TWO SHEETED DISCS AND BOUNDED ANALYTIC FUNCTIONS

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To the memory of Professor Kôtaro Oikawa

We consider the question of whether or not the space $H^{\infty}(R)$ of bounded analytic functions on a Riemann surface R separates the points of R for various two sheeted unlimited branched or unbranched covering surfaces $\tilde{D}$ of a domain D contained in the unit disc Δ. Let $\pi: \tilde{D} \to D$ be the covering map of $\tilde{D}$ onto D, and let $\{z_n\}$ be the set of images of branch points of $\tilde{D}$, i.e. the images in D of the critical points of π. Then $\{z_n\}$ is a countable discrete subset of D. If D is simply connected, then the two sheeted covering surface $\tilde{D}$ is uniquely determined by D and $\{z_n\}$.

P. J. Myrberg [7] observed that when D is the punctured unit disc and the images $\{z_n\}$ of branch points accumulate at the origin then the bounded analytic functions of $\tilde{D}$ take the same value at each of the two points of $\tilde{D}$ lying over a given

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point $z \in D$, that is,

$$H^{\infty}(D^{\sim})=H^{\infty}(D)\circ\pi. \tag{1}$$

For the case when D is the unit disc Δ and $\{z_n\}$ is a discrete subset of Δ, it is a classical result of H. L. Selberg [11] that the "Myrberg phenomenon" expressed in the equation (1) is valid if and only if

$$\sum_{n\geq 1}(1-|z_n|)=\infty. \tag{2}$$

The usual way of establishing the Myrberg phenomenon for two sheeted branched covering surface is as follows: Let f be any bounded analytic function of $D^{\sim}$, and let g(z) be the square of the differnce of the values of f at the two points of $D^{\sim}$ which lie over the point $z \in D$. Then g is a bounded analytic function in D which vanishes at all of the images of the branch points of $D^{\sim}$. In the Myrberg and Selberg cases the function g has too many zeros for a bounded holomorphic function unless it is identically zero.

For the preceding proof it might appear that the Myrberg phenomenon depends on the fact that the covering surface has "too many" branch points. We show this is not the case by constructing unbranched two sheeted covering surfaces of suitable subdomains D of Δ which exhibit the Myberg phenomenon (cf. also [4], [5], [6]).

We say that a countable subset $\{z_n\}$ of points in Δ is *admissible* if it is discrete and satisfies the equation (2). A sequence $\{\Delta_n\}$ of closed discs centered at the points $\{z_n\}$ is said to be *admissible* if the closed discs $\{\Delta_n\}$ are disjoint and contained in Δ, and we say that their radii $\{r_n\}$ form an admissible sequence of radii for $\{z_n\}$. Thus a sequence $\{r_n\}$ of positive numbers forms an admissible set of radii for $\{z_n\}$ if and only if

$$0<r_n\leq 1-|z_n|$$

and

$$r_n+r_m<|z_n-z_m|$$

for every n and m with $n\neq m$. Given an admissible set of points $\{z_n\}$ and an admissible sequence of closed discs $\{\Delta_n\}$ centered on them, we let $\Delta^{\sim}$ be the two sheeted unlimited covering surface of Δ branched over the points $\{z_n\}$, and set

$$D=\Delta\setminus\bigcup_{n\geq 1}\Delta_n \tag{3}$$

and

$$D^{\sim}=\pi^{-1}(D). \tag{4}$$

Then $D^{\sim}$ is an unbranched two sheeted covering surface of D. We call an admissible sequence $\{\Delta_n\}$ of discs *nonseparating* if the equation (1) holds when D and $D^{\sim}$ are given by equations (3) and (4). Our principal result stated in the *qualitative* form is the folowing:

THEOREM. *For any admissible sequence* $\{z_n\}$ *there exists a corresponding sequence of positive numbers* $\{\bar{r}_n\}$ *such that any admissible set* $\{\Delta_n\}$ *of discs centered at* $\{z_n\}$ *is nonseparating whenever their radii* $\{r_n\}$ *satisfy* $0<r_n\leq\bar{r}_n$.

This is a cosequence of our Main Theorem of *quantitative* nature, which gives a sufficient condition for a sequence of positive numbers $\{\bar{r}_n\}$ to have this property. The condition is expressed in terms of a convergence criterion. It is not too difficult to give a direct qualitative proof of the principal result stated above (cf. [6]). An essential problem in this regard is to give sufficiently explicit bounds so that one can determine whether a given admissible sequence of discs $\{\Delta_n\}$ is nonseparating or not. Our Main Theorem is a contribution to this problem.

An admissible sequence $\{z_n\}$ is said to be *rigid* if every choice of admissible sequence of discs $\{\Delta_n\}$ centered at $\{z_n\}$ is nonseparating. We have seen in our former paper [6] that there exist in a natural sense as many rigid admissible sequences as nonrigid ones. It is again important and interesting for us to be able to tell whether a given admissible sequence $\{z_n\}$ is rigid or not. We give a partial answer to this question so that we can give concrete examples of nonrigid admissible sequences. We also give a concrete example of rigid ones. A typical result among them is the following: The admissible sequence $\{z_n\}$ given by

$$z_n=1-1/n \quad (n\geq 1) \tag{5}$$

is nonrigid.

The paper consists of 6 sections. In the first section 1 titled *Construction of separating functions* an auxiliary function is constructed which is in the first place used to show the validity of (1) if $H^{\infty}(D^{\sim})$ does not separate the fiber $\pi^{-1}(a)$ for at least one point a in D. The essential use of the function is made in Section 2 with the title *Independence on finite parts*. Here it is shown among others that $\{z_n\}_{n\geq 1}$ is a rigid admissible sequence if and only if the admissible sequence $\{z_n\}_{n\geq k}$ is rigid for one and hence for every $k\geq 1$. In the third section 3 titled *The size of nonseparating discs* the main result of this paper of giving a concrete $\{\bar{r}_k\}$ in terms of $\{z_n\}$ will be proven. Under the title *Nonseparation criterion* an easily applicable simplification of the result in the preceding section is given here in Section 4. The result is then applied to two concrete examples in Section 5 with the title *Examples of nonseparating discs*. In Section 6 titled *Rigid and nonrigid two sheeted discs* a criterion of the nonrigidness for admissible sequences is given which is applied to concrete cases including (5). A concrete example of a rigid admissible sequence is also given in this section.

1. Construction of separating functions.

1.1. Consider a two sheeted unlimited covering surface $(\Delta^{\sim},\Delta,\pi)$ over the open unit disc $\Delta=\{|z|<1\}$ with a covering map π. Such a covering surface $(\Delta^{\sim},\Delta,\pi)$, or simply $\Delta^{\sim}$, is referred to as a two sheeted disc or more simply *2-disc*. A 2-disc $\Delta^{\sim}$ determines and is determined by a discrete sequence (or rather set) $\{z_n\}$ in Δ which is the totality of projections of branch points in $\Delta^{\sim}$ by the covering map π. The sequence $\{z_n\}$ will be referred to as a *determining sequence* (or *set*) of $\Delta^{\sim}$. As usual we denote by $H^{\infty}(R)$ the set of bounded holomorphic functions on a Riemann surface R. It is known (Selberg [11], cf. also [10], [12], [14]) that the Myrberg phenomenon ([7], cf. also [9])

$$H^{\infty}(\Delta^{\sim})=H^{\infty}(\Delta)\circ\pi$$

occurs if and only if the determining sequence $\{z_n\}$ of $\Delta^{\sim}$ satisfies

$$\sum_{n=1}^{\infty}(1-|z_n|)=\infty.$$

Such a discrete sequence $\{z_n\}$ will be referred to as an *admissible sequence* in Δ in this paper. Hereafter we will always assume that the determining sequence $\{z_n\}$ of 2-disc $\Delta^{\sim}$ is admissible unless the contrary is explicitly stated.

In terms of the Euclidean metric $r(z,w)=|z-w|$ the open (closed, resp.) Euclidean disc $\Delta(w,r)$ ($\bar{\Delta}(w,r)$, resp.) in Δ with center w and radius r in the interval $(0,1-|w|)$ is given by

$$\Delta(w,r)=\{z:r(z,w)<r\} \quad (\bar{\Delta}(w,r)=\{z:r(z,w)\leq r\}, \text{ resp.}).$$

Later we will also consider the corresponding objects with respect to the pseudohyperbolic metric on Δ.

A sequence $\{\Delta_n\}$ of pairwise disjoint closed discs $\Delta_n=\bar{\Delta}(z_n,r_n)$ in Δ, or simply the sequence $\{r_n\}$, will be said to be *admissible* for an admissible sequence $\{z_n\}$ of points in Δ.

Hence $\{\bar{\Delta}(z_n,r_n)\}$ is an admissible sequence of discs in Δ if and only if

$$0<r_n<1-|z_n|$$

for every n in N, the set of positive integers, and also

$$r_n+r_m<|z_n-z_m|$$

for every n and m in N with $n\neq m$. The former condition simply means that $\Delta_n=\bar{\Delta}(z_n,r_n)\subset\Delta$ for every n in N and the latter means that $\Delta_n\cap\Delta_m=\Phi$ for every n and m in N with $n\neq m$.

Using an admissible sequence $\{\Delta_n\}$ of discs $\Delta_n=\bar{\Delta}(z_n,r_n)$ associated with an admissible sequence $\{z_n\}$ in Δ for a 2-disc $(\Delta^{\sim},\Delta,\pi)$ we consider a region

$$D=\Delta\setminus\bigcup_{n\geq 1}\Delta_n$$

and the *smooth* covering surface $(D^{\sim},D,\pi|D^{\sim})$ naturally associated with $(\Delta^{\sim},\Delta,\pi)$ so that $D^{\sim}$ may be viewed as a subregion of $\Delta^{\sim}$. Here the smoothness of $D^{\sim}$ means that there are no branch points in $D^{\sim}$.

We say that $H^{\infty}(R)$ *separates* the points in a subset S of a Riemann surface R including the case S=R if there exists an f_{ab} in $H^{\infty}(R)$ for any pair (a,b) of distinct points a and b in S such that $f_{ab}(a)\neq f_{ab}(b)$.

1.2. Fix a 2-disc $(\Delta^{\sim},\Delta,\pi)$ with an admissible determining sequence $\{z_n\}_{n\geq 1}$ and an admissible sequence $\{\Delta_n\}$ of closed discs $\Delta_n = \overline{\Delta}(z_n, r_n)$ in Δ which defines regions

$$D_k = \Delta \setminus \left(\bigcup_{n \geq k+1} \Delta_n \right) \quad (k=0,1,2,\cdots)$$

so that $D_0 = D$ in the notation of 1.1. Take an arbitrary Jordan region W in D_k with $\overline{W} \subset D_k$ such that $\pi^{-1}(W)$ consists of disjoint copies. We maintain the following

LEMMA 1.1. *If there exists a function* f *in* $H^\infty(\pi^{-1}(D_k \setminus \overline{W}))$ *which separates the points in the fiber* $\pi^{-1}(a)$ *for some point* a *in* $D_k \setminus \overline{W}$, *then there exists a function* F *in* $H^\infty(\pi^{-1}(D_k))$ *which separates the points in the fiber* $\pi^{-1}(z)$ *for every* z *in* $\overline{W} \cup \{a\}$.

Using an $f \in H^\infty(\pi^{-1}(D_k \setminus \overline{W}))$ with $f(a^+) \neq f(a^-)$ where $\pi^{-1}(a) = \{a^+, a^-\}$, we will construct an $F \in H^\infty(\pi^{-1}(D_k))$ such that $F \circ T = -F \neq 0$ on $\pi^{-1}(\overline{W})$ where T is the cover transformation of $(\Delta^\sim, \Delta, \pi)$ so that $F(z^+) \neq F(z^-)$ for every $z \in \overline{W}$ where $\pi^{-1}(z) = \{z^+, z^-\}$. Let X be the set of points z in $D_k \setminus \overline{W}$ such that f separates the points in the fiber $\pi^{-1}(z)$ so that $a \in X$. We will actually prove more than stated in the above lemma: $F(z^+) \neq F(z^-)$ for every $z \in \overline{W} \cup X$. The construction given in 1.3 and 1.4 below is based upon a device developed in [4].

1.3. On replacing f by $f - f \circ T$ if necessary we may assume that $f\circ T=-f$ so that $f(a^+)=-f(a^-)\neq 0$. Choose a triple (J_1, J, J_2) of analytic Jordan curves J_1, J and J_2 in Δ with the following properties: the inverse image under π of the closure of the inside of J_1 consists of two disjoint copies; the inside of J_2 contains $\overline{W}$; the inside of J (J_1, resp.) contains the closure of the inside of J_2 (J, resp.); the outside of J_1 contains $\{a\}\cup(\cup_{n\geq k+1}\Delta_n)$; f does not vanish on $\pi^{-1}(\overline{A})$, where $\overline{A}$ is the closure of the annulus A bounded by J_1 and J_2. Observe that $\pi^{-1}(A)$ is the union of two annuli A^+ and A^- such that $\overline{A}^+\cap\overline{A}^-=\Phi$ and that $\pi|\overline{A}^+=\pi^+$ ($\pi|\overline{A}^-=\pi^-$, resp.) is a bijective conformal mapping between $\overline{A}^+$ ($\overline{A}^-$, resp.) and $\overline{A}$ and thus $(\pi^+)^{-1}$ ($(\pi^-)^{-1}$, resp.) is the conformal mapping of $\overline{A}$ onto $\overline{A}^+$ ($\overline{A}^-$, resp.). Then, noting that $f\circ T=-f$, we can consider the holomorphic function g on $\overline{A}$ given by

$$g=f\circ(\pi^+)^{-1}=-f\circ(\pi^-)^{-1}$$

which does not vanish on $\overline{A}$. Hence $\log g$ is defined on $\overline{A}$ as a multivalued analytic function on $\overline{A}$. The principle of persistence of functional equation assures that $e^{\log g(z)}=g(z)$. Hence the period of $\log g(z)$ along J is an integral multiple of $2\pi i$ and therefore there exists an integer m such that

$$h(z)=\log\left[(z-b)^m g(z)\right]$$

is a single valued analytic function on $\overline{A}$ where b is an arbitrarily fixed point in W. We denote by W_1 (W_2, resp.) the inside (outside, resp.) of J_1 (J_2, resp.). Then we have the decomposition

$$h(z)=h_1(z)-h_2(z)$$

of h into the difference of two holomorphic functions h_j on W_j defined by

$$h_j(z)=\frac{1}{2\pi i}\int_{J_j}\frac{h(\zeta)}{\zeta-z}d\zeta \qquad (z\in W_j;\ j=1,2).$$

It is easy to see that h_j is bounded on W_j ($j=1,2$).

1.4. Let $\pi^{-1}(W_1)$ consist of two disjoint copies W_1^+ and W_1^- such that $W_1^+\supset A^+$ and $W_1^-\supset A^-$. We define the function F(p) on $\pi^{-1}(D_k)$ by

$$F(p)=\begin{cases} (\pi(p)-b)^m f(p)e^{h_2\circ\pi(p)} & (p\in\pi^{-1}(W_2\cap\Delta)), \\ e^{h_1\circ\pi(p)} & (p\in W_1^+), \\ -e^{h_1\circ\pi(p)} & (p\in W_1^-). \end{cases}$$

Recall that $\pi=\pi^+$ on A^+ and $\pi=\pi^-$ on A^-. Observe that

$$(\pi(p)-b)^m f(p)e^{h_2\circ\pi(p)}=(\pi(p)-b)^m g\circ\pi(p)e^{h_2\circ\pi(p)}$$

$$=e^{h\circ\pi(p)}\cdot e^{h_2\circ\pi(p)}=e^{(h+h_2)\circ\pi(p)}=e^{h_1\circ\pi(p)}$$

for $p\in A^+=W_1^+\cap\pi^{-1}(W_2)$ and hence F(p) is well defined on A^+. Since $f(z^+)=-f(z^-)$ for $z^+\in A^+$ and $z^-\in A^-$ with $\pi(z^+)=\pi(z^-)$, the well definedness of $F(p)$ on $A^{-}=W_1^{-}\cap\pi^{-1}(W_2)$ is also checked. Then $F\in H^{\infty}(\pi^{-1}(D_k))$ and it is easy to see that F separates the points in the fiber $\pi^{-1}(z)$ for any $z\in W_1\cup X$. Since $\overline{W}\subset W_1$, F is the required. □

1.5. The separating function in Lemma 1.1 will be essentially made use of later in the proof of Theorem 2.1. Here we give its simple and direct application. Two points a and b in $\pi^{-1}(D_k)$ are always separated by $H^{\infty}(\pi^{-1}(D_k))$ if $\pi(a)\neq\pi(b)$ since $H^{\infty}(D_k)$ separates the points in D_k and $H^{\infty}(D_k)\circ\pi\subset H^{\infty}(\pi^{-1}(D_k))$ $(k=0,1,2,\cdots)$. Hence $H^{\infty}(\pi^{-1}(D_k))$ separates the points in $\pi^{-1}(D_k)$ if and only if $H^{\infty}(\pi^{-1}(D_k))$ separates the points in the fiber $\pi^{-1}(z)$ for *every* z in $D_k\setminus\{z_n\}_{n\leq k}$ where $\{z_n\}_{n\leq 0}=\Phi$. However we can show that $H^{\infty}(\pi^{-1}(D_k))$ separates the points in $\pi^{-1}(z)$ for every z in $D_k\setminus\{z_n\}_{n\leq k}$ if and only if $H^{\infty}(\pi^{-1}(D_k))$ separates the points in $\pi^{-1}(z)$ for a certain one point z in $D_k\setminus\{z_n\}_{n\leq k}$. Namely we have the following

FACT 1.1. *The Myrberg phenomenon* $H^{\infty}(\pi^{-1}(D_k))=H^{\infty}(D_k)\circ\pi$ *occurs if and only if* $H^{\infty}(\pi^{-1}(D_k))$ *does not separate the points in the fiber* $\pi^{-1}(a)$ *over at least one point* a *in* $D_k\setminus\{z_n\}_{n\leq k}$.

This follows from a general theorem of Forelli [2] (cf. also Royden [8]). A simple proof can be provided to such a restricted situation as above [6]. Much more simple and direct proof can now be given if we use the separating function in Lemma 1.1. The proof goes as follows. Suppose $H^\infty(\pi^{-1}(D_k))$ separates the points in the fiber $\pi^{-1}(a)$ for an $a \in D_k \setminus \{z_n\}_{n \leq k}$ and choose an arbitrary point c in $D_k \setminus \{z_n\}_{n \leq k}$. We need to show that $H^\infty(\pi^{-1}(D_k))$ separates the points in $\pi^{-1}(c)$. Take a small disc $W = \Delta(c,r)$ $(r>0)$ such that

$$\overline{W} \subset D_k \setminus (\{a\} \cup \{z_n\}_{n \leq k}).$$

Lemma 1.1 then assures the existence of a function F in $H^\infty(\pi^{-1}(D_k))$ which separates the points in every fiber $\pi^{-1}(z)$ for every $z \in W$. Hence in particular $F(c^+) \neq F(c^-)$ with $\pi^{-1}(c) = \{c^+, c^-\}$, which completes the proof. □

2. Independence on finite parts.

2.1. In addition to the 2-disc $(\Delta^\sim, \Delta, \pi) = (\pi^{-1}(\Delta), \Delta, \pi)$ of an admissible determining sequence $\{z_n\}_{n \geq 1}$ we also consider 2-discs $(\pi_k^{-1}(\Delta), \Delta, \pi_k)$ of determining sequence $\{z_n\}_{n \geq k+1}$ $(k \in N)$. Take an admissible sequence $\{\Delta_n\}$ of closed discs $\Delta_n = \overline{\Delta}(z_n, r_n)$ in Δ $(n \in N)$ and consider subregions

$$D=\Delta\setminus \bigcup_{n\geq 1}\Delta_n,\quad D_k=\Delta\setminus \bigcup_{n\geq k+1}\Delta_n \quad (k\in N).$$

We will show that surfaces $\pi^{-1}(D)$, $\pi_k^{-1}(D_k)$ and $\pi^{-1}(D_k)$ have common properties concerning the point separation by bounded analytic functions. In other word the separation property of $\{\Delta_n\}_{n\geq 1}$ does not depend on the finite part $\{\Delta_n\}_{n\leq k}$ for each k in N. Namely we have

THEOREM 2.1. *The following three conditions on the point separation by bounded analytic functions are equivalent by pairs for one and hence for every* $k\in N$:

(a) $H^\infty(\pi^{-1}(D))$ *separates the points in* $\pi^{-1}(D)$;

(b) $H^\infty(\pi_k^{-1}(D_k))$ *separates the points in* $\pi_k^{-1}(D_k)$;

(c) $H^\infty(\pi^{-1}(D_k))$ *separates the points in* $\pi^{-1}(D_k)$.

Since $H^\infty(D)$ separates the points in D, (a) is equivalent to that $H^\infty(\pi^{-1}(D))$ separates the points in the fiber $\pi^{-1}(a)$ for every $a\in D$. Here *every* may be replaced by *one* in view of Fact 1.1. The same remark as above is true both for $H^\infty(\pi_k^{-1}(D_k))$ and $H^\infty(\pi^{-1}(D_k))$. However in the latter we only have to consider those base points a that are not the projections of branch points in $\pi^{-1}(D_k)$ so that $a\notin\{z_n\}_{n\leq k}$. Hence we only have to discuss the point separation of fibers containing two distinct points. The proof will be given in 2.2-2.4.

2.2. The implication (a) from (c) is trivial since $\pi^{-1}(D)$ is a subregion of $\pi^{-1}(D_k)$.

Next we show that (b) implies (c). We denote by T_k the cover transformation of the covering surface $(\pi_k^{-1}(\Delta),\Delta,\pi_k)$. We need to find a function in $H^{\infty}(\pi^{-1}(D_k))$ which separates the points in the fiber $\pi^{-1}(a)$ for any point a in $D_k\setminus\{z_n\}_{n\leq k}$. By (b) there exists an f in $H^{\infty}(\pi_k^{-1}(D_k))$ that separates the points in the fiber $\pi_k^{-1}(a)$. Replacing f by $f-f\circ T_k$ if necessary we can assume that $f\circ T_k=-f$ on $\pi_k^{-1}(D_k)$ and $f(a^+)=-f(a^-)\neq 0$ for $\pi_k^{-1}(a)=\{a^+,a^-\}$. Define the function $g\in H^{\infty}(D_k)$ by $g(z)=f(z^+)f(z^-)$ where $\pi^{-1}(z)=\{z^+,z^-\}$ for every $z\in D_k$. From the definition it follows that every zero of g is of even order and the square root $\sqrt{g(z)}$ defines the function $\sqrt{g}$ on $\pi_k^{-1}(D_k)$ and actually $\sqrt{g}=f$. Finally take the finite Blaschke product

$$B(z)=\prod_{n\leq k}\frac{z-z_n}{1-\bar{z}_n z}$$

that has simple zeros only at $\{z_n\}_{n\leq k}$. It is seen that the square root $\sqrt{B(z)g(z)}$ defines the function $\sqrt{Bg}$ in $H^{\infty}(\pi^{-1}(D_k))$. Since we may identify fibers over a by π and π_k and

$$\sqrt{Bg}(a^+)=-\sqrt{Bg}(a^-)\neq 0,$$

$\sqrt{Bg}$ is a required function.

2.3. The equivalence proof will be complete if we show that (a) implies (b).

First we consider the case when k is *even*. Assuming (a) we need to show that (b) is valid. For the purpose it is sufficient to construct a function $F=F_a \in H^{\infty}(\pi_k^{-1}(D_k))$ for any $a \in D$ such that F separates the points in the fiber $\pi_k^{-1}(\zeta)$ for every $\zeta \in \{a\} \cup (\bigcup_{n \leq k} \Delta_n)$. Take a Jordan region W in D_k such that

$$\bigcup_{n \leq k} \Delta_n \subset W \subset \overline{W} \subset D_k \setminus \{a\}.$$

By (a) we can find an f in $H^{\infty}(\pi^{-1}(D))$ that separates the points in $\pi^{-1}(a)$. Since k is even, we can view that

$$\pi^{-1}(D \setminus \overline{W}) = \pi_k^{-1}(D_k \setminus \overline{W})$$

on which $\pi = \pi_k$ and therefore $f \in H^{\infty}(\pi_k^{-1}(D_k \setminus \overline{W}))$ which separates the points in $\pi_k^{-1}(a)$. Hence by Lemma 1.1 we can find an F in $H^{\infty}(\pi_k^{-1}(D_k))$ which separates the points in the fiber $\pi^{-1}(z)$ for every $z \in W \cup \{a\}$. Thus F is a required F_a.

2.4. The implication of (b) from (a) is shown in the preceding number 2.3 when k is even. We next consider the case when k is *odd*. Since k+1 is even, (a) implies (b) for k+1: $H^{\infty}(\pi_{k+1}^{-1}(D_{k+1}))$ separates the points in $\pi_{k+1}^{-1}(D_{k+1})$. By the implication (c) from (b) considered for the determining sequence $\{z_{k+n}\}_{n\geq 1}$, the above implies that $H^\infty(\pi_k^{-1}(D_{k+1}))$ separates the points in $\pi_k^{-1}(D_{k+1})$. Then, by the implication of (a) from (c) considered again for the determining sequence $\{z_{k+n}\}_{n\geq 1}$, the above implies that $H^\infty(\pi_k^{-1}(D_k))$ separates the points in $\pi_k^{-1}(D_k)$. Thus (a) implies (b) for odd k.

The proof of Theorem 2.1 is herewith complete. □

2.5. An admissible determining sequence $\{z_n\}$ in Δ is said to be *rigid* if every admissible sequence $\{\Delta_n\}$ of closed discs in Δ is nonseparating. Deleting a finite number of terms from a given admissible determining sequence still leaves an admissible determining sequence. Similarly adding a finite number of new points to a given admissible sequence gives rise to an admissible determining sequence. As a direct consequence of Theorem 2.1 we can say the same as above for the rigidity and nonrigidity which is very convenient in practical applications:

THEOREM 2.2. *An admissible sequence* $\{z_n\}_{n\geq 1}$ *is rigid (nonrigid, resp.) if and only if* $\{z_n\}_{n\geq k+1}$ *is rigid (nonrigid, resp.) for one and hence for every positive integer* k.

3. The size of nonseparating discs.

3.1. Fix a 2-disc $(\Delta^{\sim},\Delta,\pi)$ of an admissible determining sequence $\{z_n\}$. Then we choose an admissible sequence $\{\Delta_n\}$ of closed discs $\Delta_n=\overline{\Delta}(z_n,r_n)$ associated with $\{z_n\}$ and consider the region

$$(3.1) \qquad D=\Delta\setminus \bigcup_{n\geq 1} \Delta_n.$$

We say that $\{\Delta_n\}$, or $\{r_n\}$, is *nonseparating* (*separating*, resp.) if the so called Myrberg phenomenon (cf. Myrberg [7], also [9])

$$H^{\infty}(D^{\sim})=H^{\infty}(D)\circ\pi$$

occurs (does not occur, resp.). In this section we will give a quantitative sufficient condition for $\{\Delta_n\}$, or rather $\{r_n\}$, to be nonseparating in terms of the sequence $\{z_n\}$. We already know [6] that $\{r_n\}$ is nonseparating if we make $\{r_n\}$ convergent *enough* rapidly to zero and our concern here is to determine the rapidity quantitatively.

3.2. In addition to the Euclidean metric $r(z,w)=|z-w|$ in Δ we also consider the pseudohyperbolic metric $\rho(z,w)$ in Δ (cf. e.g. [3]) given by

$$\rho(z,w)=\left|\frac{z-w}{1-\overline{w}z}\right|.$$

The metric ρ is really a metric in Δ invariant under Möbius transformations T of Δ: $\rho(T(z),T(w))=\rho(z,w)$. The invariance of ρ under reflection of Δ is also used: $\rho(\overline{z},\overline{w})=\rho(z,w)$. The open (closed, resp.) pseudohyperbolic disc $K(w,\rho)$ ($\overline{K}(w,\rho)$, resp.)

in Δ with center w and radius ρ in $(0,1)$ is given by

$$K(w,\rho)=\{z:\rho(z,w)<\rho\} \quad (\overline{K}(w,\rho)=\{z:\rho(z,w)\leq\rho\}, \text{ resp.})$$

which is also an open (closed, resp.) Euclidean disc in Δ.

For a point w in Δ and a subset X in Δ we consider the distance $\rho(w,X)=\rho(X,w)$ between w and X given by

$$\rho(w,X)=\inf_{z\in X}\rho(w,z).$$

For a given point w in Δ and a number r in $(0,1-|w|)$ we consider the quantity $\sigma(w,r)$ defined by

$$\sigma(w,r)=\inf\{\rho\in(0,1):\ \overline{\Delta}(w,r)\subset\overline{K}(w,\rho)\}.$$

The rotation of Δ about the origin by angle $-\arg w$ gives

$$\sigma(w,r)=\sigma(|w|,r)=\rho(|w|,|w|+r)$$

so that we have

$$\sigma(w,r)=\frac{r}{1-|w|(|w|+r)}.$$

Observe that $r\to\sigma(w,r)$ is an increasing homeomorphism of $(0,1-|w|)$ onto $(0,1)$. Hence the equation $\sigma=\sigma(w,r)$ can be solved with respect to r:

$$r=\frac{(1-|w|^2)\sigma}{1+|w|\sigma}.$$

3.3. With an admissible sequence $\{z_n\}$ of points in Δ we associate the sequence $\{\tau_n\}$ given by

$$\tau_n=\frac{1}{2}\rho(z_n,\{z_m\}_{m\neq n})=\frac{1}{2}\inf_{m\in N\setminus\{n\}}\rho(z_n,z_m).$$

We now consider an admissible sequence $\{\Delta_n\}$ of closed discs $\Delta_n=\overline{\Delta}(z_n,r_n)$ for the admissible determining sequence $\{z_n\}$ in Δ. With $\{r_n\}$ we associate two sequences $\{\sigma_n\}$ and $\{\rho_n\}$ given by

$$\sigma_n=\sigma(z_n,r_n)=\frac{r_n}{1-|z_n|(|z_n|+r_n)} \quad (n\in N)$$

and

$$\rho_n=\sqrt{\sigma_n\tau_n} \quad (n\in N).$$

We will require $\{r_n\}$ to satisfy one more basic condition.

For the purpose we fix an arbitrary number a with $0<a<1$ and consider two quantities $C_n(a)$ and $c=c(a)$ given by

$$C_n(a)=\tau_n^{(1+a)/(1-a)} \quad (n\in N)$$

and

$$c=c(a)=2^{-2a/(1-a)}.$$

Note that $0<c<1$.

Our requirement for the sequence $\{r_n\}$ is the following:

$$r_n\leq\frac{(1-|z_n|^2)C_n(a)}{1+|z_n|C_n(a)} \quad (n\in N). \tag{3.2}$$

As is easily seen the condition (3.2) is equivalent to

$$\sigma_n\leq C_n(a)=\tau_n^{(1+a)/(1-a)} \quad (n\in N) \tag{3.3}$$

and in turn to

$$\left(\log\frac{\tau_n}{\rho_n}\right)/\left(\log\frac{1}{\rho_n}\right)\geq a \quad (n\in N). \tag{3.4}$$

It follows from (3.3) that

$$\sigma_n\leq\tau_n^{2a/(1-a)}\cdot\tau_n\leq2^{-2a/(1-a)}\cdot\tau_n\leq c\tau_n \quad (n\in N) \tag{3.5}$$

which in particular implies $\sigma_n<\rho_n<\tau_n$ $(n\in N)$.

Let a be in (0,1). A positive sequence $\{r_n\}$ is said to be *a-admissible* for an admissible sequence $\{z_n\}$ in Δ if (3.2) is valid for a. Since (3.5) trivially implies $0<r_n<1-|z_n|$ $(n\in N)$ and $r_n+r_m<|z_n-z_m|$ $(n\neq m)$, any a-admissible sequence $\{r_n\}$ is automatically admissible for $\{z_n\}$.

Hereafter we assume that $\{r_n\}$ is a-admissible for an a in $(0,1)$ for an admissible sequence $\{z_n\}$ of points z_n in Δ. Associated sequences $\{\tau_n\}$, $\{\sigma_n\}$ and $\{\rho_n\}$ are accordingly determined.

We denote by K_n the pseudohyperbolic closed disc $K_n=\overline{K}(z_n,\rho_n)$ with center z_n and radius ρ_n for each $n\in N$. By the choice of $\{\tau_n\}$ we see that K_n $(n\geq 1)$ are disjoint by pairs so that

$$K=\Delta\setminus\bigcup_{n\geq 1}K_n$$

is a subregion of Δ contained in D given by (3.1).

3.4. Fix a partition $\{z_n:n\in N\}=\cup_{j\in N}P_j$ of $\{z_n:n\in N\}$ into pairwise disjoint finite subsets P_j $(j\in N)$. We denote by P_j^n the set P_j less the point z_n: $P_j^n=\{z_k\in P_j:z_k\neq z_n\}$ so that $P_j^n=P_j$ if and only if $z_n\notin P_j$. Hereafter we fix a partition $\{P_j\}_{j\in N}$ of $\{z_n\}_{n\in N}$ as above. The partition $P_j=\{z_j\}$ $(j\in N)$ is said to be the identity partition which is not excluded as one of possible cases.

We introduce the final sequence $\{\mu_j\}_{j\in N}$ given by

$$\mu_j=\sup_{n\in N}\left(\sum_{z_k\in P_j^n}\log\rho(z_n,\partial K_k)\right)/\log|z_n| \quad (j\in N).$$

Here we maintain that μ_j is a finite positive constant. That $\mu_j>0$ is clear. Since

$$\mu_j\leq\sum_{z_k\in P_j}\sup_{n\in N\setminus\{k\}}\left(\log\rho(z_n,\partial K_k)\right)/\log|z_n| \quad (j\in N),$$

we only have to show that

$$\mu_k'=\sup_{n\in N\setminus\{k\}}\left(\log\rho(z_n,\partial K_k)\right)/\log|z_n|<\infty \quad (k\in N).$$

Note that the greatest real point on $\partial\overline{K}(|z_k|,\rho_k)$ is $(|z_k|+\rho_k)/(1+\rho_k|z_k|)$. Set

$$N_k=\{n\in N:|z_n|>(|z_k|+\rho_k)/(1+\rho_k|z_k|)\}.$$

Since $N\setminus N_k$ is finite and $k\in N\setminus N_k$, $\mu_k'<\infty$ if and only if

$$\mu_k''=\sup_{n\in N_k}\left(\log\rho(z_n,\partial K_k)\right)/\log|z_n|<\infty.$$

By simple geometric consideration based upon the Möbius invariance of ρ by rotation of Δ about the origin we see that

$$\rho(z_n,\partial K_k)\geqq\rho\left(|z_n|,(|z_k|+\rho_k)/(1+\rho_k|z_k|)\right)$$

for every $n\in N_k$. For simplicity set $s=(|z_k|+\rho_k)/(1+\rho_k|z_k|)$ and $\tau=\inf_{n\in N_k}|z_n|$. Then we have

$$\mu_k''\leqq\sup_{n\in N_k}\frac{\log\rho(|z_n|,s)}{\log\rho(|z_n|,0)}\leqq\sup_{\tau\leqq t<1}\frac{\log\rho(t,s)}{\log\rho(t,0)}.$$

By the l'Hospital rule we see that

$$\lim_{t\uparrow 1}\frac{\log\rho(t,s)}{\log\rho(t,0)}=\lim_{t\uparrow 1}\frac{\log\frac{t-s}{1-st}}{\log t}=\lim_{t\uparrow 1}\left(\frac{t}{t-s}+\frac{st}{1-st}\right)=\frac{1+s}{1-s}.$$

Since the function $t \to (\log\rho(t,s))/\log\rho(t,0)$ is continuous on $[\tau,1)$, we can conclude that $\mu_k'' < \infty$.

3.5. A sequence $\{\eta_n\}$ is said to be *inverse summable* in this paper if $\eta_n > 0$ ($n \in N$) and $\sum_{n\geq 1} 1/\eta_n < \infty$. We can now state

THE MAIN THEOREM. *Suppose that* $\{r_n\}$ *is* a-*admissible for an admissible* $\{z_n\}$ *in* Δ. *If* $\{r_n\}$ *satisfies*

$$\rho_k \leq |z_k|^{\mu_j \eta_j} \quad (z_k \in P_j,\ j \in N) \tag{3.6}$$

for an inverse summable sequence $\{\eta_j\}$, *then* $\{r_n\}$ *is nonseparating.*

Recall that ρ_k and μ_j were defined in 3.3 and 3.4, respectively. The proof will be given in the following 3.6 and 3.7.

3.6. By Theorem 2.1 the sequence $\{z_n\}$ may be replaced by $\{z_n\}_{n\geq k+1}$ for any $k=0,1,2,\cdots$. Therefore we may assume that $0 \in K$ and $\sum_{j\geq 1} 1/\eta_j < 1$.

Contrary to the assertion assume that there exists an f in $H^\infty(D^\sim)$ such that f is not constant on $\pi^{-1}(z)$ for some z in D. We may assume that $\sup_{D^\sim} |f| \leq (2\pi)^{-1}(1-c^{1/4})^2$, where c was defined in 3.3. Set

$$g(z)=\left(f(z^+)-f(z^-)\right)^2$$

for each $z\in D$ where $\pi^{-1}(z)=\{z^+,z^-\}$. Then $g\in H^\infty(D)\setminus\{0\}$. First we evaluate $|g|$ on ∂K_n ($n\in N$) following the procedure developed in [5]. Consider the annulus

$$A_n=\{z:\ \sigma_n<\rho(z,z_n)<\tau_n\}.$$

By the Möbius invariance of ρ we have

$$\pi^{-1}(A_n)=\{z:\ \sqrt{\sigma_n}<|z|<\sqrt{\tau_n}\}$$

in the sense of conformal equivalence. Then

$$\gamma_n=\pi^{-1}(\partial K_n)=\{z:\ |z|=(\sigma_n\tau_n)^{1/4}\}.$$

We denote by α_n and β_n the inner and outer boundary of $\pi^{-1}(A_n)$, respectively. By the Cauchy formula

$$f'(z)=\frac{1}{2\pi i}\int_{\beta_n-\alpha_n}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta \qquad (z\in\pi^{-1}(A_n)).$$

Hence we see for $z\in\gamma_n$ that

$$|f'(z)|\leq\frac{1}{2\pi}\int_{\alpha_n}\frac{|f(\zeta)|}{(|z|-|\zeta|)^2}|d\zeta|+\frac{1}{2\pi}\int_{\beta_n}\frac{|f(\zeta)|}{(|\zeta|-|z|)^2}|d\zeta|$$

$$\leq\pi^{-1}(1-c^{1/4})^2(\tau_n^{1/4}-\sigma_n^{1/4})^{-2}.$$

We denote by $|f(\gamma_n)|$ the length of the image curve $f(\gamma_n)$ of γ_n under f. Then

$$|f(\gamma_n)|=\int_{\gamma_n}|f'(z)||dz|\leq\int_{\gamma_n}\pi^{-1}(1-c^{1/4})^2(\tau_n^{1/4}-\sigma_n^{1/4})^{-2}|dz|$$

$$=2(1-c^{1/4})^2(\sigma_n\tau_n)^{1/4}(\tau_n^{1/4}-\sigma_n^{1/4})^{-2}.$$

The oscillation $\mathrm{Osc}_{\gamma_n}f$ of f over γ_n is the diameter of $f(\gamma_n)$

which is at most $|f(\gamma_n)|/2$. On the other hand $\sup_{\partial K_n}|g|\leq (\mathrm{Osc}_{\gamma_n} f)^2$ implies that

$$|g(z)|\leq(1-c^{1/4})^4(\sigma_n/\tau_n)^{1/2}(1-(\sigma_n/\tau_n)^{1/4})^{-4}\quad (z\in\partial K_n).$$

Since $\sigma_n/\tau_n\leq c$ (cf. (3.5) in 3.3) and $(\sigma_n/\tau_n)^{1/2}=\rho_n/\tau_n$, we conclude that

$$|g(z)|\leq\rho_n/\tau_n<1\quad (z\in\partial K_n). \tag{3.7}$$

3.7. Let $w_n=w(\cdot,\partial K_n;K)$ be the harmonic measure of ∂K_n with respect to K. By (3.7) we have

$$\sum_{n\geq 1}\lambda_n w_n(z)\leq\log(1/|g(z)|)$$

on each component ∂K_n $(n\in N)$ of ∂K and hence for every z in K where

$$\lambda_n=\log(\tau_n/\rho_n)>0\quad (n\in N).$$

Therefore $\sum_{n\geq 1}\lambda_n w_n$ is a positive harmonic function on K and in particular

$$\sum_{n\geq 1}\lambda_n w_n(0)<+\infty. \tag{3.8}$$

Let v_n be the harmonic function on K with boundary values $\log(|1-\bar{z}_n z|/|z-z_n|)$ on $\partial K\setminus\partial K_n$ and 0 on ∂K_n. Then

$$w_n(z)=\left(\log\left|\frac{1-\bar{z}_n z}{z-z_n}\right|-v_n(z)\right)/\log(1/\rho_n)$$

and in particular

$$\lambda_n w_n(0)=\left((\log\frac{\tau_n}{\rho_n})/(\log\frac{1}{\rho_n})\right)(\log\frac{1}{|z_n|}-v_n(0)).$$

Therefore by the inequality $(\log(\tau_n/\rho_n))/\log(1/\rho_n) \geq a$ (cf. (3.4) in 3.3) we have the estimate

$$\sum_{n\geq 1} \lambda_n w_n(0) \geq a \sum_{n\geq 1} \left(\log\frac{1}{|z_n|} - v_n(0)\right). \tag{3.9}$$

The comparison of boundary values and the maximum principle yield

$$v_n(z) \leq \sum_{k\in N\setminus\{n\}} \left((\log\rho(z_n, \partial K_k))/\log\rho_k\right) \log\left|\frac{1-z_k\bar{z}}{z-z_k}\right|$$

for z in K. Hence in particular

$$v_n(0) \leq \sum_{k\in N\setminus\{n\}} \left((\log\rho(z_n, \partial K_k))/\log\rho_k\right) \log\frac{1}{|z_k|}.$$

Since $\{z_k: k\in N\setminus\{n\}\} = \cup_{j\in N} P_j^n$, the above takes the form

$$v_n(0) \leq \sum_{j\in N}\left(\sum_{z_k\in P_j^n}\left((\log\rho(z_n, \partial K_k)^{-1})/(\log\rho_k^{-1})\right)\log\frac{1}{|z_k|}\right).$$

By (3.6), $(\log|z_k|^{-1})/(\log\rho_k^{-1}) \leq 1/\mu_j\eta_j$ $(z_k \in P_j^n)$ and thus

$$v_n(0) \leq \sum_{j\in N}\left(\sum_{z_k\in P_j^n}\left((\log\frac{1}{\rho(z_n, \partial K_k)})/(\log\frac{1}{|z_n|})\right)(\log\frac{1}{|z_n|})/\mu_j\eta_j\right).$$

By the definition of μ_j (cf. 3.4) we have

$$v_n(0) \leq \sum_{j\in N} \mu_j(\log\frac{1}{|z_n|})/\mu_j\eta_j = \left(\sum_{j\geq 1} 1/\eta_j\right)\log\frac{1}{|z_n|}.$$

Hence by (3.9) we have

$$\sum_{n\geq 1} \lambda_n w_n(0) \geq a \sum_{n\geq 1}\left(\log\frac{1}{|z_n|} - \left(\sum_{j\geq 1} 1/\eta_j\right)\log\frac{1}{|z_n|}\right).$$

By the preliminary reduction made at the beginning of 3.6, $1-\sum_{j\geq 1} 1/\eta_j = b > 0$. Hence by using $\log|z_n|^{-1} > 1-|z_n|$ we have

$$\sum_{n\geq 1} \lambda_n w_n(0) \geq ab \sum_{n\geq 1} (1-|z_n|).$$

By the very basic assumption that $\sum_{n\geq 1}(1-|z_n|)=+\infty$, we must have $\sum_{n\geq 1}\lambda_n w_n(0)=+\infty$, contradicting (3.8).

The proof of the main theorem is herewith complete. □

4. Nonseparation criterion.

4.1. We have considered a condition in 3.5 for $\{r_n\}$ to be nonseparating. The condition (3.6) given there is a bit implicit, i.e. not enough explicit for the practical application. In this section we will give in 4.4 a criterion for $\{r_n\}$ to be nonseparating easily applicable to concrete cases. First we replace ρ_k in (3.6) by r_k:

THEOREM 4.1. *If* $\{r_n\}$ *is* a*-admissible and*

$$r_k \leq (1-|z_k|)|z_k|^{\mu_j \eta_j} \quad (z_k \in P_j,\ j \in N) \tag{4.1}$$

for an inverse summable sequence $\{\eta_j\}$, *then* $\{r_n\}$ *is nonseparating.*

For the proof we observe, since $|z_k|+r_k<1$, that $\sigma_k= r_k/(1-|z_k|(|z_k|+r_k))\leq r_k/(1-|z_k|)$. By the definition of ρ_k (cf. 3.3) and $\tau_k<1$ we see that

$$\rho_k^2=\sigma_k\tau_k\leq\sigma_k\leq\frac{r_k}{1-|z_k|}\leq|z_k|^{\mu_j\eta_j} \quad (z_k\in P_j,\ j\in N)$$

so that (3.6) is valid for the inverse summable sequence $\{\eta_j/2\}_{j\in N}$. Hence the main theorem assures that $\{r_n\}$ is nonseparating. $\square$

4.2. We will further try to simplify (4.1). For the purpose and later for other purposes we will use the following three inequalities:

$$(4.2)\qquad 0<\left(\log\frac{1-bx^2}{1-b}\right)/\log\frac{1}{x}<\frac{2b}{1-b}$$

for $0<b<1$ and $0<x<1$;

$$(4.3)\qquad 0<\left(\log\left|\frac{\alpha-c}{1-c\alpha}\right|\right)/\log\alpha<\left(\log\left|\frac{\beta-c}{1-c\beta}\right|\right)/\log\beta$$

for $0<\alpha<\beta<c<1$;

$$(4.4)\qquad \left(\log\frac{\alpha-c}{1-c\alpha}\right)/\log\alpha>\left(\log\frac{\beta-c}{1-c\beta}\right)/\log\beta>\frac{1+c}{1-c}$$

for $0<c<\alpha<\beta<1$.

The inequality (4.2) follows from $\log(1/x)>1-x$ and

$$\log\frac{1-bx^2}{1-b}=\log\left(1+\frac{b}{1-b}(1-x^2)\right)\leq\frac{b}{1-b}(1-x^2).$$

To prove inequalities (4.3) and (4.4) we consider the function

$$F(x)=\left(\log\left|\frac{x-c}{1-cx}\right|\right)/\log x$$

of x in the interval $(0,1)$ for any fixed $c\in(0,1)$. Observe that $F(+0)=0$, $F(c\pm 0)=+\infty$ and $F(1-0)=(1+c)/(1-c)$. It is not difficult to see that $dF(x)/dx>0$ on $(0,c)$ which implies (4.3). A bit more may be in order to show that $dF(x)/dx<0$ on $(c,1)$ but anyhow (4.4) can also be deduced in the frame of calculus.

4.3. In the calculation of μ_j introduced in 3.4 we need to know $\rho(z_n,\partial K_k)$. The calculation would be easier if we can replace $\rho(z_n,\partial K_k)$ by $\rho(z_n,z_k)$. Hence we now associate the sequence $\{\nu_j\}$ with $\{z_n\}_{n\geq 1}=\cup_{j\in N}P_j$ as a simplified version of the sequence $\{\mu_j\}$ defined by

$$\nu_j=\sup_{n\in N}\Big(\sum_{z_k\in P_j^n}\log\rho(z_n,z_k)\Big)/\log|z_n| \quad (j\in N).$$

Assume that a sequence $\{r_n\}$ satisfies (3.2) for an a with $0<a<1$ so that $\sigma_n\leq c\tau_n$ where $c=2^{-2a/(1-a)}<1$ (c.f. (3.5) in 3.3). Then $\rho_n^2=\sigma_n\tau_n\leq c\tau_n^2$ implies that

$$\rho_n\leq\sqrt{c}\tau_n \quad (n\in N).$$

We now claim the following

$$\nu_j\leq\mu_j\leq\frac{1+\sqrt{c}}{1-\sqrt{c}}\nu_j \quad (j\in N). \tag{4.5}$$

For the proof of (4.5) we only have to show

$$1\leq\Big(\sum_{z_k\in P_j^n}\log\rho(z_n,\partial K_k)\Big)/\Big(\sum_{z_k\in P_j^n}\log\rho(z_n,z_k)\Big)\leq\frac{1+\sqrt{c}}{1-\sqrt{c}} \quad (j,n\in N)$$

for $k\neq n$. However this follows from the following:

$$1\leq(\log\rho(z_n,\partial K_k))/(\log\rho(z_n,z_k))\leq\frac{1+\sqrt{c}}{1-\sqrt{c}} \quad (n,k\in N,\ n\neq k). \tag{4.6}$$

Hence we only have to prove (4.6) for the proof of (4.5). Since $\rho(z_n,\partial K_k)\leq\rho(z_n,z_k)$, the first inequality of (4.6) is trivial. By the Möbius invariance of ρ we only have to show the last inequality of (4.6) for the case $z_n=\beta>0$ and $z_k=0$. Since $\tau_k\leq\beta/2<\beta$, we have

$$0<\rho_k<\sqrt{c}\tau_k<\tau_k<\beta<1.$$

Observe that $\rho(z_n,z_k)=\rho(\beta,0)=\beta$ and $\rho(z_n,\partial K_k)=\rho(\rho_k,\beta)>\rho(\sqrt{c}\tau_k,\beta)$. Thus we have

$$(\log\rho(z_n,\partial K_k))/(\log\rho(z_n,z_k))\le(\log\rho(\sqrt{c}\tau_k,\beta))/\log\beta.$$

On setting $A=\sqrt{c}\tau_k$, $\alpha=\tau_k$ we see that $0<A<\alpha<\beta<1$ and a fortiori (4.4) and then (4.2) implies that

$$(\log\rho(\sqrt{c}\tau_k,\beta))/\log\beta=\left(\log\frac{\beta-A}{1-A\beta}\right)/\log\beta$$

$$\le\left(\log\frac{\alpha-A}{1-A\alpha}\right)/\log\alpha \;=1+\left(\log\frac{1-\sqrt{c}\tau_k^2}{1-\sqrt{c}}\right)/\left(\log\frac{1}{\tau_k}\right)$$

$$\le 1+2\sqrt{c}/(1-\sqrt{c})=(1+\sqrt{c})/(1-\sqrt{c}).$$

Thus (4.6) and hence (4.5) is established.

4.4. Using ν_j introduced in the preceding 4.3 instead of μ_j we obtain the following simplified version of Theorem 4.1:

THEOREM 4.2. *If* $\{r_n\}$ *is a-admissible and*

(4.7) $$r_k\le(1-|z_k|)|z_k|^{\nu_j\eta_j}\quad(z_k\in P_j,\ j\in N)$$

for an inverse summable sequence $\{\eta_j\}$, *then* $\{r_n\}$ *is nonseparating.*

For the proof we use $C\mu_j\le\nu_j$ with $C=(1-\sqrt{c})/(1+\sqrt{c})$ (cf. (4.5)). Since $|z_k|<1$, $|z_k|^{\nu_j\eta_j}\le|z_k|^{C\mu_j\eta_j}$. Hence (4.7) implies

$$r_k\le(1-|z_k|)|z_k|^{\mu_j(C\eta_j)}\quad(z_k\in P_j,\ j\in N)$$

which means that (4.1) is valid for the inverse summable sequence $\{C\eta_j\}$. Therefore Theorem 4.1 assures that $\{r_n\}$ is nonseparating. □

4.5. In concrete cases ν_j introduced in 4.3 is still hard to compute. But it often occurs that ν_j is estimated in terms of τ_j introduced at the beginning of 3.3. The calculation of τ_j is in general much easier than that of ν_j. Thus we view the following as the final form of our simplification of the main theorem:

NONSEPARATION CRITERION. *Suppose that* $\{z_n\}_{n\geq 1}=\bigcup_{j\in N}P_j$ *satisfies*

(4.8) $$\nu_j\leq q_j(\log\tau_k)/\log|z_k| \quad (z_k\in P_j,\ j\in N)$$

for a positive sequence $\{q_j\}$ *and that* $\{r_n\}$ *is* a-*admissible. If* $\{r_n\}$ *satisfies*

(4.9) $$r_k\leq(1-|z_k|)\tau_k^{q_j\eta_j} \quad (z_k\in P_j,\ j\in N),$$

or if $\{q_j\}$ *is moreover bounded and* $\{r_n\}$ *satisfies*

(4.10) $$r_k\leq(1-|z_k|)\tau_k^{\eta_j} \quad (z_k\in P_j,\ j\in N)$$

for an inverse summable $\{\eta_j\}$, *then* $\{r_n\}$ *is nonseparating.*

For the proof we use the identity $\tau_k=|z_k|^{(\log\tau_k)/\log|z_k|}$, (4.9) and (4.8) to deduce

$$r_k \leq (1-|z_k|)\tau_k^{q_j \eta_j}$$

$$=(1-|z_k|)|z_k|^{(q_j(\log\tau_k)/\log|z_k|)\eta_j}$$

$$\leq (1-|z_k|)|z_k|^{\nu_j \eta_j}$$

so that (4.7) follows. Theorem 4.2 implies that $\{r_n\}$ is nonseparating. If moreover $0<q_j \leq q$ $(j \in N)$ for some finite constant q and (4.10) is valid, then we see that

$$r_k \leq (1-|z_k|)\tau_k^{\eta_j} = (1-|z_k|)\tau_k^{q(\eta_j/q)} \leq (1-|z_k|)\tau_k^{q_j(\eta_j/q)}$$

so that (4.9) is valid for the inverse summable sequence $\{\eta_j/q\}$. By the first part, $\{r_n\}$ is concluded to be nonseparating. □

4.6. In the case of the *identity partition* of $\{z_n\}$ conditions in the above criterion of course take the following simpler forms. To check (4.8) is reduced to see whether

(4.8)′ $$(\log\rho(z_n,z_k))/\log|z_n| \leq q_k(\log\tau_k)/\log|z_k| \quad (k,n \in N,\ k \neq n)$$

is valid for a positive sequence $\{q_k\}$. The condition (4.9) takes the form

(4.9)′ $$r_k \leq (1-|z_k|)\tau_k^{q_k \eta_k} \quad (k \in N)$$

which implies that the a-admissible sequence $\{r_k\}$ is nonseparating. If the sequence $\{q_k\}$ can be chosen to be bounded in (4.8)′, then we only have to see whether the following reduced version of (4.10$'$) is valid to conclude that $\{r_k\}$ is nonseparating :

$$(4.10)' \qquad r_k \leq (1-|z_k|)\tau_k^{\eta_k} \quad (k \in N).$$

4.7. At the end of this section we remark that the condition (4.10) in 4.4 for any sequence $\{r_n\}$ only assumed to be admissible implies that $\{r_n\}_{n\geq k}$ for sufficiently large k is a-admissible. Thus the last part of the nonseparation criterion is simplified: $\{r_n\}$ need only to be required to be admissible for the case where $\{q_j\}$ is bounded and (4.10) is postulated.

5. Examples of nonseparating discs

5.1. In this section we will explicitly give bounds for $\{r_n\}$ to be nonseparating when admissible determining sequences $\{z_n\}$ are concretely given. Two kinds of $\{z_n\}$ will be treated. The first is a $\{z_n\}$ whose set of accumulation points consists of a single point on $\partial\Delta$ and the second is a $\{z_n\}$ whose set of accumulation points is the whole $\partial\Delta$. We start with the former case:

EXAMPLE 5.1. *For the admissible determining sequence* $\{z_n\}_{n\geq 2}$ *of points* z_n *in* Δ *given by*

$$z_n = 1 - 1/n \quad (n \geq 2),$$

any admissible sequence $\{r_n\}$ *of radii* r_n *is nonseparating if*

(5.1) $$r_n \leq (1/n)^{\eta_n} \quad (n \geq 2)$$

for an inverse summable sequence $\{\eta_n\}$.

The proof is given in 5.2.

5.2. By (4.3) and (4.4) we have

$$\begin{cases} (\log\rho(z_n,z_k))/\log|z_n| \leq (\log\rho(z_{k-1},z_k))/\log|z_{k-1}| & (2\leq n\leq k-1), \\ (\log\rho(z_n,z_k))/\log|z_n| \leq (\log\rho(z_{k+1},z_k))/\log|z_{k+1}| & (k+1\leq n). \end{cases}$$

Since $\rho(z_{k+1},z_k)=1/2k$ and $\rho(z_{k-1},z_k)=1/2(k-1)$, we see that $\tau_k=$ $1/4k$. Hence $\rho(z_{k\pm1},z_k)\geq\tau_k$ or

$$\log\rho(z_{k\pm1},z_k)^{-1}\leq\log\tau_k^{-1}.$$

It is readily seen that

$$\log|z_{k\pm1}|^{-1}\geq\tfrac{1}{2}\log|z_k|^{-1}.$$

Therefore $(\log\rho(z_{k\pm1},z_k))/\log|z_{k\pm1}|\leq 2(\log\tau_k)/\log|z_k|$ and thus

$$(\log\rho(z_n,z_k))/\log|z_n|\leq 2(\log\tau_k)/\log|z_k| \quad (n,k\geq 2,\ n\neq k)$$

which shows that (4.8)′ is valid for $q_k=2$ $(k\geq2)$.

By Theorem 2.1 we may assume that $\eta_k>1$ $(k\geq2)$ in (5.1). Let $\eta_k'=(\eta_k-1)/(1+(\log4)/\log k)$ $(k\geq2)$. Then

$$r_k\leq(1/k)^{\eta_k}=(1/k)^{1+(1+(\log 4)/\log k)\eta_k'}$$

$$=(1/k)(1/4k)^{\eta_k'}=(1-|z_k|)\tau_k^{\eta_k'},$$

i.e. (4.10)′ is valid for the inverse summable $\{\eta_k'\}$.

The right hand side of (3.2) is not less than $(4n)^{-2/(1-a)}$. On the other hand $r_n \leq (1/n)^{\eta_n}$ with $\eta_n \to +\infty$ and therefore (3.2) is satisfied for sufficiently large n. By theorem 2.1 we may suppose (5.1) implies that $\{r_n\}$ is a-admissible (cf. also 4.7).

By Nonseparation criterion for the identity partition we now conclude that $\{r_n\}$ is nonseparating. □

5.3. We turn to the case when the set of accumulation points of the determining sequence $\{z_n\}$ is the whole boundary $\partial\Delta$:

EXAMPLE 5.2. *For the admissible sequence*

$$\{z_{nj}\colon\ n \in N,\ 0 \leq j < 2^n\}$$

of points z_{nj} *in* Δ *given by*

$$z_{nj} = (1-2^{-n})e^{2\pi j 2^{-n} i} \quad (n \in N,\ 0 \leq j < 2^n),$$

any admissible sequence $\{r_{nj}\}$ *of radii* r_{nj} *of* $\Delta(z_{nj}, r_{nj})$ *is nonseparating if*

$$r_{nj} \leq 2^{-\eta_n} \quad (n \in N,\ 0 \leq j < 2^n) \tag{5.2}$$

for an inverse summable sequence $\{\eta_n\}$ *with* $1/\eta_n = o(1/n)$.

The proof will be given in 5.4–5.7.

5.4. Let $\{z_{nj}\} = \cup_{n \in N} P_n$ be the partition of $P = \{z_{nj}\}$ such that

$$P_n=\{z_{nj}:\ 0\le j<2^n\}\quad (n\in N).$$

The sequence $\{\tau_{nj}\}$ associated with P takes the form

$$\tau_{nj}=\frac{1}{2}\inf_{z_{mk}\in P,\, z_{mk}\ne z_{nj}}\rho(z_{nj},z_{mk})\quad (n\in N,\ 0\le j<2^n).$$

The subfamily P_n^{mk} of P is given by

$$P_n^{mk}=\{z_{nj}\in P_n:\ z_{nj}\ne z_{mk}\}.$$

Then the sequence $\{\nu_n\}$ is given by

$$\nu_n=\sup_{z_{mk}\in P}\Big(\sum_{z_{nj}\in P_n^{mk}}\log\rho(z_{nj},z_{mk})\Big)/\log|z_{mk}|.$$

We use the following three inequalities which will be proven later:

(5.3) $$\nu_n\le q\cdot 2^n\quad (n\in N,\ q=24+\sum_{t=1}^{\infty}1/t^2),$$

(5.4) $$2^{-n}\le\log(1/|z_{nj}|)\le(e/(e-1))\cdot 2^{-n}\quad (n\in N,\ 0\le j<2^n),$$

(5.5) $$1/6\le\tau_{nj}\le 1/5\quad (n\in N,\ 0\le j<2^n).$$

Then we proceed as follows:

$$\begin{aligned}\nu_n&\le q\cdot 2^n\le q\cdot(e/(e-1))/(\log|z_{nj}|^{-1})\\&\le q\cdot\Big((e/(e-1))/(\log|z_{nj}|^{-1})\Big)\cdot\Big((\log\tau_{nj}^{-1})/\log 5\Big)\\&=\Big((qe)/((e-1)\log 5)\Big)\cdot\Big((\log\tau_{nj})/\log|z_{nj}|\Big).\end{aligned}$$

Thus (4.8) is valid for $q_n=(qe)/((e-1)\log 5)$ $(n\in N)$ so that $\{q_n\}$ is a bounded positive sequence.

For any inverse summable sequence $\{\eta_n\}$ with $1/\eta_n=o(1/n)$, i.e. $n/\eta_n=o(1)$, by Theorem 2.1, we can view $\{\eta_n'\}$ to be inverse summable if

$$\eta_n'=\eta_n(1-n/\eta_n)/\log_2 6 \quad (n\in N).$$

The condition (5.2) with (5.5) implies that

$$r_{nj}\leq 2^{-\eta_n}=2^{-n-\eta_n'\log_2 6}=2^{-n}\cdot(1/6)^{\eta_n'}\leq(1-|z_{nj}|)\tau_{nj}^{\eta_n'}.$$

Thus (4.10) is valid for the inverse summable $\{\eta_n'\}$.

The right hand side of (3.2) is not less than $C\cdot 2^{-n}$ for some positive constant C. Since $r_{nj}\leq 2^{-\eta_n}$ with $n/\eta_n= o(1)$, by Theorem 2.1, we may consider that (3.2) is satisfied so that $\{r_{nj}\}$ with (5.2) is a-admissible (cf. also 4.7).

Thus Nonseparation criterion assures that $\{r_{nj}\}$ is nonseparating.

5.5. The inequality (5.4) follows from the inequalities

$$1-x\leq\log(1/x)\leq(e/(e-1))(1-x) \quad (1/e\leq x\leq 1).$$

We next prove (5.5). By the Möbius invariance of ρ we have

$$\tau_{nj}=\tau_{n0} \quad (n\in N,\ 0\leq j<2^n).$$

By the direct computation

$$\rho(z_{n0},z_{n+1,0})=1/(3-2^{-n})\leq 2/5 \quad (n\in N).$$

By setting $\theta=2\pi/2^n$ we have

$$|1-e^{i\theta}|>\sin\theta>(2/\pi)\theta=4/2^n \quad (n\geq 2).$$

Thus we see that

$$|1-e^{i\theta}|\geq 4/2^n \quad (n\in N)$$

including the trivial case of $n=1$. Hence we see that

$$1/\rho(z_{n0},z_{n1})^2=\left|\frac{1-\bar{z}_{n0}z_{n1}}{z_{n0}-z_{n1}}\right|^2$$

$$=1+\frac{(1-|z_{n0}|^2)(1-|z_{n1}|^2)}{|z_{n0}-z_{n1}|^2}$$

$$=1+\frac{2^{-2n}(2-2^{-n})^2}{(1-2^{-n})^2|1-e^{i\theta}|^2}$$

$$\leq 1+\frac{2^{-2n}(2-2^{-n})^2}{(1-2^{-n})^2\cdot 16\cdot 2^{-2n}}<2$$

so that we have

$$\rho(z_{n0},z_{n1})>1/\sqrt{2}>2/5>\rho(z_{n0},z_{n+1,0})=1/(3-2^{-n}).$$

By a pseudohyperbolic geometric consideration we conclude from the above relation that $2\tau_{n0}=\rho(z_{n0},z_{n+1,0})=1/(3-2^{-n})$. Hence we obtain

$$\tau_{nj}=1/2(3-2^{-n}) \quad (n\in N,\ 0\leq j<2^n)$$

from which (5.5) follows.

5.6. Finally we prove (5.3). For simplicity we set $\rho(z_{ns},z_{kt})=\rho_{ns,kt}$ and

$$\nu_{k,ns}=\Big(\sum_{z_{kt}\in P_k^{ns}}\log\rho_{ns,kt}\Big)/\log|z_{ns}|.$$

Then $\nu_k=\sup_{z_{ns}\in P}\nu_{k,ns}$. Hence we need to evaluate $\nu_{k,ns}$ for every $z_{ns}\in P$. We also set

$$I_{kt,ns}=(\log\rho_{ns,kt})/\log|z_{ns}|$$

so that $\nu_{k,ns}=\sum_{z_{kt}\in P_k^{ns}} I_{kt,ns}$. In order to evaluate $\nu_{k,ns}$ we thus need to evaluate $I_{kt,ns}$ and for the purpose first we wish to evaluate $\log\rho_{ns,kt}^{-1}$. Set $\theta=2\pi(s\cdot 2^{-n}-t\cdot 2^{-k})$. By the rotational and reflectional invariance of ρ we may without loss of generality assume that $|\theta|\leq\pi$. Then

$$1-\cos\theta=2\sin^2(|\theta|/2)\geq 2\left(\frac{2}{\pi}\cdot\frac{\theta}{2}\right)^2=8(s\cdot 2^{-n}-t\cdot 2^{-k})^2.$$

In addition to the above we use $\log\xi\leq\xi-1$ $(\xi>0)$ to evaluate $\log\rho_{ns,kt}^{-1}$ as follows:

$$\log\rho_{ns,kt}^{-1}=\frac{1}{2}\log\rho_{ns,kt}^{-2}\leq\frac{1}{2}(\rho_{ns,kt}^{-2}-1)$$

$$=\frac{1}{2}\ \frac{(1-|z_{ns}|^2)(1-|z_{kt}|^2)}{|z_{ns}-z_{kt}|^2}$$

$$=\frac{1}{2}\ \frac{2^{-n-k}(2-2^{-n})(2-2^{-k})}{(|z_{ns}|-|z_{kt}|)^2+2|z_{ns}||z_{kt}|(1-\cos\theta)}$$

$$\leq\frac{2\cdot 2^{-n-k}}{(2^{-n}-2^{-k})^2+4(s\cdot 2^{-n}-t\cdot 2^{-k})^2}.$$

We also have that $\log|z_{ns}|^{-1}=\log(1-2^{-n})^{-1}\geq 2^{-n}$ and thus

$$I_{kt,ns}\leq\frac{2\cdot 2^{-k}}{(2^{-n}-2^{-k})^2+4(s\cdot 2^{-n}-t\cdot 2^{-k})^2}.$$

5.7. We now evaluate $\nu_{k,ns}$ in the following three cases separately: k=n, k>n and k<n. In the first two cases $k\geq n$ we use the rotational invariance of ρ to see that $I_{kt,ns}=I_{kt',n0}$.

Here $z_{kt'}$ exhausts P_k^{n0} if z_{kt} exhausts P_k^{ns}. Hence

$$\nu_{k,ns}=\nu_{k,n0}=\sum_{z_{kt}\in P_k^{n0}} I_{kt,n0}$$

and we have the estimate

$$I_{kt,n0}\leq\frac{2\cdot 2^k}{(2^{k-n}-1)^2+4t^2}.$$

To begin with we consider the case k=n. By the reflectional invariance of ρ and $z_{k0}\notin P_k^{n0}$ we see that

$$\nu_{k,ns}=\nu_{k,n0}=\sum_{z_{kt}\in P_k^{n0}} I_{kt,n0}\leq 2\sum_{t=1}^{2^{k-1}} I_{kt,n0}\leq 2\sum_{t=1}^{2^{k-1}} 2^k/2t^2\leq\left(\sum_{t=1}^{\infty} 1/t^2\right)2^k.$$

Next let k>n. Similarly as above but noting $z_{k0}\in P_k^{n0}$ this time we see that

$$\nu_{k,ns}=\nu_{k,n0}=\sum_{z_{kt}\in P_k^{n0}} I_{kt,n0}\leq I_{k0,n0}+2\sum_{t=1}^{2^{k-1}} I_{kt,n0}$$

$$\leq 2\cdot 2^k+2\sum_{t=1}^{2^{k-1}} 2^k/2t^2\leq\left(2+\sum_{t=1}^{\infty} 1/t^2\right)2^k.$$

Finally we consider the case k<n. Observe that the circular arc $z_{k0}z_{k1}$ corresponds to the circular arc $z_{n0}z_{n2^{n-k}}$ by the radial projection. Hence, by the rotational and reflectional invariance of ρ, we only have to evaluate $\nu_{k,ns}$ for the case $0\leq s<2^{n-k}$. Then

$$I_{kt,ns}\leq\frac{2\cdot 2^{-k}}{2^{-2k}(2^{k-n}-1)^2}\leq 8\cdot 2^k$$

for general t in $0\leq t<2^k$ and in particular for t=0 and 1. If

$2 \leq t \leq 2^{k-1}$, then, since $s \cdot 2^{k-n} \leq 1$, we see that

$$I_{kt,ns} \leq \frac{2 \cdot 2^{-k}}{4 \cdot 2^{-2k}(s \cdot 2^{k-n}-t)^2} \leq \frac{2^k}{2(t-1)^2}.$$

Hence again by the reflectional invariance of ρ we see that

$$\nu_{k,ns} \leq I_{k0,ns} + 2I_{k1,ns} + 2\sum_{t=2}^{2^{k-1}} I_{kt,ns}$$

$$\leq 8 \cdot 2^k + 2 \cdot 8 \cdot 2^k + 2\sum_{t=2}^{2^{k-1}} 2^k/2(t-1)^2$$

$$\leq \left(24 + \sum_{t=1}^{\infty} 1/t^2\right) 2^k.$$

Therefore, for the choice of $q = 24 + \sum_{t=1}^{\infty} 1/t^2$, we see that $\nu_{k,ns} \leq q \cdot 2^k$ for every $z_{ns} \in P$ and a fortiori (5.3) is established. □

6. Rigid and nonrigid two sheeted discs.

6.1. An admissible determining sequence $\{z_n\}$ in Δ, or its 2-disc, is said to be *rigid* if every admissible sequence $\{\Delta_n\}$ of discs $\Delta_n = \bar{\Delta}(z_n, r_n)$ in Δ is nonseparating, and $\{z_n\}$, or its 2-disc, is said to be *nonrigid* if it is not rigid. Thus $\{z_n\}$ is nonrigid if and only if there exists an admissible sequence $\{\Delta_n\}$ of discs $\Delta_n = \bar{\Delta}(z_n, r_n)$ in Δ which is separating. It is known (see [6]; cf. also 6.7-6.8 below) that there exists a nonrigid (rigid, resp.) sequence as close to each rigid (nonrigid, resp.) one as we wish in an appropriate sense.

Hence we may say that there are as many rigid sequences as nonrigid ones in a natural sense. It is therefore both important and interesting for us to be able to tell whether a concretely given admissible sequence is rigid or not. In this section a nonrigidness criterion is given and then it is applied to exhibit two concrete examples of nonrigid two sheeted discs. First we give a nonrigidness criterion in the following form:

THEOREM 6.1. *Suppose an admissible determining sequence* $\{\zeta_n\}_{n\in N}$ *in* Δ *is expressed as a disjoint union of an admissible determining sequence* $\{z_n\}_{n\in N}$ *in* Δ *and a discrete set* Z *in* Δ *such that either* Z *is empty or* $\sum_{\zeta\in Z}(1-|\zeta|)<\infty$. *The admissible sequence* $\{\zeta_n\}_{n\in N}=\{z_n\}_{n\in N}\cup Z$ *is nonrigid if there exists a sequence* $\{W_n\}$ *of pairwise disjoint discs* $W_n=\Delta(z_n,R_n)$ *in* $\Delta\setminus Z$ *such that* $\overline{W}_{2k-1}\cap\overline{W}_{2k}$ *consists of a single point for every positive integer* k.

The proof will be given in 6.2-6.4. In the actual application of this theorem later in 6.5 and 6.6 the set Z is taken to be empty. However, as a general criterion, the presence of Z widely enlarges the applicability of the criterion. In connection with this there is an open problem whether deleting (adding, resp.) a sequence $\{w_n\}$ in Δ with $\sum_{n\geq 1}(1-|w_n|)<\infty$ from (to, resp.) a nonrigid sequence gives rise to a new nonrigid sequence or not. When $\{w_n\}$ is a finite sequence we have seen in Theorem 2.2 that the answer is in the affirmative.

6.2. We denote by $(\Delta^{\sim},\Delta,\pi)$ the two sheeted disc whose determining set is $\{z_n\}_{n\in N}\cup Z$. We denote by b_k the common single point of $\overline{W}_{2k-1}$ and $\overline{W}_{2k}$: $\overline{W}_{2k-1}\cap\overline{W}_{2k}=\{b_k\}$ $(k\in N)$. Recalling the notation $r(z,S)=\inf_{\zeta\in S}r(z,\zeta)$ for any $z\in C$ and any subset S of C with $r(z,\zeta)=|z-\zeta|$, we consider the sequence $\{d_k\}_{k\in N}$ of positive numbers

$$d_k=\inf\{r(b_k,W_n):\ n\in N\setminus\{2k-1,2k\}\}\quad (k\in N)$$

and one more sequence $\{\varepsilon_k\}_{k\in N}$ of positive numbers

$$\varepsilon_k=\min\{d_k/(2^{k+1}+1),\ R_{2k-1},R_{2k}\}\quad (k\in N).$$

Choose the point a_{2k-1} on the line segment $[z_{2k-1},b_k]$ connecting two points z_{2k-1} and b_k and the point $a_{2k}\in[z_{2k},b_k]$ such that

$$|a_{2k-1}-b_k|=|a_{2k}-b_k|=\varepsilon_k\quad (k\in N).$$

Then consider the admissible sequence $\{\Delta_n\}_{n\in N}\cup\{\Delta_\zeta\}_{\zeta\in Z}$ of closed discs Δ_n and Δ_ζ given by

$$\begin{cases} \Delta_{2k-1}=\overline{\Delta}(z_{2k-1},R_{2k-1}-\varepsilon_k/2) & (k\in N),\\ \Delta_{2k}=\overline{\Delta}(z_{2k},R_{2k}-\varepsilon_k/2) & (k\in N),\\ \Delta_\zeta=\overline{\Delta}(\zeta,r(\zeta,(\bigcup_{n\in N}\Delta_n)\cup(Z\setminus\{\zeta\}))/3) & (\zeta\in Z).\end{cases}$$

The proof will be complete if we find a function $f\in H^\infty(D^{\sim})$ such that f separates the fiber $\pi^{-1}(z)$ for every $z\in D$ where

$D^{\sim}=\pi^{-1}(D)$ and

$$D=\Delta\setminus\Big((\bigcup_{n\in N}\Delta_n)\cup(\bigcup_{\zeta\in Z}\Delta_\zeta)\Big).$$

6.3. For the purpose first consider the infinite product

$$p(z)=\prod_{n\geq 1}(1-q_n(z))$$

where

$$q_n(z)=\frac{a_{2n-1}-a_{2n}}{z-a_{2n}}$$

so that

$$p(z)=\prod_{n\geq 1}\frac{z-a_{2n-1}}{z-a_{2n}}.$$

To see the convergence of $p(z)$ in Δ observe that

$$|q_n(z)|\leq 2\varepsilon_n/\delta_m(z)\leq 2^{-n+1}/\delta_m(z) \quad (n\geq m)$$

where $\delta_m(z)=r(z,\{a_{2k}: k\geq m\})$. Since $\delta_m(z)$ is continuous on Δ and strictly positive on $\Delta\setminus\{a_{2k}: k\geq 1\}$, we see that $\sum_{n\geq m}q_n(z)$ converges absolutely and uniformly on each compact subset of $\Delta\setminus\{z_{2k}: k\geq m\}$ for any fixed $m\in N$. Therefore $p(z)$ defines a meromorphic function on Δ such that $\{a_{2k-1}: k\in N\}$ ($\{a_{2k}: k\in N\}$, resp.) is the set of zeros (poles, resp.) of $p(z)$ on Δ.

Let B be the Blaschke product with the zero set Z (cf. e.g. [1], [3], [13]):

$$B(z)=\prod_{\zeta\in Z}\frac{-\bar{\zeta}}{|\zeta|}\cdot\frac{z-\zeta}{1-\bar{\zeta}z},$$

where we may assume $0\notin Z$. Finally we consider the function $F(z)=p(z)B(z)$ which is meromorphic on Δ with the zero set

$\{a_{2k-1}\}_{k\in N}\cup Z$ and the pole set $\{a'_{2k}\}_{k\in N}$. We denote by $(\Delta_*^\sim, \Delta, \pi_*)$ the 2-disc with the determining sequence $\{a_n\}_{n\in N}\cup Z$. Then the square root $\sqrt{F(z)}$ defines a single valued meromorphic function $\sqrt{F}$ on $\Delta_*^\sim$ and hence the restriction f of $\sqrt{F}$ on $\pi_*^{-1}(D)$ is a holomorphic function on $\pi_*^{-1}(D)$ without zeros. Since $D^\sim=\pi^{-1}(D)$ may be identified with $\pi_*^{-1}(D)$ as covering surfaces $(D^\sim, D, \pi)$ and $(\pi_*^{-1}(D), D, \pi_*)$ over D so that $D^\sim$ may be viewed as a subsurface of $\Delta_*^\sim$ (cf. e.g. [6]), we can consider that f is a holomorphic function on $D^\sim$ without zeros. Since

$$f(z^+)=\sqrt{F}(z^+)=-\sqrt{F}(z^-)=-f(z^-)\neq 0$$

for any $z\in D$ where $\pi^{-1}(z)=\{z^+,z^-\}$, we see that f separates the points in the fiber $\pi^{-1}(z)$ for every z in D.

6.4. The proof will be over if we show that $f\in H^\infty(D^\sim)$, or what amounts to the same that $\sqrt{F}$ is bounded on $D^\sim$ or equivalently F is bounded on D. Since $|B|\leq 1$ on Δ, we only have to show that p is bounded on

$$X=\Delta\setminus\bigcup_{k\geq 1}\Delta_{2k}$$

since $X\supset D$. Fix an arbitrary $a\in X$, and we will show that

$$p(a)\leq A=5\prod_{n\geq 1}(1+2^{-n})$$

which will complete the proof.

Fix an arbitrary $m\in N$ and set

$$Y_m = C^\wedge \setminus \bigcup_{1\leq n\leq m}'' \Delta_{2n}$$

where $C^\wedge$ is the extended complex plane. We also set

$$p_m(z) = \prod_{1\leq n\leq m} (1-q_n(z))$$

which is holomorphic on the closure $\overline{Y}_m$ of Y_m in $C^\wedge$.

Take an arbitrary z in $\partial\Delta_{2k}$ $(1\leq k\leq m)$. If $n=k$, then by the choice of ε_k

$$|q_n(z)| = |q_k(z)| \leq (2\varepsilon_k)/(\varepsilon_k/2) = 4$$

and therefore

$$|1-q_k(z)| \leq 5 \leq 5(1+2^{-k}) \quad (z\in\partial\Delta_{2k}).$$

If $n\neq k$ $(1\leq n\leq m)$, then

$$|q_n(z)| \leq 2\varepsilon_n/(d_n-\varepsilon_n) \leq 2(d_n/(2^{n+1}+1))/(d_n-d_n/(2^{n+1}+1))$$

$$=2^{-n} \quad (z\in\partial\Delta_{2k})$$

and a fortiori

$$|1-q_n(z)| \leq 1+2^{-n} \quad (z\in\partial\Delta_{2k},\ 1\leq n\leq m,\ n\neq k).$$

Thus we see that

$$|p_m(z)| = \prod_{1\leq n\leq m} |1-q_n(z)| = |1-q_k(z)|\cdot\Big(\prod_{1\leq n\leq m,\, n\neq k} |1-q_n(z)|\Big)$$

$$\leq 5(1+2^{-k})\cdot \prod_{1\leq n\leq m,\, n\neq k} (1+2^{-n}) = 5\prod_{1\leq n\leq m} (1+2^{-n}) \leq A$$

for any z in $\partial\Delta_{2k}$ $(1\leq k\leq m)$. Since $\partial Y_m = \bigcup_{1\leq k\leq m}\partial\Delta_{2k}$, we have

$$\max_{z\in\partial Y_m} |p_m(z)| \leq A.$$

By the maximum modulus principle, $|p_m(z)| \leq A$ for all $z\in Y_m$ and in particular $|p_m(a)| \leq A$. In view of the fact that $p_m(a)\to p(a)$

($m\to\infty$), we can conclude that $|p(a)'|\leq A$, completing the proof of Theorem 6.1. □

6.5. We next show that two 2-discs in Example 5.1 and 5.2 are nonrigid. The 2-disc in Example 5.1 has its determining sequence lying on the positive real line to which Theorem 6.1 is most conveniently applied:

EXAMPLE 6.1. *The sequence* $\{z_n\}$ *given by* $z_n=1-1/n$ $(n\geq 2)$ *is nonrigid. More generally any admissible increasing positive sequence* $\{z_n\}$ *satisfying*

$$z_{n+1}-z_n>z_{n+2}-z_{n+1} \quad (n\in N)$$

is nonrigid.

It is sufficient to prove the second part since $z_n=1-1/n$ satisfies the condition of the second part. Let

$$\begin{aligned} R_{2k}&=r(z_{2k},\Delta(z_{2k+1},z_{2k+2}-z_{2k+1})) \\ &=(z_{2k+1}-z_{2k})-(z_{2k+2}-z_{2k+1}) \quad (k\in N) \end{aligned}$$

which is positive by the condition on $\{z_n\}$ in the second part and then let

$$R_{2k-1}=(z_{2k}-z_{2k-1})-R_{2k} \quad (k\in N)$$

which is also positive by the above reason. Then set

$$\begin{cases} W_{2k-1}=\Delta(z_{2k-1},R_{2k-1}), \\ W_{2k}=\Delta(z_{2k},R_{2k}) \end{cases}$$

for every $k\in N$. It is easy to see that $\{W_n\}_{n\in N}$ satisfies the condition of Theorem 6.1 with $Z=\Phi'$ which assures that $\{z_n\}$ is nonrigid. □

6.6. Next we show that the 2-disc in example 5.2 is nonrigid. This comes from the fact that each radial distiribution of $\{z_n\}$ satisfies the condition in the second part of Example 6.1 and each circular distribution of $\{z_n\}$ is much more sparse than radial one.

EXAMPLE 6.2. *The sequence* $\{z_{nj}\}$ *given by the following is nonrigid*:

$$z_{nj}=(1-2^{-n})e^{2\pi j2^{-n}i} \quad (n\in N,\ 0\leq j<2^n).$$

For the proof let $W_{nj}=\Delta(z_{nj},2^{-n}/3)$ $(n\in N,\ 0\leq j<2^{-n})$. Then it is verified that $\{W_{nj}\}$ is a pairwise disjoint sequence of discs in Δ. Observe that

$$\{W_{nj}:\ n\in N,\ 0\leq j<2^n\}$$

$$=\{W_{m0}:\ m\in N\}\cup\Big(\bigcup_{n\in N}\Big(\bigcup_{1\leq k\leq 2^{n-1}}\{W_{n+m-1,(2k-1)2^{m-1}}:\ m\in N\}\Big)\Big)$$

is the radial partition of $\{W_{nj}\}$. According to this partition the totality of required pairings of $\{W_{nj}\}$ is given as follows:

$$\{\langle W_{2m-1,0},W_{2m,0}\rangle:\ m\in N\}\cup$$

$$\{\langle W_{n+2p-2,(2k-1)2^{2p-2}},W_{n+2p-1,(2k-1)2^{2p-1}}\rangle:\ n,p\in N,\ 1\leq k\leq 2^{n-1}\}.$$

Theorem 6.1 with $Z=\Phi$ then assures that $\{z_{n_j}\}$ is nonrigid. □

6.7. We do not have any practical criterion for the rigidity such as Theorem 6.1 for the nonrigidity but we have a device developed in [6] to produce a rigid sequence by adding each even number of new points suitably close to each point of any admissible sequence of points given in advance. The device was used in [6] to prove the existence of a rigid 2-disc but only implicitly. By virtue of Nonseparation criterion we can now give a concrete example of rigid two sheeted discs. It is constructed from Example 5.1. It is of course possible to produce a rigid one from Example 5.2 by exactly the same fashion as in the case of Example 5.1. Therefore we only state the former:

EXAMPLE 6.3. *The sequence* $\{z_n\}_{n\in N}$ *given by the following is rigid:*

$$\begin{cases} z_{3k-5}=1-1/k-\alpha_k/k^{\eta_k} \\ z_{3k-4}=1-1/k \qquad\qquad (k\geq 2) \\ z_{3k-3}=1-1/k+\beta_k/k^{\eta_k} \end{cases}$$

for an arbitrary inverse summable sequence $\{\eta_k\}$ *with* $\eta_k>2$ $(k\geq 2)$ *and arbitrary sequences* $\{\alpha_k\}$ *and* $\{\beta_k\}$ *in* $(0,1/2)$.

6.8. For the proof of the assertion in Example 6.3 let $(\tilde{\Delta}_*,\Delta,\pi_*)$ be the two sheeted disc with $\{z_{3k-4}\colon k\geq 2\}$ its determining sequence, $\Delta_{*k}=\overline{\Delta}(z_{3k-4}, 1/k^{n_k})$ $(k\geq 2)$ and

$$D_*=\Delta\setminus\bigcup_{k\geq 2}\Delta_{*k}.$$

By Example 5.1, we have $H^\infty(\pi_*^{-1}(D_*))=H^\infty(D_*)\circ\pi_*$.

Next we consider the two sheeted disc $(\Delta^\sim,\Delta,\pi)$ whose determining sequence is $\{z_n\}_{n\in N}$. Since $\{z_n\}_{n\in N}\subset\bigcup_{k\geq 2}\Delta_{*k}$, we may make the following identification:

$$\pi^{-1}(D_*)=\pi_*^{-1}(D_*).$$

We now show that $(\Delta^\sim,\Delta,\pi)$ is rigid. For the purpose take an arbitrary admissible sequence $\{\Delta_n\}$ of closed discs in Δ associated with $(\Delta^\sim,\Delta,\pi)$. We have to show that $\{\Delta_n\}$ is nonseparating. We see that

$$\Delta_{3k-5}\cup\Delta_{3k-4}\cup\Delta_{3k-3}\subset\Delta_{*k}\quad(k\geq 2)$$

no matter how we choose $\{\Delta_n\}$. This extremely simple fact is, however, the key point that makes $(\Delta^\sim,\Delta,\pi)$ rigid. Let

$$D=\Delta\setminus\bigcup_{n\in N}\Delta_n.$$

Since $D\supset D_*$, we have

$$H^\infty(D^\sim)=H^\infty(\pi^{-1}(D))\subset H^\infty(\pi^{-1}(D_*)).$$

Take an arbitrary f in $H^\infty(D^\sim)$ and set $g(z)=(f(z^+)-f(z^-))^2$ $(z\in D)$ where $\pi^{-1}(z)=\{z^+,z^-\}$. Since $f\in H^\infty(\pi^{-1}(D_*))=H^\infty(D_*)\circ\pi$, $g\equiv 0$ on D_*. By $D_*\subset D$, we then have $g\equiv 0$ on D. Hence $H^\infty(D^\sim)=H^\infty(D)\circ\pi$ and $\{\Delta_n\}$ is nonseparating. □

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