary to this expectation, we shall show that the converse of the part (B), including in the case of the above generalization, is not true; namely, the Myrberg phenomenon unfortunately does not imply the uniqueness theorem.

In § 2, the next section, we shall give a necessary and sufficient condition in order that a particular holomorphic function on a Zalcman domain $R(c_n, r_n)$ is bounded. In § 3, we shall examine this necessary and sufficient condition from the point of view how a sequence $\{r_n\}$ depends on a sequence $\{c_n\}$. In § 4, we shall generalize the part (A) of the above theorem. In § 5, the final section, we shall prove that the converse of the part (B) is false by constructing an example. The method used in this construction can be applied to any unlimited two-sheeted covering $(\varphi, \widetilde{D}, D)$ of an arbitrary plane domain D with a nonconstant bounded holomolphic function, which we shall mention at the end of the section.

2 A bounded holomorphic function on $R(c_n, r_n)$

2.1

In this section we give a necessary and sufficient condition in order that the following function

$$p(z) = \prod_{n=1}^{\infty} \frac{z}{z - c_n} = \prod_{n=1}^{\infty} \left(1 + \frac{c_n}{z - c_n}\right) \tag{2.1}$$

is bounded on a Zalcman domain $R = R(c_n, r_n)$ satisfying (1.4).

Suppose $n \geq m$ and $|z| \geq c_{m-1}$. We have

$$\left|\frac{c_n}{z - c_n}\right| \leq \frac{c_n}{c_{m-1} - c_m} \ .$$

Assumption (1.4) implies that $\sum_{n=m}^{\infty} c_n < \infty$. Thus, p is meromorphic on $\overline{\mathbf{C}} \setminus \{0\}$ and holomorphic on $\overline{R} \setminus \{0\}$, where $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ denotes the Riemann sphere.

Now, we estimate the bound of $|p|$ on $R(c_n, r_n)$. For simplicity, we denote $\Delta_n = \overline{\Delta(c_n, r_n)}$ and $R = R(c_n, r_n)$. Since p is holomorphic on $\overline{R} \setminus \{0\}$,

$$M_n := \max_{z \in \partial\Delta_n} |p(z)| \tag{2.2}$$

is finite for each $n \in \mathbf{N}$. We will describe $\sup_{z \in R} |p(z)|$ with respect to the sequence $\{M_n\}_{n=1}^{\infty}$. Consider the maximum value of $|p|$ on the circle $\gamma_n = \{z \in \mathbf{C} : |z| = c_n - r_n\}$. Setting $q_m(z) = c_m/(z - c_m)$, $p(z) = \prod_{m \in \mathbf{N}}(1 + q_m(z))$. For each factor $1 + q_m(z)$ we see that

$$\begin{aligned}
\max_{z \in \gamma_n} |1 + q_m(z)| &= \max_{z \in \gamma_n} \left|\frac{z}{z - c_m}\right| = \frac{c_n - r_n}{\min_{z \in \gamma_n} |z - c_m|} \\
&= \frac{c_n - r_n}{|c_n - r_n - c_m|} = |1 + q_m(c_n - r_n)| \ .
\end{aligned}$$

Since $\gamma_n \cap \partial\Delta_n = \{c_n - r_n\}$ (one point set), we have

$$\max_{z\in\gamma_n} |p(z)| = |p(c_n - r_n)| \leq M_n = \max_{z\in\partial\Delta_n} |p(z)| \ . \tag{2.3}$$

Set

$$R_n = \overline{R} \cap \{z \in \mathbf{C} : |z| > c_n - r_n\} \ ,$$

$$R'_n = \{z \in \overline{\mathbf{C}} : |z| > c_n - r_n\} \setminus \bigcup_{k=1}^{n} \Delta_k \ .$$

Then, $R_n \subset R'_n$ and the function p is holomorphic on R'_n. By (2.2) and (2.3), it follows that

$$\sup_{z\in R_n} |p(z)| = \sup_{z\in R'_n} |p(z)| = \max\{M_1, \ldots, M_n\} \ .$$

Hence we have

$$\sup_{z\in R} |p(z)| = \sup_{n\in\mathbf{N}} M_n \ . \tag{2.4}$$

2.2

Also, we need the follwing simple lemma.

Lemma 2.1 *Let $\{E_n\}_{n\in\mathbf{N}}$ be a family of subsets of $\mathbf{N}$ and $\{\delta_{m,n} : n \in \mathbf{N}, m \in E_n\}$ be a set of positive numbers. If $\sup_{n\in\mathbf{N}} \sum_{m\in E_n} \delta_{m,n} < \infty$, then*

$$\sup_{n\in\mathbf{N}} \prod_{m\in E_n} (1 + \delta_{m,n}) < \infty \ .$$

Proof. Since $\log(1 + x) < x$ for $x > 0$, we have

$$\sum_{m\in E_n} \log(1 + \delta_{m,n}) \leq \sum_{m\in E_n} \delta_{m,n} \ .$$

Therefore

$$\sup_{n\in\mathbf{N}} \prod_{m\in E_n} (1 + \delta_{m,n}) \leq \exp(\sup_{n\in\mathbf{N}} \sum_{m\in E_n} \delta_{m,n}) < \infty \ .$$

□

2.3

We now prove the main theorem of this section.

Theorem 2.1 *Let* $\{c_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *be sequences satisfying* (1.1) *and* (1.4). *The holomorphic function* p *is bounded on* $R(c_n, r_n)$ *if and only if*

$$\sup_{n\in\mathbf{N}} \frac{c_n^n}{c_1\cdots c_{n-1}r_n} < \infty \ . \tag{2.5}$$

Proof. (The "only if" part) Set $\Delta_n = \overline{\Delta(c_n, r_n)}$. For $z \in \partial\Delta_n$

$$\log|p(z)| = \sum_{m\in\mathbf{N}\setminus\{n\}} \log\left|\frac{z}{z-c_m}\right| + \log|z| - \log r_n \ .$$

Since

$$\frac{1}{2\pi}\int_{\partial\Delta_n} \log|p(z)|d\arg z \leq \log M_n \ ,$$

applying the Gauss mean value theorem to the left hand side of the last inequality, we have

$$\sum_{m\in\mathbf{N}\setminus\{n\}} \log\left|\frac{c_n}{c_n-c_m}\right| + \log c_n - \log r_n \leq \log M_n \ ,$$

or equivalently

$$\left(\prod_{m=1}^{n-1}\frac{c_n}{c_m-c_n}\right)\left(\prod_{m=n+1}^{\infty}\frac{c_n}{c_n-c_m}\right)\frac{c_n}{r_n} \leq M_n \ . \tag{2.6}$$

Note

$$\begin{aligned}\prod_{m=1}^{n-1}\frac{c_n}{c_m-c_n} &= \prod_{m=1}^{n-1}\left(\frac{c_n}{c_m}\frac{1}{1-c_nc_m^{-1}}\right)\\ &> \prod_{m=1}^{n-1}\frac{c_n}{c_m} = \frac{c_n^{n-1}}{c_1\cdots c_{n-1}}\end{aligned} \tag{2.7}$$

and

$$\prod_{m=n+1}^{\infty}\frac{c_n}{c_n-c_m} > 1 \ . \tag{2.8}$$

By (2.6), (2.7) and (2.8), it follows that

$$\frac{c_n^n}{c_1\cdots c_{n-1}r_n} \leq M_n \ .$$

Therefore

$$\sup_{n\in\mathbf{N}}\frac{c_n^n}{c_1\cdots c_{n-1}r_n} \leq \sup_{n\in\mathbf{N}} M_n = \sup_{z\in R}|p(z)| < \infty.$$

(The "if" part) Condition (2.5) implies that there exists $\rho_0 > 0$ such that

$$r_n \geq \rho_0 \frac{c_n^n}{c_1 \cdots c_{n-1}} =: r'_n$$

for any $n \in \mathbf{N}$. Since $R(c_n, r_n) \subset R(c_n, r'_n)$, it sufficies to show that p is bounded on $R(c_n, r'_n)$. For the proof, we use this r'_n in place of r_n and write r'_n as r_n for simplicity. By (1.4), there exist constants $0 < \delta_0 < 1$ and $n_0 \in \mathbf{N}$ such that $c_n/c_{n-1} \leq \delta_0$ for all $n \geq n_0$. Since the function p is meromorphic on $\overline{\mathbf{C}} \setminus \{0\}$, M_n is finite for each $n \in \mathbf{N}$. By (2.4), we have only to show that $\sup_{n > n_0} M_n < \infty$. Set $D_0 = \Delta \setminus \cup_{k=1}^{n_0} \Delta_k$ and

$$M_0 = \sup_{z \in D_0} \left| \prod_{m=1}^{n_0} \frac{1}{z - c_m} \right| .$$

Suppose $n > n_0$. Noting

$$p(z) = \left(\prod_{m=1}^{n_0} \frac{1}{z - c_m} \right) \cdot z^{n_0} \cdot \left(\prod_{m=n_0+1}^{\infty} \frac{z}{z - c_m} \right),$$

we have

$$\begin{aligned}
M_n &= \sup_{z \in \partial \Delta_n} \left| \left(\prod_{m=1}^{n_0} \frac{1}{z - c_m} \right) \cdot z^{n_0} \cdot \left(\prod_{m=n_0+1}^{\infty} \frac{z}{z - c_m} \right) \right| \\
&\leq M_0 (c_n + r_n)^{n_0} \prod_{m=n_0+1}^{\infty} \sup_{z \in \partial \Delta_n} \left| 1 + \frac{c_m}{z - c_m} \right| \\
&\leq M_0 (c_n + r_n)^{n_0} \left(\prod_{m=n_0+1}^{n-1} \frac{c_n + r_n}{c_m - (c_n + r_n)} \right) \\
&\qquad \times \frac{c_n + r_n}{r_n} \cdot \prod_{m=n+1}^{\infty} \left(1 + \frac{c_m}{c_n - r_n - c_m} \right) \\
&= M_0 \frac{(c_n + r_n)^n}{r_n} \prod_{m=n_0+1}^{n-1} \left(\frac{1}{c_m} \cdot \frac{1}{1 - (1 + r_n c_n^{-1}) c_n c_m^{-1}} \right) \\
&\qquad \times \prod_{m=n+1}^{\infty} \left(1 + \frac{c_m c_n^{-1}}{(1 - r_n c_n^{-1}) - c_m c_n^{-1}} \right).
\end{aligned}$$

Set $\varepsilon_n = r_n c_n^{-1}$. Then, we have

$$\begin{aligned}
M_n &\leq M_0 (1 + \varepsilon_n)^n \frac{c_n^n}{c_1 \cdots c_{n-1} r_n} \prod_{m=n_0+1}^{n-1} \left(1 + \frac{(1 + \varepsilon_n) c_n c_m^{-1}}{1 - (1 + \varepsilon_n) c_n c_m^{-1}} \right) \\
&\qquad \times \prod_{m=n+1}^{\infty} \left(1 + \frac{c_m c_n^{-1}}{(1 - \varepsilon_n) - c_m c_n^{-1}} \right). \qquad (2.9)
\end{aligned}$$

Now, it sufficies to show the following three assertions (2.10), (2.11) and (2.12).

$$\sup_{n>n_0} (1+\varepsilon_n)^n < \infty \ . \tag{2.10}$$

$$\sup_{n>n_0} \prod_{m=n_0+1}^{n-1} \left(1 + \frac{(1+\varepsilon_n)c_n c_m^{-1}}{1-(1+\varepsilon_n)c_n c_m^{-1}}\right) < \infty \ . \tag{2.11}$$

$$\sup_{n>n_0} \prod_{m=n+1}^{\infty} \left(1 + \frac{c_m c_n^{-1}}{(1-\varepsilon_n) - c_m c_n^{-1}}\right) < \infty \ . \tag{2.12}$$

Proof of (2.10): Since $n > n_0$, we see that

$$\varepsilon_n = r_n c_n^{-1} = \rho_0 \frac{c_n^{n-1}}{c_1 \cdots c_{n-1}} \leq \rho_0 \left(\frac{c_n}{c_{n-1}}\right)^{n-1} \leq \rho_0 \delta_0^{n-1}.$$

Since $\delta_0 < 1$, we have $\varepsilon_n < 1/n$ for sufficiently large n. This implies (2.10).

Proof of (2.11): Since $\varepsilon_n \to 0$ as $n \to \infty$, $\varepsilon_0 = \max_{n>n_0} \varepsilon_n$ exists. We have

$$\delta_{m,n} := \frac{(1+\varepsilon_n)c_n c_m^{-1}}{1-(1+\varepsilon_n)c_n c_m^{-1}} \leq \frac{(1+\varepsilon_0)\delta_0^{n-m}}{1-(1+\varepsilon_0)\delta_0} \ .$$

Therefore,

$$\sup_{n>n_0} \sum_{m=n_0+1}^{n-1} \delta_{m,n} < \infty$$

and (2.11) follows by Lemma 2.1.

Proof of (2.12): Since $\varepsilon_n \to 0$ as $n \to \infty$, replacing n_0 by a larger one if necessary, we may assume that there is a constant δ_1 with $\varepsilon_n < \delta_1 < 1 - \delta_0$ for $n > n_0$. Since

$$\delta'_{m,n} := \frac{c_m c_n^{-1}}{(1-\varepsilon_n) - c_m c_n^{-1}} \leq \frac{\delta_0^{m-n}}{1-\delta_1-\delta_0},$$

we have

$$\sup_{n>n_0} \sum_{m=n+1}^{\infty} \delta'_{m,n} < \infty$$

and hence, (2.12) follows by Lemma 2.1. □

3 Dependence of a sequence $\{r_n\}$ on a sequence $\{c_n\}$

3.1

Corresponding to a pair of sequences $\{c_k\}_{k=1}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$, let us consider two sequences $\{\nu_n\}_{n=1}^{\infty}$ and $\{N(n)\}_{n=1}^{\infty}$ determined by the relations $c_n = 2^{-\nu_n}$

and $r_n = 2^{-\nu_n N(n)}$. Since $\{c_n\}_{n=1}^{\infty}$ is strictly decreasing and converging to 0, $\{\nu_n\}_{n=1}^{\infty}$ is strictly increasing and $\lim_{n\to\infty} \nu_n = \infty$. Then, conditions (1.4) and (2.5) are equivalent to the conditions

$$\liminf_{n\to\infty}(\nu_{n+1} - \nu_n) > 0 \tag{3.1}$$

and

$$\sup_{n\in\mathbf{N}} \nu_n \left\{ N(n) - \left(n - \frac{\nu_1 + \cdots + \nu_{n-1}}{\nu_n} \right) \right\} < \infty \ , \tag{3.2}$$

respectively. In the case $c_n = 2^{-n}$ and $r_n = 2^{-nN(n)}$, (3.2) is writen as

$$\sup_{n\in\mathbf{N}} n \left(N(n) - \frac{n+1}{2} \right) < \infty \tag{3.3}$$

(cf. [3]).

3.2

As we have seen, (3.2) is a necessary and sufficient condition in order that the function $p(z)$ is bounded on the domain $R(c_n, r_n)$. We are interested in how small r_n's (or, how large $N(n)$'s) can be chosen depending on $\{c_n\}$. From (3.2), we see that an approximate size of $N(n)$ is given by

$$\nu_n^* = n - \frac{\nu_1 + \cdots + \nu_{n-1}}{\nu_n} \ . \tag{3.4}$$

In order that $p(z) \in H^\infty(R(c_n, r_n))$, the next proposition shows that the sequence $\{N(n)\}_{n=1}^{\infty}$ can be chosen always as $N(n) \to \infty$ $(n \to \infty)$; and that $N(n)$ can be chosen almost equal to n (the maximum order) for a sequence $\{c_n\}_{n=1}^{\infty}$, while $N(n)$ should increase very slowly for another sequence $\{c_n\}_{n=1}^{\infty}$. Note that (3.1) is obvious when $\{\nu_n\}$ is a strictly increasing sequence of positive integers, which is the case we shall consider in the proof of parts (b) and (c) below.

Proposition 3.1 (a) *For any strictly increasing sequence $\{\nu_n\}_{n=1}^{\infty}$ with $\nu_n > 0$ and $\lim_{n\to\infty} \nu_n = \infty$, the sequence $\{\nu_n^*\}_{n=1}^{\infty}$ is strictly increasing, $\nu_n^* \le n$ and $\lim_{n\to\infty} \nu_n^* = \infty$.*

(b) *For any sequence $\{\sigma_n\}_{n=1}^{\infty}$ with $0 < \sigma_n < 1$, there exists a strictly increasing sequence $\{\nu_n\}_{n=1}^{\infty}$ of positive integers such that $\nu_n^* \ge \sigma_n n$ $(n \in \mathbf{N})$.*

(c) *For any increasing sequence $\{\beta_n\}_{n=1}^{\infty}$ no matter how slowly increasing it may be as far as $\lim_{n\to\infty} \beta_n = \infty$, there exist a strictly increasing sequence $\{\nu_n\}_{n=1}^{\infty}$ of positive integers and a subsequence $\{n_i\}_{i=1}^{\infty} \subset \mathbf{N}$ such that $\nu_{n_i}^* < \beta_{n_i}$ $(i \in \mathbf{N})$.*

Proof. (a) Since

$$\begin{aligned}\nu_{n+1}^* - \nu_n^* &= n+1-\frac{\nu_1+\cdots+\nu_n}{\nu_{n+1}} - n + \frac{\nu_1+\cdots+\nu_{n-1}}{\nu_n} \\ &= 1-\frac{\nu_n}{\nu_{n+1}} + (\nu_1+\cdots+\nu_{n-1})\left(\frac{1}{\nu_n}-\frac{1}{\nu_{n+1}}\right) \\ &> 0\ ,\end{aligned}$$

$\{\nu_n^*\}_{n=1}^\infty$ is a strictly increasing sequence. For any $m \in \mathbf{N}$ there exists an $n(m) \in \mathbf{N}$ such that $\nu_n > 2\nu_{m-1}$ for all $n \geq n(m)$. Then, $\nu_j/\nu_n < 1/2$ for $n \geq n(m)$ and $j = 1, \cdots, m-1$. Hence,

$$\begin{aligned}\nu_n^* &= 1+\left(1-\frac{\nu_1}{\nu_n}\right)+\cdots+\left(1-\frac{\nu_{m-1}}{\nu_n}\right)+\cdots+\left(1-\frac{\nu_{n-1}}{\nu_n}\right) \\ &\geq \frac{1}{2}m\ .\end{aligned}$$

Thus, $\lim_{n\to\infty} \nu_n^* = \infty$.

(b) Set $\nu_1 = 1$. Then, $\nu_1^* = 1 > \sigma_1 1$. Suppose that we have already chosen $\{\nu_\ell\}_{\ell=1}^n$ so that $\nu_\ell^* \geq \sigma_\ell \ell$. We can choose a positive integer ν_{n+1} such that $\nu_n/\nu_{n+1} < 1-\sigma_{n+1}$. Then,

$$\begin{aligned}\nu_{n+1}^* &= 1+\left(1-\frac{\nu_1}{\nu_{n+1}}\right)+\cdots+\left(1-\frac{\nu_n}{\nu_{n+1}}\right) \\ &\geq (n+1)\left(1-\frac{\nu_n}{\nu_{n+1}}\right) \\ &> (n+1)\sigma_{n+1}\ .\end{aligned}$$

(c) Suppose that we have already chosen $n_{\ell-1}$ and $\{\nu_n\}_{n=1}^{n_{\ell-1}}$. We are going to choose an n_ℓ $(> n_{\ell-1})$ and $\{\nu_n\}_{n=n_{\ell-1}+1}^{n_\ell}$. Fix an $\varepsilon > 0$. Since $\lim_{n\to\infty} \beta_n = \infty$, there exists an $n_\ell \in \mathbf{N}$ such that

$$n_\ell > n_{\ell-1} \quad \text{and} \quad \beta_{n_\ell} > 1 + n_{\ell-1} + \varepsilon\ .$$

For $m = n_\ell - n_{\ell-1} - 1$ there exists $\nu \in \mathbf{N}$ such that

$$\nu > \max\left\{m, \frac{m^2}{\varepsilon}\right\}.$$

We have

$$\begin{aligned}&\left(1-\frac{\nu-m}{\nu}\right)+\cdots+\left(1-\frac{\nu-2}{\nu}\right)+\left(1-\frac{\nu-1}{\nu}\right) \\ &\qquad < m\left(1-\frac{\nu-m}{\nu}\right) = \frac{m^2}{\nu} < \varepsilon\ .\end{aligned}$$

Now we define $\nu_n = \nu - (n_\ell - n)$ for n $(n_{\ell-1} < n \le n_\ell)$. Then,

$$\begin{aligned}\nu^*_{n_\ell} &= 1 + \left(1 - \frac{\nu_1}{\nu_{n_\ell}}\right) + \cdots + \left(1 - \frac{\nu_{n_{\ell-1}}}{\nu_{n_\ell}}\right) \\ &\quad + \left(1 - \frac{\nu_{n_{\ell-1}+1}}{\nu_{n_\ell}}\right) + \cdots + \left(1 - \frac{\nu_{n_\ell - 1}}{\nu_{n_\ell}}\right) \\ &< 1 + n_{\ell-1} + \varepsilon\ .\end{aligned}$$

Therefore, it follows that $\nu^*_{n_\ell} \le 1 + n_{\ell-1} + \varepsilon < \beta_{n_\ell}$. □

4 A necessary condition for the uniqueness theorem

4.1

First we show the next lemma.

Lemma 4.1 *Suppose that $\{\nu_n\}_{n=1}^\infty$ is a strictly increasing sequence of positive numbers satisfying the property (3.1). Let $R = R(2^{-\nu_n}, 2^{-\nu_n N(n)})$ be a Zalcman domain. If $\{N(n)\}_{n=1}^\infty$ satisfies $\lim_{n\to\infty}(\nu^*_n - N(n)) = \infty$, then the uniqueness theorem is not valid for $H^\infty(R)$ at $z = 0$, i.e., there exists an $f \in H^\infty(R)$ such that*

$$f \not\equiv 0, \qquad \lim_{z\to 0, z<0} f^{(m)}(z) = 0 \quad (m = 0, 1, 2, \cdots)\ .$$

Proof. We set

$$f(z) = \prod_{n=1}^\infty \frac{z}{z - c_n}, \qquad g_k(z) = \prod_{n=k+1}^\infty \frac{z}{z - c_n} \quad (k \in \mathbf{N})\ .$$

Applying Theorem 2.1, we have $f \in H^\infty(R)$ and $g_k \in H^\infty(R)$. In fact, for $n > k$

$$\begin{aligned}&\nu_n \left\{N(n) - \left(n - k - \frac{\nu_{k+1} + \cdots + \nu_{n-1}}{\nu_n}\right)\right\} \\ &= \nu_n \left\{N(n) - \nu^*_n + k - \frac{\nu_1 + \cdots + \nu_k}{\nu_n}\right\}\ .\end{aligned}$$

Since $\lim_{n\to\infty}(\nu_1 + \cdots + \nu_k)/\nu_n = 0$, the assumption $\lim_{n\to\infty}(\nu^*_n - N(n)) = \infty$ implies

$$\sup_{n>k} \nu_n \left\{N(n) - \left(n - k - \frac{\nu_{k+1} + \cdots + \nu_{n-1}}{\nu_n}\right)\right\} < \infty \quad k = 0, 1, 2, \cdots\ .$$

This shows that $g_k \in H^\infty(R(c_n, r_n)_{n>k})$. In particular, we have $f,\ g_k \in H^\infty(R)$. Setting

$$f_k(z) = g_k(z) \prod_{n=1}^{k} \frac{1}{z - c_n} \ ,$$

we have $f_k \in H^\infty(R(c_n, r_n))$, and $f(z) = z^k f_k(z)$ $(k = 0, 1, 2, \cdots)$. Also,

$$f^{(m)}(z) = \frac{d^m}{dz^m}\{z^{m+1} f_{m+1}(z)\} = \sum_{k=0}^{m} {}_mC_k \frac{(m+1)!}{(k+1)!}\ z^{k+1} f_{m+1}^{(k)}(z) \ , \tag{4.1}$$

where ${}_mC_k$ denote the binomial coefficients. If we show that $f_k^{(m)}(z)$ is bounded on $[-1/2, 0)$ for any $m,\ k \in \mathbf{N} \cup \{0\}$, then it follows from (4.1) that

$$\lim_{z \to 0, z<0} f^{(m)}(z) = 0 \quad (m = 0, 1, 2, \cdots) \ .$$

Set $D = \{z \in \mathbf{C} : |z + 1/2| < 1/2\}$. Since $f_k \in H^\infty(D)$, we have, for $z \in [-1/2, 0)$ and $k, m \in \mathbf{N} \cup \{0\}$,

$$\begin{aligned} |f_k^{(m)}(z)| &= \left| \frac{m!}{2\pi i} \int_{\partial D} \frac{f_k(\zeta)}{(\zeta - z)^{m+1}} d\zeta \right| \\ &= \left| \frac{m!}{2\pi i} \int_{\partial D} \frac{\zeta^{m+1} f_{k+m+1}(\zeta)}{(\zeta - z)^{m+1}} d\zeta \right| \\ &\le \frac{m!}{2\pi} \sup_{\zeta \in D} |f_{k+m+1}(\zeta)| \int_{\partial D} \left| \frac{\zeta}{\zeta - z} \right|^{m+1} |d\zeta| \ . \end{aligned}$$

Set $\psi_z(\zeta) = \zeta/(\zeta - z)$. Since ψ_z maps ∂D onto the circle with radius $1/2(1+z)$ centered at $1/2(1+z)$, we have

$$\sup_{\zeta \in \partial D, z \in [-1/2, 0)} \left| \frac{\zeta}{\zeta - z} \right| = 2 \ .$$

Thus $f_k^{(m)}(z)$ is bounded on $[-1/2, 0)$. □

Theorem 4.1 *Let* $\{c_n\}_{n=1}^\infty$ *and* $\{r_n\}_{n=1}^\infty$ *be any sequences satisfying* (1.1) *and* (1.4) *with* $r_n = c_n^{N(n)}$. *If the uniqueness theorem is valid for* $H^\infty(R(c_n, r_n))$ *at* $z = 0$, *then* $\lim_{n\to\infty} N(n) = \infty$.

Proof. To the contrary, we assume that $\liminf_{n\to\infty} N(n) < \infty$. This implies that there exist a positive constant μ and a strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers with $N(n_k) \le \mu$ for all $k \in \mathbf{N}$. We set $c'_k := c_{n_k} = 2^{-\nu'_k}$, and $N'(k) := \mu$. By Proposition 3.1 (a), we see that $\lim_{k\to\infty} ({\nu'}_k^* - N'(k)) = \infty$ and

$$N'(k) = \mu \ge N(n_k) \ . \tag{4.2}$$

Applying Lemma 4.1, we find a function $f \in H^\infty(R(c'_k, {c'_k}^{N'(k)}))$ such that $\lim_{z \to 0, z<0} f^{(m)}(z) = 0$ for all $m \in \mathbf{N} \cup \{0\}$ and $f \not\equiv 0$. From (4.2), it follows that $R(c_n, r_n) \subset R(c'_k, {c'_k}^{N'(k)})$. Thus f belongs to $H^\infty(R(c_n, r_n))$. This contradicts the assumption that the uniqueness theorem is valid for $H^\infty(R(c_n, r_n))$ at $z = 0$. □

5 The Myrberg phenomenon

5.1

Let $\varphi : \widetilde{\Delta}_0 \to \Delta_0$ be an unlimited two-sheeted covering with the branch points $\{\varphi^{-1}(c_n)\}$. From Theorem 4.1, we have the following theorem.

Theorem 5.1 *Let $\{c_n\}_{n=1}^\infty$ and $\{r_n\}_{n=1}^\infty$ be a pair of sequences satisfying* (1.1) *and* (1.4)*, and let $R = R(c_n, r_n)$ and $\widetilde{R} = \varphi^{-1}(R)$. If the uniqueness theorem holds for $H^\infty(R)$ at $z = 0$, then the Myrberg phenomenon $H^\infty(R) \circ \varphi = H^\infty(\widetilde{R})$ occurs.*

The proof of Theorem 5.1 is the same as in [3]. Here we only sketch its outline. Let $g \in H^\infty(\widetilde{R})$. Set $f(z) = (g(z^+) - g(z^-))^2$, where $\varphi^{-1}(z) = \{z^+, z^-\}$ for $z \in R$. By Theorem 4.1, $N(n) \to \infty$ $(n \to \infty)$. Using this fact, we can prove

$$\lim_{z \to 0, z<0} f^{(m)}(z) = 0$$

for every $m \in \mathbf{N} \cup \{0\}$ in the same way as [3]. The uniqueness theorem implies $f \equiv 0$. This implies that $g = G \circ \varphi$ for $G(z) = (g(z^+) + g(z^-))/2 \in H^\infty(R)$.

5.2

Now we prove, as one of our main purpose of this paper, that the converse of Theorem 5.1 is false. More generally, the following theorem holds.

Theorem 5.2 *Let $\{c_n\}_{n=1}^\infty$ be any strictly decreasing sequence with $0 < c_n < 1$ satisfying* (1.4)*. Then, there exists a Zalcman domain $R = R(c_n, r_n)$ such that the Myrberg phenomenon occurs for $(\widetilde{R}, R, \varphi)$ but the uniqueness theorem fails for $H^\infty(R)$ at $z = 0$.*

Proof. For simplicity, we only prove the case $c_n = 2^{-n}$. (The general case can be proved in the same way.) For any strictly increasing sequence $\{n_k\}_{k=1}^\infty$ of positive integers, we set

$$N(n) = \begin{cases} n_k & \text{for } n_{k-1} < n < n_k \\ 4 & \text{for } n = n_k \ . \end{cases} \tag{5.1}$$

Then, $\liminf_{n\to\infty} N(n) = 4 < \infty$. Therefore, the uniqueness theorem is not valid for $H^\infty(R(2^{-n}, 2^{-nN(n)}))$ at $z = 0$ by Theorem 4.1. In what follows, we inductively choose such a sequence $\{n_k\}_{k=1}^\infty$ that the Myrberg phenomenon occurs for $(\widetilde{R}(2^{-n}, 2^{-nN(n)}), R(2^{-n}, 2^{-nN(n)}), \varphi)$. Set

$$\Delta_n = \overline{\Delta(2^{-n}, 2^{-nN(n)})}, \ \Delta_m^* = \overline{\Delta(0, 2^{-m} - 2^{-mN(m)})}$$

and

$$R_k = \Delta_0 \setminus \left(\cup_{n=1}^{n_k} \Delta_n \cup \Delta_{n_k}^*\right).$$

Fixing a point a with $-1 < a < -1/2$, we denote $\varphi^{-1}(a) = \{a^+, a^-\}$. For a subdomain W of Δ_0 with $a \in W$, we define

$$\alpha(W, a) = \sup\{|f(a^+) - f(a^-)| : f \in H^\infty(\widetilde{W}), \ \|f\|_\infty \leq 1\}.$$

Set $\alpha_k = \alpha(R_k, a)$. By a normal family argument, we see that there exists a $f_k \in H^\infty(\widetilde{R}_k)$ such that $\|f\|_\infty = 1$ and $\alpha_k = |f_k(a^+) - f_k(a^-)|$. Set $n_1 = 1$. Note that $\alpha_1 \leq 2 = 2/1$. Suppose that $n_1 < \cdots < n_k$ have been chosen so that $\alpha_j \leq 2/j$ $(1 \leq j \leq k)$ and $\{N(n)\}_{n=1}^{n_k}$ is defined by (5.1). For an integer m $(> n_k)$, we define $N(n) = m$ $(n_k < n < m)$, $N(m) = 4$, and set

$$R_{k+1}^{(m)} = \Delta_0 \setminus \left(\bigcup_{n=1}^{m} \Delta_n \cup \Delta_m^*\right)$$

and

$$\alpha_{k+1}^{(m)} = \alpha(R_{k+1}^{(m)}, a).$$

There exists $g_m \in H^\infty(\widetilde{R}_{k+1}^{(m)})$ such that $\|g_m\|_\infty = 1$ and

$$\alpha_{k+1}^{(m)} = |g_m(a^+) - g_m(a^-)|.$$

From the definition of $R_{k+1}^{(m)}$, it follows that as $m \to \infty$,

$$R_{k+1}^{(m)} \nearrow R'_k := \Delta_0 \setminus \left(\left(\bigcup_{n=1}^{n_k} \Delta_n\right) \cup \{2^{-n} : n > n_k\}\right).$$

Thus, we can find a subsequence $\{g_{m_\ell}\}_{\ell=1}^\infty$ and a function $g_0 \in H^\infty(\widetilde{R}'_k)$ such that $\{g_{m_\ell}\}_{\ell=1}^\infty$ converges to g_0 uniformly on every compact subset of $\widetilde{R}'_k$. In particular,

$$\alpha_{k+1}^{(m_\ell)} = |g_{m_\ell}(a^+) - g_{m_\ell}(a^-)| \to |g_0(a^+) - g_0(a^-)| \quad (\ell \to \infty).$$

Since $(\widetilde{R}'_k, R'_k, \varphi)$ have branch points $\{2^{-n}\}_{n > n_k}$, the classical Myrberg argument implies that the Myrberg phenomenon occurs for $(\widetilde{R}'_k, R'_k, \varphi)$. That is, $g_0(a^+) = g_0(a^-)$. Therefore there exists an m such that $m > n_k$ and $\alpha_{k+1}^{(m)} \leq 2/(k+1)$. We set $n_{k+1} = m$, $N(n) = m$ ($n_k < n < m = n_{k+1}$) and $N(n_{k+1}) = 4$. In this way, we define $\{n_k\}_{k=1}^{\infty}$ and $\{N(n)\}_{n=1}^{\infty}$. As above, we now find functions $f_k \in H^\infty(\widetilde{R}_k)$ such that $\|f_k\|_\infty = 1$ and

$$|f_k(a^+) - f_k(a^-)| = \alpha(R_k, a) \leq \frac{2}{k}$$

for $k \in \mathbf{N}$. Since $R_k \subset R = R(2^{-n}, 2^{-nN(n)})$, $H^\infty(\widetilde{R})|_{R_k} \subset H^\infty(\widetilde{R}_k)$ and

$$\alpha(R, a) \leq \alpha_k \leq \frac{2}{k}$$

for all k. Therefore, $\alpha(R, a) = 0$ and we obtain $f(a^+) = f(a^-)$ for all $f \in H^\infty(\widetilde{R})$. By Forelli's theorem ([1], cf. also [4]), this implies that $f(z^+) \equiv f(z^-)$ for all $z \in R$, $\varphi^{-1}(z) = \{z^+, z^-\}$. Hence, the Myrberg phenomenon occurs for $(\widetilde{R}, R, \varphi)$. □

5.3

One may see from the proof of Theorem 5.2 that the order of r_n with respect to c_n is not so restrictive for the Myrberg phenomenon, comparing with the uniqueness theorem. The next theorem may also emphasizes this fact in a slightly different flavor.

Theorem 5.3 *Let D be an arbitrary plane domain such that $H^\infty(D)$ contains a nonconstant function. Let $\varphi : \widetilde{D} \to D$ be any unlimited (branched) two-sheeted covering for which the Myrberg phenomenon $H^\infty(D) \circ \varphi = H^\infty(\widetilde{D})$ occurs. Let $\{K_n\}_{n=1}^{\infty}$ be a family of mutually disjoint compact subsets of D such that*

- (a) *$\{K_n\}_{n=1}^{\infty}$ clusters to the boundary of D, that is, only a finite number of K_n intersect with each compact subset of D;*
- (b) *$D \setminus \cup_{n=1}^{\infty} K_n$ is connected.*

Then, there exists a subsequence $\{K_{n_k}\}_{k=1}^{\infty}$ such that the Myrberg phenomenon occurs for $(\widetilde{D} \setminus \cup_{k=1}^{\infty} \widetilde{K}_{n_k}, D \setminus \cup_{k=1}^{\infty} K_{n_k}, \varphi)$, where $\widetilde{K}_n = \varphi^{-1}(K_n)$.

If we apply this theorem to the case $\varphi : \widetilde{\Delta}_0 \to \Delta_0$ with the branch points $\{\varphi^{-1}(2^{-n})\}$, then we may choose, for instance,

$$K_n = \{2^{-n} + iy : y \in [\ 0\ ,\ 2^{-\log n}]\}.$$

The diameter of K_n is $2^{-\log n} = 2^{-nN(n)}$, where $N(n) = \frac{\log n}{n} \to 0$ $(n \to \infty)$. In a sense, this shows that a part of, because only a subsequence remains in the sequel, K_n's can be chosen very large for the Myrberg phenomenon.

Before the proof of Theorem 5.3, we remark about the compact sets K_n mentioned in the theorem. By the assumptions, we can find, on considering an exhaution of $D \setminus \cup_{n=1}^{\infty} K_n$ by relatively compact smooth domains, relatively compact connected open subsets E_m of D such that $K_m \subset E_m \subset \overline{E}_m \subset D$ and $E_m \cap K_n = \phi$ $(m \neq n)$. Thus, each point $z \in K_m$ can be joined with a point in $E_m \setminus K_m$ by an arc in E_m ($\subset D \setminus K_n$ for $m \neq n$). Therefore, $D \setminus K_n$ is connected for every n. This further implies that $\mathbf{C} \setminus K_n$ is connected for every n, or equivalently, that $\mathbf{C} \setminus K_n$ has no bounded components.

Looking at the proof of Theorem 5.2, it may be obvious that Theorem 5.3 follows from the next proposition by a similar argument.

Proposition 5.1 *Let D be an arbitrary plane domain such that $H^\infty(D)$ contains a nonconstant function. Let $\varphi : \widetilde{D} \to D$ be any unlimited (branched) two-sheeted covering. Let K be any compact subset of D. For a connected component D_0 of $D \setminus K$, write $\widetilde{D}_0 = \varphi^{-1}(D_0)$. Then, Myrberg phenomenon occurs for $(\widetilde{D}, D, \varphi)$ if and only if so does for some of $(\widetilde{D}_0, D_0, \varphi)$.*

Proof. (The "if" part) Suppose that the Myrberg phenomenon $H^\infty(D_0) \circ \varphi = H^\infty(\widetilde{D}_0)$ occurs for a connected component D_0 of $D \setminus K$. Let a be a point in D_0 such that a is not the projection of a branch point of φ. Then, $a^+ \neq a^-$, where $\varphi^{-1}(a) = \{a^+, a^-\}$. Since the points a^+ and a^- are not separated by $H^\infty(\widetilde{D}_0)$, by the assumption, $f(a^+) = f(a^-)$ for every $f \in H^\infty(\widetilde{D}_0)$. Since $H^\infty(\widetilde{D})|_{\widetilde{D}_0} \subset H^\infty(\widetilde{D}_0)$, the points a^+ and a^- are not separated by $H^\infty(\widetilde{D})$. For $f \in H^\infty(\widetilde{D})$, $g(z) = (f(z^+) - f(z^-))^2$ defines an element in $H^\infty(D)$, where $z \in D$ and $\varphi^{-1}(z) = \{z^+, z^-\}$. Now, we have $g(a) = 0$ for any points $a \in D_0$ such that a is not the projection of a branch point of φ. Since such points a are dense in D_0, we have $g \equiv 0$. This shows that $H^\infty(D) \circ \varphi = H^\infty(\widetilde{D})$, the Myrberg phenomenon for $(\widetilde{D}, D, \varphi)$.

(The "only if" part) We shall prove the contraposition. Namely, we assume that the Myrberg phenomenon does not occur for any $(\widetilde{D}_0, D_0, \varphi)$. By Forelli's theorem ([1]), this implies that $H^\infty(\widetilde{D} \setminus \widetilde{K})$ is point separating, where $\widetilde{K} = \varphi^{-1}(K)$. Thus, for any pair of distinct points p, q in $\widetilde{D} \setminus \widetilde{K}$, there exists a function f in $H^\infty(\widetilde{D} \setminus \widetilde{K})$ such that $f(p) \neq f(q)$. We find a relatively compact open set Ω such that

$$K \subset \Omega \subset \overline{\Omega} \subset D$$

and such that the boundary $\partial\Omega$ consists of a finite number of mutually disjoint closed Jordan curves $\Gamma_1, \cdots, \Gamma_\ell$. For the proof, we may replace K by a larger compact subset if necessary. Replacing K by $\overline{\Omega}$ and Ω by a larger one, we may assume that K and $D \setminus K$ consists of a finite number of connected components. Then, connecting the components of K by arcs in D, we may assume that K is connected. In addition, attaching all relatively compact components of $D \setminus K$ with respect to D to K, we may assume that $D \setminus K$ has no relatively compact components in D. It can be also assumed that $D \setminus K$ consists of ℓ components each of which contains only one Γ_j $(j = 1, \cdots, \ell)$. Now we may assume that Γ_ℓ, renumbering Γ_j's if necessary, surrounds all other curves $\Gamma_1, \cdots \Gamma_{\ell-1}$. (For the proof of Theorem 5.3, we only need the case when $\mathbf{C} \setminus K$ is connected. In this case, the proof becames a little simpler by setting $\ell = 1$ below. Because the proposition may have its own interest, we shall prove this proposition in the present form.) Next, we choose an annular neighborhood A_j of each Γ_j such that

$$\overline{A}_j \cap K = \phi \quad \text{and} \quad \overline{A}_j \subset D$$

and such that the boundary ∂A_j consists of two closed Jordan curves Γ_j^+ and Γ_j^-. Here we may assume that Γ_ℓ^+ surrounds Γ_ℓ^- and that Γ_j^- surrounds Γ_j^+ for $j = 1, \cdots, \ell - 1$. Let Ω^+ be the domain surrounded by $\Gamma^+ := \cup_{j=1}^{\ell} \Gamma_j^+$. For each $j = 1, \cdots, \ell$, let Ω_j^* (resp., Ω_j^-) be the component of $D \setminus K$ (resp., $D \setminus \Gamma_j^-$) that contains Γ_j. Then, $\Omega_j^- \subset \Omega_j^*$. Choose a point a_0 from K and a point a_j from $\Omega_j^- \setminus \overline{A}_j$ for $j = 1, \cdots, \ell$. Let $\tau : \widetilde{D} \to \widetilde{D}$ be the cover transformation of φ defined by $\tau(z^+) = z^-$, where $\varphi^{-1}(z) = \{z^+, z^-\}$ for $z \in D$. Since $H^\infty(\widetilde{D} \setminus \widetilde{K})$ separates the points a_j^+ and a_j^-, there exists a function $f \in H^\infty(\widetilde{D} \setminus \widetilde{K})$ such that $f(a_j^+) \neq f(a_j^-)$ for all $j = 1, \cdots, \ell$. Replacing f by $f - f \circ \tau$, we may assume that

$$f(z^+) = -f(z^-)$$

for all $z \in D \setminus K$. Since $\widetilde{\Omega}^+ = \varphi^{-1}(\Omega^+)$ is a finite bordered Riemann surface, we can find a function $q \in H^\infty(\widetilde{\Omega}^+)$ such that $q(a_0^+) \neq q(a_0^-)$. Replacing q by $q - q \circ \tau$, we also assume

$$q(z^+) = -q(z^-)$$

for all $z \in \Omega^+$. Set $A = \cup_{j=1}^{\ell} A_\ell$. It follows that

$$\frac{f}{q}(z^+) = \frac{f}{q}(z^-)$$

for all $z \in A$. If necessary, deforming $\Gamma := \cup_{j=1}^{\ell} \Gamma_j$ slightly, we may assume that both f and q have no zero on $\widetilde{\Gamma} = \varphi^{-1}(\Gamma)$. Thus, shrinking A_j's if necessary, we may further assume that f and q have no zero on $\widetilde{\overline{A}} = \varphi^{-1}(\overline{A})$. Now, f/q is holomorphic on a neighborhood of $\widetilde{\overline{A}}$, and hence, there is a holomorphic function g on a neighborhood of $\overline{A}$ such that

$$\frac{f}{q} = g \circ \varphi. \tag{5.2}$$

Now, $|g| > 0$ on $\overline{A}$. Thus $\log g$ is a multi-valued holomorphic function on $\overline{A}$. The periods of $\log g$ along Γ_j is an integer multiple of $2\pi i$. Note that $\widetilde{\Omega}_j^*$'s are mutually disjoint and $\cup_{j=1}^{\ell}\widetilde{\Omega}_j^* = \widetilde{D} \setminus \widetilde{K}$. Choosing suitable integers n_j, and replacing f by the function defined by

$$((z-a_j)^{n_j} \circ \varphi) f \qquad \left(\text{resp.}, \left(\left(\frac{z-a_0}{z-a_\ell}\right)^{n_\ell} \circ \varphi\right) f\right)$$

on $\widetilde{\Omega}_j^*$ for $j = 1, \cdots, \ell-1$ (resp., on $\widetilde{\Omega}_\ell^*$), we can make the function $\log g$ to be single-valued on $\overline{A}$, while the function f may have poles at $\varphi^{-1}(a_j)$ ($j = 1, \cdots, \ell$). Set $h = \log g$ and

$$h^{\pm}(z) = \frac{1}{2\pi i}\int_{\Gamma^{\pm}} \frac{h(\zeta)}{\zeta - z} d\zeta,$$

where $\Gamma^{\pm} = \cup_{j=1}^{\ell}\Gamma_j^{\pm}$. Then,

$$h(z) = h^+(z) - h^-(z) \tag{5.3}$$

for $z \in A$. By Cauchy's integral theorem, the functions h^+ and h^- are unchanged on A even if we move the integral paths slightly. Thus, we may assume that $h^{\pm}$ belong to $H^{\infty}(\Omega^{\pm})$, where $\Omega^- = \cup_{j=1}^{\ell}\Omega_j^-$. We define a function F on $\widetilde{D}$ by

$$F = \begin{cases} q \exp(h^+ \circ \varphi) & \text{on } \widetilde{\Omega}^+ \\ f \exp(h^- \circ \varphi) & \text{on } \widetilde{\Omega}^- \end{cases}. \tag{5.4}$$

By (5.3) and (5.4), we have

$$\frac{f}{q} = g \circ \varphi = \exp(h \circ \varphi) = \exp(h^+ \circ \varphi - h^- \circ \varphi)$$

on $\widetilde{A}$. Hence, F is a well-defined meromorphic function on $\widetilde{D}$. Since $H^{\infty}(D)$ contains a nonconstant element by the assumption on the domain D, we can find nonconstant functions $f_j \in H^{\infty}(D)$ with $f_j(a_j) = 0$. Multiplying certain powers of $f_j \circ \varphi$ to F, we obtain a holomorphic function F_0 on $\widetilde{D}$. Since the functions $z - a_j$ on Ω_j^* ($j = 1, \cdots, \ell-1$) and $(z-a_0)/(z-a_\ell)$ on Ω_ℓ^* are bounded on ∂D, the maximum principle yields $F_0 \in H^{\infty}(\widetilde{D})$. Note that F_0 separates the fiber $\varphi^{-1}(a)$ for some point a of D. Using Forelli's theorem again, we conclude that $H^{\infty}(\widetilde{D})$ separates the points of $\widetilde{D}$. □

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