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How does a choice of Markov partition affect the resultant symbolic dynamics?

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The mutual relationship among Markov partitions is investigated for one-dimensional piecewise monotonic map. It is shown that if a Markov partition is regarded as a map-refinement of the other Markov partition, that is, a concept we newly introduce in this article, one can uniquely translate a set of symbolic sequences by one Markov partition to those by the other or vice versa. However, the set of symbolic sequences constructed using Markov partitions is not necessarily translated with each other if there exists no map-refinement relation among them. By using a roof map we demonstrate how the resultant symbolic sequences depend on the choice of Markov partitions.
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Given a scalar time series, what can one learn from it concerning the underlying dynamical system? The question of whether it is possible to symbolize the time series, resulting in a symbolic dynamics, hopefully unique in some sense, still remains as one of the most challenging subjects in contemporary sciences. We address this question by taking a one-dimensional roof map as an example and show that, even though we restrict ourselves to Markov partition, which is regarded as the most natural mean to symbolize the dynamical system, the ways of the symbolization in the Markov partition and, hence, its resultant symbolic dynamics are not unique. We also discuss in what condition two different Markov partitions generate the same symbolic dynamics.

I. INTRODUCTION

The question of how one can symbolize a given dynamical system without losing any information of complexity in dynamics is one of the most intriguing subjects in analyzing information processing in dynamical systems. Once the dynamical system can be symbolized properly, various techniques such as ε-machine[1] can be applied to reveal the process of information in the underlying dynamical system. Among several symbolization schemes, Markov partition provides one of the most natural means to symbolize the dynamical system. The concept of Markov partition is dated back to Sinai.[2] By constructing the Markov partition, one can symbolize the original dynamical system and construct its shift space, that is, a set of all possible symbolic sequences constructed from a given Markov partition. The shift space enables us to extract several important properties of the dynamical system such as Kolmogorov–Sinai entropy[3] and topological entropy;[4] Kolmogorov–Sinai entropy provides a lower bound of the sum of all the positive Lyapunov exponents[5] and positive topological entropy implies Li–Yorke chaos[6] (note that there exists a system which exhibits Li–Yorke chaos while it has zero topological entropy[7]).

However, the general properties of Markov partition have not been fully understood yet. For instance, Bowen[8] showed that the boundaries of Markov partition for Anosov automorphisms of three-dimensional tori cannot be smooth. In case of four- or higher-dimensional systems, Cawley[9] showed that the boundaries cannot be smooth for Anosov automorphisms on odd-dimensional tori, while for those on even-dimensional tori, the boundary can be smooth in some limited cases. The properties of shift space constructed from *non*-Markov partition are also highly nontrivial even for one-dimensional map.[10] Reference 10 demonstrated how the symbolic dynamics changes by choosing different positions of the partition (not Markovian) in the case of two symbols by using a tent map, and provided an algorithm to construct the corresponding sofic-shift. The question of whether dynamical systems that admit Markov partition exist generically is also one of the nontrivial subjects, except some families of one-dimensional maps to admit Markov partition exist densely in the system parameter space.[11–13]

The definition for Markov partition for Anosov systems of arbitrary dimension is presented in Ref. 8. Systems that admit only finite types of Markov partition must have zero topological entropy (as can be proved by using Theorem 2.2 in Ref. 14 and the definition of topological entropy[4]). Since most dynamical systems in interest are chaotic, they can have infinitely many Markov partitions. However, the mutual relationship among different Markov partitions has not been well-revealed.[15] In this article, for one-dimensional piecewise monotonic map, we investigate the properties of mutual relationship among Markov partitions. We show that if a Markov partition has a certain relationship we call "map-refinement of the other Markov partition," the shift spaces corresponding to these two Markov partitions are topologi-

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cally the same. If this relationship does not hold, the Markov partitions are not necessarily the case. By using a roof map as an illustrative, typical example of one-dimensional piecewise linear map, we demonstrate how the choice of Markov partitions affects the resultant shift space.

The outline of this paper is as follows. In Sec. II, we briefly introduce the concept of Markov partition in case of one-dimensional piecewise monotonic map. In Sec. III, we introduce a concept of "map-refinement" to classify Markov partitions. We prove that if a Markov partition is the map-refinement of another Markov partition, the two shift spaces are topologically the same. In Sec. IV we discuss the case that two Markov partitions do not have the map-refinement relation. Finally, we give the conclusion in Sec. V.

II. MARKOV PARTITION

We briefly introduce the definition of Markov partition in the case of one-dimensional piecewise monotonic map.[15] Let $I=[0,1]$ and a piecewise monotonic map τ be $\tau:[0,1]\to[0,1]$. Let $\mathcal{P}$ be a partition of I given by points $\{a_i\}$ such that $a_0<a_1<\cdots<a_i<\cdots<a_{n-1}<a_n$, where $a_0=0$ and $a_1=1$. Hereinafter, we denote such a point set as $\{a_0<a_1<\cdots<a_i<\cdots<a_{n-1}<a_n\}$ and $I_i=(a_{i-1},a_i)$ $(i=1,\ldots,n)$. Suppose that τ_i is the restriction of τ to I_i, that is, a map whose domain of the definition is restricted to I_i, satisfying $\forall x\in I_i, \tau_i(x)=\tau(x)$. If all τ_i are homeomorphism from I_i onto a connected union of intervals of $\mathcal{P}$, we call τ "Markov with respect to $\mathcal{P}$" and $\mathcal{P}$ "Markov partition."

The incidence matrix A is useful to characterize the topology of the shift space, which is defined by

$$A_{ij}=\begin{cases}1 & \text{if } I_j\subseteq\tau(I_i)\\ 0 & \text{otherwise}\end{cases}\qquad (i,j=1,2,\ldots,n).$$

In order to visualize the matrix, we introduce a directed graph $G_{\mathcal{P}}=(\mathcal{V},\mathcal{E})$ which has n vertices $\mathcal{V}=\{v_1,\ldots,v_n\}$, corresponding to the n intervals $\{I_1,\ldots,I_n\}$, and directed edges emanating from v_i to v_j when $A_{ij}=1$ [denoted by $(v_i\to v_j)\in\mathcal{E}$]. Here we regard a graph whose vertices and edges are denoted by $(\mathcal{V}',\mathcal{E}')$ as essentially equivalent to $(\mathcal{V},\mathcal{E})$ when there is a bijection $g:\mathcal{V}\to\mathcal{V}'$ and $(v_i\to v_j)\in\mathcal{E}$ if and only if $(g(v_i)\to g(v_j))\in\mathcal{E}'$ for all i and j $(i,j=1,\ldots,n)$.

Suppose that the inverse image of $\mathcal{P}$, $\tau^{-1}(\mathcal{P})$, is given by

$$\begin{aligned}\tau^{-1}(\mathcal{P})=\{&a_0<a_0^1<\cdots<a_0^{k_1-1}<a_1<a_1^1<\cdots\\ &<a_1^{k_2-1}<a_2<\cdots<a_{i-1}<a_{i-1}^1<\cdots<a_{i-1}^{k_i-1}\\ &<a_i<\cdots<a_{n-1}<a_{n-1}^1<\cdots<a_{n-1}^{k_n-1}<a_n\},\end{aligned}\tag{1}$$

where k_i-1 means the number of all elements of the set $I_i\cap\tau^{-1}(\mathcal{P})$. By using the notations of k_i and a_i^j $(j=0,\ldots,k_i)$ with $a_i^0=a_i(i=0,\ldots,n-1)$ and $a_{i-1}^{k_i}=a_i(i=1,\ldots,n)$, we can express the corresponding graph $G_{\mathcal{P}}$ more explicitly as follows: let $I_i^j=(a_{i-1}^{j-1},a_{i-1}^j)$ and then $\overline{I}_i=\overline{\cup_{j=1}^{k_i}I_i^j}$ holds, where $\overline{I}$ denotes the closure of the set I. Under these settings, for each I_i^j, there exists a positive integer ζ_i^j satisfying

$$\tau(I_i^j)=I_{\zeta_i^j},\tag{2}$$

where $1\le\zeta_i^j\le n$. One can prove this as follows: if $\tau(I_i^j)$ is a connected union of several intervals of I_i, there must be an element of $a\in\mathcal{P}$ such that $a\in\tau(I_i^j)$. Since $\mathcal{P}$ is a Markov partition, τ_i and its restriction to I_i^j should be homeomorphism, and, thus, there exists $b\in I_i^j$ such that $\tau(b)=a$. However, it contradicts the condition that $\tau^{-1}(\mathcal{P})$ does not contain any other elements besides the points defined by Eq. (1). Therefore, Eq. (2) holds and

$$\tau(I_i)\supseteq\tau(\bigcup_{j=1}^{k_i}I_i^j)=\bigcup_{j=1}^{k_i}I_{\zeta_i^j},\tag{3}$$

implying $\mathcal{E}=\{(v_i\to v_{\zeta_i^j})\,|\,1\le i\le n, 1\le j\le k_i\}$.

Given a Markov partition $\mathcal{P}$, we can symbolize a trajectory (a successive sequence of τ-mapping) $\cdots x_{-1}x_0x_1\cdots$, where $x_i\in I(i\in\mathbb{Z})$ and $x_{i+1}=\tau(x_i)(i\in\mathbb{Z})$ as $\cdots s_{-1}s_0s_1\cdots$, where $s_i\in\{1,\ldots,n\}(i\in\mathbb{Z})$. The set of all possible symbolic sequences can be characterized by using the set of forbidden blocks $\mathcal{F}$ defined by $\mathcal{F}=\{s_is_{i+1}|A_{s_i,s_{i+1}}=0\}$: all possible symbolic sequences constitute a set of all blocks excluding such forbidden blocks. Here, we regard this set as "the shift space constructed from $\mathcal{P}$" (see the rigorous definition of the shift space in Appendix A). Shift space can be classified in terms of conjugacy.[16] Suppose that we have two symbolic sequences. If and only if one can uniquely translate symbolic sequences in one shift space to those in the other shift space or vice versa, without referring the infinite past and future, we regard such two shift spaces as being "conjugate with each other." (In Appendix A, we define the concept more precisely.)

III. CLASSIFICATION OF MARKOV PARTITIONS OF ONE-DIMENSIONAL PIECEWISE MONOTONIC MAP

At first, in order to classify the Markov partitions of a piecewise monotonic map, we prove that if $\mathcal{P}$ is a Markov partition that consists of finite elements, $\mathcal{P}$ has at least one periodic orbit. We prove this statement using "proof by contradiction." Suppose that a Markov partition $\mathcal{P}$ that consists of finite elements has no any periodic orbit. Since $\mathcal{P}$ is a Markov partition, $\tau(\mathcal{P})\subseteq\mathcal{P}$ because the boundary of Markov partition maps into the boundary. If $a\in\mathcal{P}$, $\tau^n(a)\in\mathcal{P}$ for all $n\in\mathbb{N}$. Since we assume that $\mathcal{P}$ does not have periodic orbit, $\tau^n(a)\ne\tau^m(a)$ for all $n\ne m\in\mathbb{N}$ but it contradicts to the condition that $\mathcal{P}$ consists of finite elements.

The above fact allows us to divide a Markov partition of a certain finite number of elements $\mathcal{P}$ into two parts, one part that consists of periodic orbits and the other part that consists of all the rest.[17] We denote the subset of $\mathcal{P}$ that consists of periodic orbits as $\mathcal{Q}$ and the other as $\mathcal{R}$. By adding new element(s) to a given partition $\mathcal{P}(=\mathcal{Q}\cup\mathcal{R})$, in most cases (see Appendix B), it is possible to create a new Markov partition $\mathcal{P}'$, i.e., $\mathcal{P}\nsubseteq\mathcal{P}'$, which consists of periodic orbits and the rest, denoted $\mathcal{Q}'$ and $\mathcal{R}'$, respectively (i.e., $\mathcal{P}'=\mathcal{Q}'\cup\mathcal{R}'$, $\mathcal{Q}\subseteq\mathcal{Q}'$, $\mathcal{R}\subseteq\mathcal{R}'$). One can see that $\mathcal{P}'$ is also a Markov partition as far as $\mathcal{P}$ is Markov and $\tau(\mathcal{P}')\subseteq\mathcal{P}'$ is satisfied (see Appendix B). In this section, we focus on the case of adding new element(s) to $\mathcal{R}$ [in Sec. IV, we discuss the case of adding element(s) to $\mathcal{Q}$].

In the following, first, we introduce a concept of map-refinement and describe the concept of outsplitting graph by following Ref. 16. Next, by using these two concepts, we prove that if the partition $\hat{\mathcal{P}}$ is the map(τ)-refinement of $\mathcal{P}$, the shift space constructed by using these two different partitions $\mathcal{P}$ and $\hat{\mathcal{P}}$ is conjugate with each other, irrespective of how many elements are added to $\mathcal{R}$.

A. map(τ)-refinement

Suppose that there exist two Markov partitions $\mathcal{P}$ and $\hat{\mathcal{P}}$. We call $\hat{\mathcal{P}}$ as τ refinement of $\mathcal{P}$ if $\mathcal{P}\nsubseteq\hat{\mathcal{P}}$ and $\tau(\hat{\mathcal{P}})=\mathcal{P}$.

B. Outsplitting graph

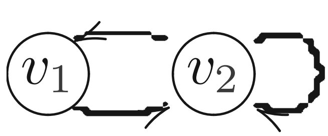
Let G be a graph with a vertex set $\mathcal{V}=\{v_1,\ldots,v_n\}$ and an edge set $\mathcal{E}$, which is a subset of $\{(v_i\to v_j)\,|\,1\le i,j\le n\}$, i.e., $G=(\mathcal{V},\mathcal{E})$. We denote a set of edges that emanates from v_i by $\mathcal{E}_i$. Suppose that we have a mutually exclusive decomposition of $\mathcal{E}_i$ defined by $\mathcal{E}_i=\cup_{j=1}^{k_i}\mathcal{E}_i^j$ (k_i is the total number of the decomposition of $\mathcal{E}_i$), where $\mathcal{E}_i^j$ are a subset of $\mathcal{E}_i$ satisfying $\mathcal{E}_i^j\cap\mathcal{E}_i^{j'}=\emptyset$ $(j\neq j',j,j'=1,2,\ldots,k_i)$.

Associated with each decomposition of $\mathcal{E}_i$, let each vertex v_i divide into $\{v_i^1,v_i^2,\ldots,v_i^{k_i}\}$. One can then construct a graph whose vertices $\mathcal{V}'$ are defined by $\mathcal{V}'=\{v_1^1,v_1^2,\ldots,v_1^{k_1}\}\cup\{v_2^1,\ldots,v_2^{k_2}\}\cup\cdots\cup\{v_n^1,\ldots,v_n^{k_n}\}$ and edges $\mathcal{E}'$ are by

$$\mathcal{E}'=\{(v_i^j\to v_{i'}^{j'})|(v_i\to v_{i'})\in\mathcal{E}_i^j,1\le i,i'\le n,1\le j\le k_i,1\le j'\le k_{i'}\}. \tag{4}$$

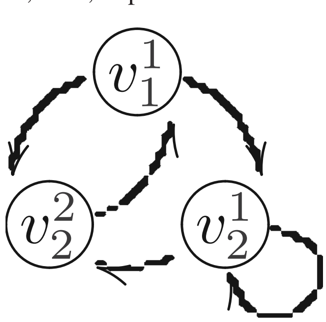
All possible graphs that are essentially the same as this graph are called "outsplitting graph" of G with respect to the decomposition $\mathcal{E}_i=\cup_{j=1}^{k_i}\mathcal{E}_i^j$, which is here denoted by G'.

For example, let $G=(\mathcal{V},\mathcal{E})$ be a graph defined by $\mathcal{V}=\{v_1,v_2\}$, $\mathcal{E}=\mathcal{E}_1\cup\mathcal{E}_2$, $\mathcal{E}_1=\{(v_1\to v_2)\}$, and $\mathcal{E}_2=\{(v_2\to v_1),(v_2\to v_2)\}$ depicted as



□

One of the outsplitting graphs with respect to a decomposition $\mathcal{E}_1^1=\mathcal{E}_1$, $\mathcal{E}_2=\mathcal{E}_2^1\cup\mathcal{E}_2^2$ where $\mathcal{E}_2^1=\{(v_2\to v_2)\}$ and $\mathcal{E}_2^2=\{(v_2\to v_1)\}$ is, then, depicted as



□

Note that, for example, a graph obtained by renaming such as $(v_1^1,v_2^1,v_2^2)\to(a,b,c)$ is also G' with respect to the decomposition of $\mathcal{E}_i$.

In terms of these two concepts, we prove that the shift space constructed by using $\mathcal{P}$ and $\hat{\mathcal{P}}$ is conjugate with each other if the partition $\hat{\mathcal{P}}$ is the τ-refinement of $\mathcal{P}$. The outline of the proof is as follows. First, we construct the graph $G_{\hat{\mathcal{P}}}$, and then we show that one can choose a decomposition of the edges $\mathcal{E}$ of $G_{\mathcal{P}}$ so that the outsplitting graph of $G_{\mathcal{P}}$ with respect to the decomposition is essentially the same as $G_{\hat{\mathcal{P}}}$. This means the two shift spaces constructed from $\mathcal{P}$ and $\hat{\mathcal{P}}$ are conjugate with each other, since the shift space corresponding to the outsplitting graph $G'_{\mathcal{P}}$ is conjugate to that corresponding to $G_{\mathcal{P}}$, as shown in Ref. 16. The diagram of their mutual relationship is shown below:

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "\tau\text{-refinement}"] \arrow[d, "\text{graph}"'] & & \hat{\mathcal{P}} \arrow[d, "\text{graph}"] \\
G_{\mathcal{P}} \arrow[dr, "\text{out-splitting}"'] & & G_{\hat{\mathcal{P}}} \arrow[dl, equal, "\text{the main claim of the theorem}"] \\
& G'_{\mathcal{P}} &
\end{tikzcd}$$

□

First, we construct $G_{\hat{\mathcal{P}}}$ as follows. Since $\hat{\mathcal{P}}$ is τ-refinement of $\mathcal{P}$, $\mathcal{P}\nsubseteq\hat{\mathcal{P}}$ and $\tau(\hat{\mathcal{P}})=\mathcal{P}$. Suppose that the elements of $\hat{\mathcal{P}}$ partition I_i into p_i intervals: $\hat{\mathcal{P}}\cap I_i=\{a_{i-1}^{r_i^1}<a_{i-1}^{r_i^2}<\cdots<a_{i-1}^{r_i^{p_i-1}}\}\subseteq\tau^{-1}(\mathcal{P})$, where r_i^j is a positive integer satisfying $1\le r_i^1<r_i^2<\cdots<r_i^{p_i-1}\le k_i-1$. For the sake of convenience, we set $r_i^0=0$ and $r_i^{p_i}=k_i$. Let $\hat{I}_i^j=(a_{i-1}^{r_i^{j-1}},a_{i-1}^{r_i^j})$ for $1\le i\le n$ and $1\le j\le p_i$.

Since $\hat{I}_i^j\supseteq\cup_{l=r_i^{j-1}+1}^{r_i^j}(a_{i-1}^{l-1},a_{i-1}^l)=\cup_{l=r_i^{j-1}+1}^{r_i^j}I_i^l$, we get

$$\tau(\hat{I}_i^j)\supseteq\tau\Big(\bigcup_{l=r_i^{j-1}+1}^{r_i^j}I_i^l\Big)=\bigcup_{l=r_i^{j-1}+1}^{r_i^j}I_{\xi_i^l}\supseteq\bigcup_{l=r_i^{j-1}+1}^{r_i^j}\bigcup_{m=1}^{p_{\xi_i^l}}\hat{I}_{\xi_i^l}^m, \tag{5}$$

where the second equality comes from Eq. (2) and the third $\supseteq$ is from $I_{\xi_i^l}\supseteq\cup_{m=1}^{p_{\xi_i^l}}\hat{I}_{\xi_i^l}^m$. This means that the vertex $\hat{v}_i^j$ corresponding to $\hat{I}_i^j$ has outgoing edges bound for the vertices $\hat{v}_{\xi_i^l}^m$ corresponding to $\hat{I}_{\xi_i^l}^m$. Here we denote the set of the vertices and the edges as $G_{\hat{\mathcal{P}}}=(\hat{\mathcal{V}},\hat{\mathcal{E}})$.

Then, we show below that $G_{\hat{\mathcal{P}}}$ is an outsplitting graph of $G_{\mathcal{P}}$ when one chooses the decomposition $\mathcal{E}_i$ as $\mathcal{E}_i=\cup_{j=1}^{p_i}\mathcal{E}_i^j$, where $\mathcal{E}_i^j=\{(v_i\to v_{\xi_i^l})\,|\,r_i^{j-1}+1\le l\le r_i^j\}$.

According to each decomposition $\mathcal{E}_i$, each vertex v_i is divided into $\{v_i^1,v_i^2,\ldots,v_i^{p_i}\}$. We denote the outsplitting graph $G'_{\mathcal{P}}$ as $G'_{\mathcal{P}}=(\mathcal{V}',\mathcal{E}')$ where $\mathcal{V}'=\{v_1^1,v_1^2,\ldots,v_1^{p_1}\}\cup\{v_2^1,\ldots,v_2^{p_2}\}\cup\cdots\cup\{v_n^1,\ldots,v_n^{p_n}\}$, and

$$\mathcal{E}'=\{(v_i^j\to v_{i'}^{j'})|(v_i\to v_{i'})\in\mathcal{E}_i^j,1\le i,i'\le n,1\le j\le p_i,1\le j'\le p_{i'}\}. \tag{6}$$

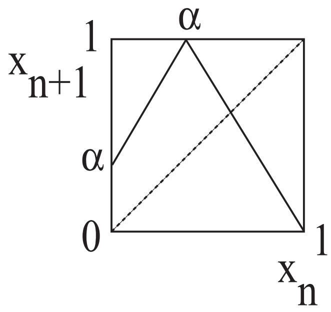


FIG. 1. A roof map.

Let us suppose $(v_i^j \to v_{i'}^{j'}) \in \mathcal{E}'$ and ask to what they correspond. The condition $(v_i^j \to v_{i'}^{j'}) \in \mathcal{E}'$ is equivalent to $(v_i \to v_{i'}) \in \mathcal{E}_i^j$ and it holds if and only if $i' \in \{\zeta_i^l \mid r_i^{j-1}+1 \leq l \leq r_i^j\}$ by the definition of $\mathcal{E}_i^j$. Then,

$$\hat{I}^{j'}_{i'} \subseteq \bigcup_{l=r_i^{j-1}+1}^{r_i^j} \hat{I}^{j'}_{\zeta_i^l} \subseteq \bigcup_{l=r_i^{j-1}+1}^{r_i^j} \bigcup_{m=1}^{p_{\zeta_i^l}} \hat{I}^{m}_{\zeta_i^l} \subseteq \tau(\hat{I}_i^j), \tag{7}$$

where the second $\subseteq$ is due to the fact that $1 \leq j' \leq p_{\zeta_i^l}$, and the last equality comes from Eq. (5). Equation (7) tells us that $(v_i^j \to v_{i'}^{j'}) \in \mathcal{E}'$ corresponds to $(\hat{v}_i^j \to \hat{v}_{i'}^{j'}) \in \hat{\mathcal{E}}$ for $1 \leq i, i' \leq n$ and $1 \leq j \leq p_i, 1 \leq j' \leq p_{i'}$. [One can easily prove that $(\hat{v}_i^j \to \hat{v}_{i'}^{j'}) \in \hat{\mathcal{E}}$ also corresponds to $(v_i^j \to v_{i'}^{j'}) \in \mathcal{E}'$ since all steps of this proof are equivalent transformation.] Therefore, we can conclude that $G_{\hat{\mathcal{P}}}$ has the same structure as an out-splitting graph with respect to this chosen decomposition (a simple illustration of this proof is given for a roof map in Appendix C).

IV. DEPENDENCIES OF GRAPH TOPOLOGY ON MARKOV PARTITIONS

In this section, using a roof map as an illustrative example, we discuss about how the resultant shift space changes when we add new periodic orbits to a Markov partition $\mathcal{P}$. That is, how a shift space constructed from $\mathcal{P}' = \mathcal{Q}' \cup \mathcal{R}$ is different from a shift space constructed from $\mathcal{P} = \mathcal{Q} \cup \mathcal{R}$ where $\mathcal{Q} \nsubseteq \mathcal{Q}'$. In this case, $\mathcal{P}'$ cannot be a τ-refinement of $\mathcal{P}$ and we show that the shift space constructed from $\mathcal{P}'$ is not necessarily conjugate to $\mathcal{P}$. This means that, depending on how we symbolize the dynamical system, the structure of the resultant shift space such as number of periodic orbits can be different.

Here we consider a roof map,[18] a one-dimensional piecewise linear map $F:[0,1] \to [0,1]$ defined by

$$x_{n+1} = F(x_n) = \begin{cases} \dfrac{1-\alpha}{\alpha} x_n + \alpha & \text{if } 0 \leq x < \alpha \\ \dfrac{1-x_n}{1-\alpha} & \text{otherwise,} \end{cases}$$

where $0 < \alpha < \frac{1}{2}$, showing that all the periodic orbits are unstable. A schematic picture of F is shown in Fig. 1. The topological entropy h_F of this system and the number of the root $\{x \mid F^n(x) = x\}$ (corresponding to the number of the periodic orbits) $\wp_n$ are given, respectively, by $h_F = \log \mu$ and $\wp_n = \mu^n + \lambda^n$. Here $\mu = 1 + \sqrt{5}/2$ and $\lambda = 1 - \sqrt{5}/2$ (see Appendix D for more details).

One can see that the Markov partition composed of the minimum number of elements is given by $\mathcal{P}_3 = \{0, \alpha, 1\}$, where the subscript of $\mathcal{P}$ means the period of periodic orbits used in the Markov partition. It is because, if we do not choose α as a boundary, the restriction of F to an interval that includes α cannot be injective, contradicting the definition of Markov partition. The boundary consists of a period-3 periodic orbit since $F(\alpha) = 1$ and $F(1) = 0$, $F(0) = \alpha$. The corresponding incidence matrix of this partition is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and the number of period-n periodic symbolic sequences $\mu^n + \lambda^n$ coincides with $\wp_n$.

Then, *how about another Markov partition to symbolize the piecewise linear map?* As we showed in Sec. III, adding new element(s) in the nonperiodic part does not change the corresponding shift space topologically.

Let us consider another Markov partition having the second least number of elements composed of periodic orbits defined by $\mathcal{P}_{1+3} = \{0, \alpha, 1/(2-\alpha), 1\}$, where the subscript $1+3$ means that the boundary of the Markov partition consists of a period-1 periodic orbit and a period-3 periodic orbit. One can easily see that this is not an F-refinement of $\mathcal{P}_3$ and consists of a period-1 fixed point and a period-3 periodic orbit. In this case, the corresponding incidence matrix is

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

with eigenvalues μ, λ, and -1. The graph $G_{\mathcal{P}_{1+3}}$ is not topologically equivalent to $G_{\mathcal{P}_3}$ since each corresponding incidence matrix has different nonzero eigenvalues.[16] In fact, the number of the periodic orbits of the shift space constructed from the partition $\mathcal{P}_{1+3}$ is given by $\mu^n + \lambda^n + (-1)^n$, which is different from the actual total number of periodic orbits $\wp_n$ the original system F has. For example, F has one fixed point but the shift space of $\mathcal{P}_{1+3}$ does not have a period-1 periodic orbit.

Note that although the number of periodic orbits depends on the way of Markov partitioning, all the Markov partitions listed in Table I have the same topological entropy as that of the original map F, as is expected from Theorem 2.2 in Ref. 14.

V. SUMMARY

In this article, we revisited Markov partitions for the case of one-dimensional piecewise monotonic map. Even in this simplest case, the map can have several Markov partitions whose shift spaces are not conjugate with each other. In order to clarify the condition that two Markov partitions have conjugate shift spaces, by introducing a new concept, map-refinement, we proved that if one Markov partition is the map-refinement of the other, the two corresponding shift spaces are conjugate with each other. We investigated, by using a roof map as an example, what happens when two Markov partitions do not have this relation. The results show that in general, the shift spaces corresponding to two Markov

TABLE I. Some incidence matrices and the number of the periodic orbits of the corresponding shift space of several Markov partitions. Note that for $m=5$, there exist two period-5 periodic orbits, which are denoted by A and B, respectively.

Partition	Incidence matrix	No. of period-n orbits
$\mathcal{P}_3$	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	$\lambda^n+\mu^n$
$\mathcal{P}_{1+3}$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\lambda^n+\mu^n+(-1)^n$
$\mathcal{P}_{2+3}$	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$	$\lambda^n+\mu^n+i^n+(-i)^n$
$\mathcal{P}_{4+3}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$	$\lambda^n+\mu^n$ $+e^{n\pi i/4}+e^{3n\pi i/4}$ $+e^{5n\pi i/4}+e^{7n\pi i/4}$
$\mathcal{P}^A_{5+3}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$	$\lambda^n+\mu^n+(-1)^n$ $+e^{n\pi i/5}+e^{3n\pi i/5}$ $+e^{7n\pi i/5}+e^{9n\pi i/5}$
$\mathcal{P}^B_{5+3}$	$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\lambda^n+\mu^n+1$ $+e^{2n\pi i/5}+e^{4n\pi i/5}$ $+e^{6n\pi i/5}+e^{8n\pi i/5}$

partitions are not necessarily conjugate with each other, resulting in various different shift spaces that cannot be translated with each other.

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APPENDIX A: SHIFT SPACE AND CONJUGACY

In this section, we introduce the concept of shift space, following Ref. 16. Let $\mathcal{A}$ be a set of some symbols.

Definition 1: (full $\mathcal{A}$-shift) *A full $\mathcal{A}$-shift is the collection of all bi-infinite sequences of the symbols $\mathcal{A}$.*

We denote the full $\mathcal{A}$-shift by $\mathcal{A}^{\mathbb{Z}}$.

Definition 2: (shift map) *The shift map $\sigma:\mathcal{A}^{\mathbb{Z}}\to\mathcal{A}^{\mathbb{Z}}$ is a map that maps a point $\cdots s_{-1}s_0s_1s_2s_3\cdots$ to $\cdots s_0s_1s_2s_3s_4\cdots$, where $s_i\in\mathcal{A}(i\in\mathbb{Z})$.*

Definition 3: (shift space) *The shift space is a subset X of full $\mathcal{A}$-shift in which sequences of X do not contain any blocks of forbidden blocks over $\mathcal{A}$.*

Definition 4: (sliding block code) *Suppose that there are two sets of symbols $\mathcal{A},\hat{\mathcal{A}}$ and shift spaces over them, that is, $X\subseteq\mathcal{A}^{\mathbb{Z}}$ and $\hat{X}\subseteq\hat{\mathcal{A}}^{\mathbb{Z}}$. A sliding block code from X to $\hat{X}$,*

$$\phi:s=\cdots s_{-1}s_0s_1\cdots\mapsto\phi(s)=\cdots\phi(s)_{-1}\phi(s)_0\phi(s)_1\cdots \tag{A1}$$

[where s_k and $\phi(s)_k$ denote the kth symbol of the symbolic sequence s and $\phi(s)$, respectively], is a map that satisfies $\phi\circ\sigma_X=\sigma_{\hat{X}}\circ\phi$ and there is a certain positive integer N such that $\phi(s)_i$ depends only on $s_{-N+i}s_{-N+i+1}\cdots s_i\cdots s_N+i$, where $\sigma_X,\sigma_{\hat{X}}$ are the shift operators of each shift space.

Definition 5: (conjugacy) *Suppose that there are two shift spaces X,Y. A sliding block code $\phi:X\to Y$ is called conjugacy if it is bijective. Two shift spaces X and Y are called conjugate if there is conjugacy from X to Y.*

APPENDIX B: EXTENDABILITY OF MARKOV PARTITIONS AND THEIR PROPERTY

In this section, we show that if a partition $\mathcal{P}$ is a Markov partition, any partition $\mathcal{P}'$ constructed by adding any element(s) to $\mathcal{P}$ so as to satisfy $\tau(\mathcal{P}')\subseteq\mathcal{P}'$ is also a Markov partition. We then show in which case there exists at least one $\mathcal{P}'$ such that $\mathcal{P}\nsubseteq\mathcal{P}'$.

Let I' be an interval with respect to the partition $\mathcal{P}'$. Note that τ restricted to I' is homeomorphism from I' onto $\tau(I')$: since there exists $i\in\{1,\ldots,n\}$ such that $I'\subseteq I_i$ and the restriction of τ to I_i is homeomorphism $\tau(I_i)$, the restriction of τ to I' should be also homeomorphism onto $\tau(I')$. In addition, the restriction of τ to I' should be monotonic, since, otherwise, it cannot be homeomorphism. Therefore, τ maps the two boundary points of the interval I' to those of $\tau(I')$, which are also elements of $\mathcal{P}'$ because of the fact that $\tau(\mathcal{P}')\subseteq\mathcal{P}'$. This means that $\tau(I')$ is a connected union of the intervals of $\mathcal{P}'$ between the two boundary points of the interval $\mathcal{P}'$.

In the following, given a Markov partition $\mathcal{P}$, we would like to point out in which cases there is a Markov partition $\mathcal{P}'$ such that $\mathcal{P}\nsubseteq\mathcal{P}'$ and $\tau(\mathcal{P}')\subseteq\mathcal{P}'$. The Markov partition of a certain finite number of elements $\mathcal{P}$ can be divided into two parts, i.e., $\mathcal{Q}$ composed of periodic orbits and $\mathcal{R}$ composed of all the rest.

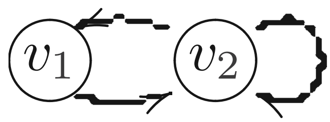
By adding element(s) of periodic orbit(s) to the $\mathcal{Q}$ part, one can construct $\mathcal{P}'$ if there is (are) a periodic orbit(s) that is (are) not taken as the element(s) of $\mathcal{Q}$. In turn, if the set $\tau^{-1}(\mathcal{P})\setminus\mathcal{P}$ is not empty, one can add any element(s) of the set to the $\mathcal{R}$ part. The resultant partition $\mathcal{P}'$ forms a Markov partition for both cases.

APPENDIX C: AN ILLUSTRATION OF THE PROOF IN SECTION III

In this section, we demonstrate the proof of Sec. III using a roof map $F:[0,1]\to[0,1]$ (Fig. 1). We choose $\mathcal{P}=\{0$

$=a_0, \alpha=a_1, 1=a_2\}$ as a Markov partition and $\hat{\mathcal{P}}=\{0,\alpha,\beta=1-\alpha+\alpha^2,1\}$ as an F-refinement of it. It is straightforward to check that $\hat{\mathcal{P}}$ is an F-refinement of $\mathcal{P}$ since $F(\beta)=\alpha$, and thus $F(\hat{\mathcal{P}})\subseteq\mathcal{P}$.

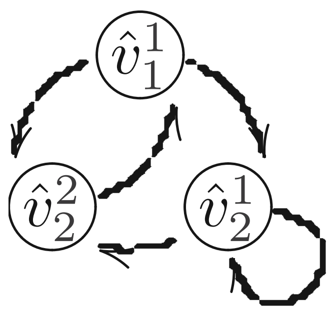
First, note that $G_{\mathcal{P}}$ has the same structure as a graph of $\mathcal{V}=\{v_1,v_2\}$, $\mathcal{E}=\mathcal{E}_1\cup\mathcal{E}_2$ where $\mathcal{E}_1=\{(v_1\to v_2)\}$ and $\mathcal{E}_2=\{(v_2\to v_1),(v_2\to v_2)\}$,



□

It is because $\tau(I_1)=I_2, \tau(I_2)\supseteq I_1\cup I_2$, where $I_1=(0,\alpha)$, $I_2=(\alpha,1)$.

Next, we construct $G_{\hat{\mathcal{P}}}$ as follows: let us rewrite $\mathcal{P}$ and $\hat{\mathcal{P}}$ by $\mathcal{P}=\{0,\alpha,1\}=\{a_0,a_1,a_2\}$ and, according to the notation of Eq. (1), $\hat{\mathcal{P}}=\tau^{-1}(\mathcal{P})=\{0,\alpha,\beta,1\}=\{a_0^0,a_0^1=a_1^0,a_1^1,a_1^2\}$. Compared to the notations of Sec. III, this corresponds to $k_1=1$, $k_2=2$, $p_1=1$, $p_2=2$, $r_1^0=0$, $r_1^1=1$, and $r_2^0=0, r_2^1=1, r_2^2=2$, i.e., $\hat{\mathcal{P}}=\{a_0^{r_1^0},a_0^{r_1^1}=a_1^{r_2^0},a_1^{r_2^1},a_1^{r_2^2}\}$. Each interval of the partition $\hat{\mathcal{P}}$ can be written as $\hat{I}_1^1=(0,\alpha)=(a_0^{r_1^0},a_0^{r_1^1})$, $\hat{I}_2^1=(\alpha,\beta)=(a_1^{r_2^0},a_1^{r_2^1})$, and $\hat{I}_2^2=(\beta,1)=(a_1^{r_2^1},a_1^{r_2^2})$. As a result, $G_{\hat{\mathcal{P}}}$ is depicted as



□

because of the fact that $\tau(\hat{I}_1^1)=I_2=\hat{I}_2^1\cup\hat{I}_2^2$, $\tau(\hat{I}_2^1)=I_2=\hat{I}_2^1\cup\hat{I}_2^2$, and $\tau(\hat{I}_2^2)=I_1=\hat{I}_1^1$.

Then, we construct the outsplitting graph $G'_{\mathcal{P}}$ of $G_{\mathcal{P}}$ by following the recipes in Sec III. According to the recipes, the decomposition of the edges $\mathcal{E}_i=\cup_{j=1}^{p_i}\mathcal{E}_i^j$ is chosen as $\mathcal{E}_i^j=\{(v_i\to v_{\zeta_i^l})\,|\,r_i^{j-1}+1\le l\le r_i^j\}$. In this case, $\mathcal{E}_1^1=\{(v_1\to v_{\zeta_1^l})\,|\,r_1^0+1\le l\le r_1^1\}=\{(v_1\to v_2)\}$, $\mathcal{E}_2^1=\{(v_2\to v_2)\}$, and $\mathcal{E}_2^2=\{(v_2\to v_1)\}$, where $\zeta_1^1=2, \zeta_2^1=2, \zeta_2^2=1$. Therefore, the set of all vertices of the outsplitting graph is $\mathcal{V}'=\{v_1^1\}\cup\{v_2^1,v_2^2\}$ and that of the edges is

$$\mathcal{E}'=\{(v_1^1\to v_2^1),(v_1^1\to v_2^2),(v_2^1\to v_2^1),\\ (v_2^1\to v_2^2),(v_2^2\to v_1^1)\}. \tag{C1}$$

One can see that the structure of the outsplitting graph $G_{\mathcal{P}}$ is manifestly the same as $G_{\hat{\mathcal{P}}}$ by renaming v_i^j as $\hat{v}_i^j$.

APPENDIX D: THE NUMBER OF PERIODIC ORBITS OF THE ROOF MAP AND ITS TOPOLOGICAL ENTROPY

We present here the analytical expression of the number of the root of the equation $x=F^n(x)$ and the topological entropy of the roof map presented in Sec. IV. The graph $y=F^n(x)$ of n-times mapping is also piecewise linear as $F(x)$, which consists of two types of lines: one is a line starting from $y=0$ to $y=1$ and the other from $y=\alpha$ to $y=1$. Here we call the former type of line as "long line" and the latter as "short line." For example, the graph of F has one short line from $(x,y)=(0,\alpha)$ to $(\alpha,1)$ and one long line from $(\alpha,1)$ to $(1,0)$. Since F has a "bending point" at $x=\alpha$ and the bending point is mapped to a bending point of the graph $y=F^2(x)$ by F, and by repeating this argument the graph $y=F^n(x)$ has also a bending point at that point. Due to this fact, every short line and long line that appeared in F^n exist either on the domain $0\le x<\alpha$ or $\alpha\le x\le 1$. One can easily confirm that every short line and every long line appeared in F^n are mapped to one long line, and one short and one long lines, respectively. For example, at F^2, one short line at F^1 is mapped to a long line from (0,1) to $(\alpha,0)$, and one long line is mapped to a short line from $(1/(2-\alpha),1)$ to $(1,\alpha)$ and a long line from $(\alpha,0)$ to $(1/(2-\alpha),1)$, respectively.

Since such long lines run through the whole range of the value of F whose absolute value of gradient is always greater than unity when $0<\alpha<\frac{1}{2}$, every long line that appeared in F^n must have one intersecting point with $y=x$, corresponding to the root of $x=F^n(x)$. However, as for the short lines, every short line does not necessarily have an intersecting point with $y=x$ because short lines run through only the range $\alpha\le y\le 1$. Thus, when they are on the domain $0\le x<\alpha$, it cannot have an intersecting point with $y=x$. Note also that $y=F^n(x)$ and $y=x$ cannot be tangential with each other because the absolute value of the gradients of all linear pieces of F^n is always greater than unity for $0<\alpha<\frac{1}{2}$. The above facts allow us to write down the number of the root $x=F^n(x)$, $\wp_n$, as $\wp_n=d_n+e_n+f_n$, where d_n is the number of the long lines located on the domain $0\le x<\alpha$, e_n is that of the short lines on $\alpha\le x\le 1$, and f_n is that of the long lines on $\alpha\le x\le 1$.

Next, we derive the recurrence formula for d_n, e_n, f_n, and c_n, which is the number of the short lines of F^n located on the domain $0\le x<\alpha$. Due to the structure of the map F, every short line is mapped to one long line and every long line is mapped to one short and one long lines. Thus, the recurrence formulas are

$$c_{n+1}=d_n, \tag{D1}$$

$$d_{n+1}=c_n+d_n, \tag{D2}$$

$$e_{n+1}=f_n, \tag{D3}$$

$$f_{n+1}=e_n+f_n, \tag{D4}$$

with the boundary condition $c_1=1, d_1=0, e_1=0, f_1=1$. The solution of the recurrence formula with the boundary condition is

$$c_n=\frac{1}{\sqrt{5}}(\lambda^n-\mu^n), \tag{D5}$$

$$d_n=\frac{1}{\sqrt{5}}(\lambda^n-\mu^n), \tag{D6}$$

$$e_n = \frac{1}{\sqrt{5}}(\lambda^n - \mu^n), \tag{D7}$$

$$f_n = \frac{1}{\sqrt{5}}(\lambda^{n-1} - \mu^{n-1}), \tag{D8}$$

and finally we can get the analytical expression of the number of period-n periodic orbits in the graph F^n by

$$\wp_n = d_n + e_n + f_n = \mu^n + \lambda^n. \tag{D9}$$

Here, μ and λ correspond to the eigenvalues of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ which are $\mu = 1+\sqrt{5}/2$ and $\lambda = 1-\sqrt{5}/2$, respectively.

The topological entropy h_F can be calculated using the formula in Ref. 19,

$$h_F = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{lap}(F^n), \tag{D10}$$

where lap(F^n) is the smallest number of intervals on each of which F^n is monotonic. In this case, lap$(F^n)$$=c_n+d_n+e_n+f_n$ $=2/\sqrt{5}(\mu^{n+1}-\lambda^{n+1})$ because $c_n+d_n+e_n+f_n$ is the total number of the piecewise linear intervals of F^n, which is rationalized as the smallest one because F^n cannot be monotonic on the interval if we connect adjacent two of those intervals to get an interval. Therefore, we have $h_F = \log\, \mu$.

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