Strong Time Operators in Algebraic Quantum Mechanics and Quantum Field Theory

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Abstract

Some aspects of strong time operators in an abstract algebraic quantum mechanics (including quantum statistical mechanics) and in quantum field theory (QFT) are described. As for the facts presented in QFT, they have essentially been established in a previous paper (A. Arai, *Rev. Math. Phys.* **17**(2005), 1071–1109).

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1 Introduction: Strong Time Operators and Their Fundamental Properties

This paper is concerned with time operators in algebraic quantum mechanics (including quantum statistical mechanics) and quantum field theory. The concept of time operators itself is a general one and there are some types on them [1, 2, 5, 9]. In the present paper we concentrate our attention on strong time operators as defined below and their realizations in algebraic quantum mechanics and quantum field theory.

Let $\mathcal{H}$ be a complex Hilbert space. We denote the inner product and the norm of $\mathcal{H}$ by $\langle \cdot\,,\,\cdot \rangle_{\mathcal{H}}$ (anti-linear in the first variable) and $\|\cdot\|_{\mathcal{H}}$ respectively. If there is no danger

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of confusion, then the subscript $\mathcal{H}$ in $\langle \cdot\,,\,\cdot \rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$ is omitted. For a linear operator A on a Hilbert space, $\mathrm{D}(A)$ (resp. $\mathrm{Ran}(A)$) denotes the domain (resp. range) of A. If A is closable, then we denote its closure by $\bar{A}$.

Let H be a self-adjoint operator on $\mathcal{H}$ and T be a symmetric operator on $\mathcal{H}$. If there exists a bounded self-adjoint operator $C \neq 0$ on $\mathcal{H}$ such that $\mathrm{D}(C) = \mathcal{H}$,

$$e^{-itH}\mathrm{D}(T) \subset \mathrm{D}(T), \quad \forall t \in \mathbb{R}, \tag{1}$$

and

$$Te^{-itH}\psi = e^{-itH}(T + tC)\psi, \quad \forall \psi \in \mathrm{D}(T), \forall t \in \mathbb{R}, \tag{2}$$

then we call T a *strong time operator of H with non-commutative factor C*. The operator C is uniquely determined by the pair (T, H), because (1) and (2) are equivalent to the operator equality

$$e^{itH}Te^{-itH} = T + tC, \quad \forall t \in \mathbb{R}. \tag{3}$$

and $\mathrm{D}(T)$ is dense in $\mathcal{H}$.

The name "non-commutative factor" for C comes from the following fact:

Proposition 1.1 *If T is a strong time operator of H with non-commutative factor C, then*

$$\langle T\psi, H\phi\rangle - \langle H\psi, T\phi\rangle = \langle \psi, iC\phi\rangle\,, \quad \forall \psi, \phi \in \mathrm{D}(T) \cap \mathrm{D}(H). \tag{4}$$

Proof. For all $t \in \mathbb{R}$ and $\psi, \phi \in D(T) \cap D(H)$, we have

$$\left\langle T\psi, e^{-itH}\phi\right\rangle = \left\langle e^{itH}\psi, T\phi\right\rangle + t\,\langle \psi, C\phi\rangle\,.$$

Differentiating the both sides in t at $t = 0$, we obtain (4). □

Remark 1.2 In the case $C = I$ (the identity on $\mathcal{H}$), T is simply called a *strong time operator* of H. In this case, (4) shows that (T, H) is a symmetric representation of the canonical commutation relation (CCR) with one degree of freedom in the *weak sense*.

Remark 1.3 The converse of Proposition 1.1 is not true, i.e., (4) does not necessarily imply that T is a strong time operator of H with non-commutative factor C.

Domain properties of a pair (T, H) satisfying (3) can be found in an explicit way. Let $f \in C_0^\infty(\mathbb{R})$ (the set of all the infinitely differentiable functions on $\mathbb{R}$ with compact support) and $\psi \in \mathcal{H}$. Then one can define a vector

$$\psi_f := \int_{\mathbb{R}} f(t)e^{-itH}\psi dt, \tag{5}$$

where the integral on the right hand side is taken in the sense of the strong Riemann integral. For a subset $\mathcal{D} \neq \emptyset$ of $\mathcal{H}$, we define a subspace $\mathcal{D}_H$ by

$$\mathcal{D}_H := \mathcal{L}(\{\psi_f | f \in C_0^\infty(\mathbb{R}), \psi \in \mathcal{D}\}), \tag{6}$$

where $\mathcal{L}(S)$ means the subspace algebraically spanned by all the vectors in the set S.

Proposition 1.4

(i) $\mathcal{D}_H \subset \mathrm{D}(H)$ *and*

$$H\psi_f = -i\psi_{f'}, \quad \forall \psi_f \in \mathcal{D}_H, \tag{7}$$

where f' is the derivative of f. In particular, $H\mathcal{D}_H \subset \mathcal{D}_H$.

(ii) *If $\mathcal{D}$ is dense in $\mathcal{H}$, then so is $\mathcal{D}_H$.*

Proof. (i) For all $\phi \in \mathrm{D}(H)$ and $\psi_f \in \mathcal{D}_H$, we have

$$\begin{aligned} \langle H\phi, \psi_f \rangle &= i \int_{\mathbb{R}} f(t) \frac{d}{dt} \left\langle e^{itH}\phi, \psi \right\rangle dt \\ &= -i \int_{\mathbb{R}} f'(t) \left\langle \phi, e^{-itH}\psi \right\rangle dt \quad \text{(integration by parts)} \\ &= \langle \phi, (-i)\psi_{f'} \rangle . \end{aligned}$$

Hence $\psi_f \in \mathrm{D}(H^*) = \mathrm{D}(H)$ and $H\psi_f = -i\psi_{f'}$.

(ii) It is sufficient to show that $\mathcal{D} \subset \overline{\mathcal{D}}_H$. Take a function $\rho \in C_0^\infty(\mathbb{R})$ such that $\rho(t) \geq 0, \forall t \in \mathbb{R}$, $\operatorname{supp} \rho \subset [-1, 1]$ and $\int_{\mathbb{R}} \rho(t)dt = 1$, where, for a function f on $\mathbb{R}$, $\operatorname{supp} f$ means the support of f. Let $\psi \in \mathcal{D}$ and $\phi_n := \psi_{\rho_n}$, where $\rho_n(t) := n\rho(nt), t \in \mathbb{R}$. Then $\phi_n \in \mathcal{D}_H$ and

$$\|\psi - \phi_n\| = \left\| \int_{\mathbb{R}} \rho(t)(\psi - e^{-itH/n}\psi)dt \right\| \leq \int_{[-1,1]} \rho(t)\|\psi - e^{-itH/n}\psi\|dt.$$

It is easy to see that one can apply the Lebesgue dominated convergence theorem to the last integral, obtaining $\lim_{n\to\infty} \int_{[-1,1]} \rho(t)\|\psi - e^{-itH/n}\psi\|dt = 0$. Hence $\lim_{n\to\infty} \phi_n = \psi$. Thus the desired result follows. □

Proposition 1.5 *Let T be a strong time operator of H with non-commutative factor C. Then:*

(i) *One has*

$$\mathrm{D}(T)_H \subset \mathrm{D}(\bar{T}) \cap \mathrm{D}(H) \tag{8}$$

and

$$\bar{T}\psi_f = (T\psi)_f + (C\psi)_{tf}, \quad \forall \psi_f \in \mathrm{D}(T)_H. \tag{9}$$

In particular, $\mathrm{D}(\bar{T}) \cap \mathrm{D}(H)$ is dense in $\mathcal{H}$.

(ii) *One has*

$$\mathrm{D}(T)_H \subset \mathrm{D}(\bar{T}H) \cap \mathrm{D}(H\bar{T}) \tag{10}$$

and

$$[\bar{T}, H] = iC \quad \text{on } \mathrm{D}(T)_H, \tag{11}$$

where $[X, Y] := XY - YX$. In particular, $\mathrm{D}(\bar{T}H) \cap \mathrm{D}(H\bar{T})$ is dense in $\mathcal{H}$.

Proof. (i) Let $\psi_f \in \mathrm{D}(T)_H$. Then $f(t)e^{-itH}\psi \in \mathrm{D}(T)$ for all $t \in \mathbb{R}$ and

$$T(f(t)e^{-itH}\psi) = f(t)e^{-itH}T\psi + tf(t)e^{-itH}C\psi.$$

Hence $\int_{\mathbb{R}} T(f(t)e^{-itH}\psi)dt$ exists and is equal to $(T\psi)_f + (C\psi)_{tf}$. Therefore $\psi_f \in \mathrm{D}(\bar{T})$ and (9) holds. In particular, we have (8). Since $\mathrm{D}(T)$ is dense, so is $\mathrm{D}(T)_H$ by Proposition 1.4-(ii).

(ii) It follows from Proposition 1.4-(i) and (9) that $\bar{T}\psi_f \in \mathrm{D}(H)$ and

$$H\bar{T}\psi_f = -i(T\psi)_{f'} - i(C\psi)_f - i(C\psi)_{tf'}.$$

Similarly $H\psi_f = -i\psi_{f'} \in \mathrm{D}(\bar{T})$ and

$$\bar{T}H\psi_f = -i(T\psi)_{f'} - i(C\psi)_{tf'}.$$

Hence (11) follows. □

Remark 1.6 In the case $C = I$, (11) means that $(\bar{T}, H)$ is a symmetric representation of the CCR with one degree of freedom. Hence, for a general C, $(\bar{T}, H)$ may be regarded as a symmetric representation of a *deformed* CCR with one degree of freedom.

Remarkable properties of a strong time operator are summarized in the next theorem [1].

Theorem 1.7 *Let T be a strong time operator of H with non-commutative factor C.*

(i) *Let H be semi-bounded (bounded below or bounded above) and $CT \subset TC$. Then T is not essentially self-adjoint.*

(ii) *H is reduced by $\overline{\mathrm{Ran}(C)}$ and the reduced part $H|\overline{\mathrm{Ran}(C)}$ to $\overline{\mathrm{Ran}(C)}$ is purely absolutely continuous.*

(iii) *Let H be bounded below. Then, for all $\beta > 0$, $e^{-\beta H}\mathrm{D}(\overline{T}) \subset \mathrm{D}(\overline{T})$ and*

$$\overline{T}e^{-\beta H}\psi - e^{-\beta H}\overline{T}\psi = -i\beta e^{-\beta H}C\psi, \quad \psi \in \mathrm{D}(\overline{T}).$$

(iv) *For $\psi, \phi \in \mathrm{D}(T^n)$ ($n \in \mathbb{N}$), we define constants $d_k^T(\phi, \psi)$ ($k = 1, \cdots, n$) by the following recursion relation:*

$$d_1^T(\phi, \psi) := \|T\phi\|\|\psi\| + \|\phi\|\|T\psi\|, \tag{12}$$

$$d_n^T(\phi, \psi) := \|T^n\phi\|\|\psi\| + \|\phi\|\|T^n\psi\| + \sum_{r=1}^{n-1} {}_nC_r d_{n-r}^T(\phi, T^r\psi), \quad n \geq 2, \tag{13}$$

where ${}_nC_r := n!/(n-r)!r!$. Assume that

$$CT \subset TC. \tag{14}$$

Then, for all $t \in \mathbb{R} \setminus \{0\}$,

$$|\left\langle \phi, e^{-itH}C^n\psi \right\rangle| \leq \frac{d_n^T(\phi, \psi)}{|t|^n}, \quad \phi, \psi \in \mathrm{D}(T^n). \tag{15}$$

(v) *For each* $n \in \mathbb{N}$*, there exists a subspace* $\mathcal{D}_n(T, C)$ *such that, for all* $\phi \in \mathrm{D}(T^n)$ *and* $\psi \in \mathcal{D}_n(T, C)$ *(*$\phi, \psi \neq 0$*),*

$$|\left\langle \phi, e^{-itH} C^n \psi \right\rangle| \leq \frac{d_n(\phi, \psi)}{|t|^n}, \quad t \in \mathbb{R} \setminus \{0\}$$

where $d_n(\phi, \psi) > 0$ *is a constant independent of* t*.*

2 Strong Time Operators in Algebraic Quantum Mechanics

In this section we show how strong time operators appear in the context of abstract algebraic quantum mechanics described by a density operator.

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{L}_2(\mathcal{H})$ be the Hilbert space of the Hilbert-Schmidt class on $\mathcal{H}$ with inner product

$$\langle A, B \rangle_2 := \mathrm{Tr}\,(A^* B), \quad A, B \in \mathcal{L}_2(\mathcal{H}), \tag{16}$$

where Tr means trace. Let H be a self-adjoint operator on $\mathcal{H}$ and, for a linear operator A on $\mathcal{H}$,

$$A(t) := e^{itH} A e^{-itH}, \quad t \in \mathbb{R}, \tag{17}$$

the Heisenberg operator of A with H. If H represents the Hamiltonian of a quantum system, then $A(t)$ is interpreted as the time development of a quantum object A with the Hamiltonian H, where t denotes time in the classical sense. Let ρ be a nonnegative trace class operator on $\mathcal{H}$ such that $\mathrm{Tr}\,\rho = 1$. Such an operator ρ is called a *density operator* or a *density matrix* on $\mathcal{H}$. Then a two-point correlation function with density operator ρ is defined by

$$W_{AB}(t) := \mathrm{Tr}\,(\rho A B(t)) = \langle A^* \rho, B(t) \rangle_2, \quad A, B \in \mathcal{L}_2(\mathcal{H}), t \in \mathbb{R}. \tag{18}$$

Remark 2.1 If $\rho = \langle \psi, \cdot \rangle_{\mathcal{H}} \psi$ with $\|\psi\|_{\mathcal{H}} = 1$ ($\psi \in \mathcal{H}$), then

$$W_{AB}(t) = \langle \psi, AB(t)\psi \rangle_{\mathcal{H}},$$

the standard expectation value of $AB(t)$ with the state vector ψ.

Let J be a conjugation on $\mathcal{H}$, i.e., J is an anti-linear operator on $\mathcal{H}$ such that $J^2 = I$ and $\|J\psi\| = \|\psi\|, \forall \psi \in \mathcal{H}$. It is well known [4, Chapter 1, 1.3.3] that there exists a unique unitary operator $U_J : \mathcal{L}_2(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}$ such that

$$U_J P_{\phi,\psi} = \psi \otimes J\phi, \quad \psi, \phi \in \mathcal{H}, \tag{19}$$

where

$$P_{\phi,\psi} := \langle \phi, \cdot \rangle\, \psi. \tag{20}$$

Moreover, under the condition that

$$J\mathrm{D}(H) \subset \mathrm{D}(H), \tag{21}$$

one has [4, Chapter 4]

$$W_{AB}(t) = \left\langle U_J(A^*\rho), e^{it\mathcal{L}_H} U_J(B) \right\rangle_{\mathcal{H}\otimes\mathcal{H}}, \quad A, B \in \mathcal{L}_2(\mathcal{H}), \tag{22}$$

where the operator

$$\mathcal{L}_H := \overline{H \otimes I - I \otimes JHJ} \tag{23}$$

is called the *Liouvillian* of H. It would be natural to ask if $\mathcal{L}_H$ has a strong time operator.

Proposition 2.2 *Let T be a strong time operator of H with non-commutative factor C. Then, for all $a, b \in \mathbb{R}$ such that $a + b \neq 0$, the operator*

$$\mathcal{T}_{a,b} := aT \otimes I + bI \otimes JTJ \tag{24}$$

with domain $\mathrm{D}(\mathcal{T}_{a,b}) := D(T)\hat{\otimes}J\mathrm{D}(T)$ ($\hat{\otimes}$ means algebraic tensor product) is a strong time operator of $\mathcal{L}_H$ with non-commutative factor

$$C_{a,b} := aC \otimes I + bI \otimes JCJ \neq 0. \tag{25}$$

Proof. One has

$$e^{-it\mathcal{L}_H} = e^{-itH} \otimes (Je^{-itH}J).$$

Using this formula, one can directly see that $(T \otimes I)|(D(T)\hat{\otimes}JD(T))$ (resp. $(I \otimes JTJ)$ $|(D(T)\hat{\otimes}JD(T))$) is a strong time operator of $\mathcal{L}_H$ with non-commutative factor $C \otimes I$ (resp. $I \otimes (JCJ)$). The desired result immediately follows from these facts. It is easy to see that the condition $a + b \neq 0$ ensures that $C_{a,b} \neq 0$, which implies that $\mathcal{T}_{a,b} \neq 0$. □

By Proposition 2.2, we can apply Theorem 1.7-(iv) to obtain a decay property (in time) of $W_{AB}(t)$:

Theorem 2.3 *Let T be a strong time operator of H with non-commutative factor C and $n \in \mathbb{N}$. Assume (21). Let $A \in \mathcal{L}_2(\mathcal{H})$ be such that $U_J(A^*\rho) \in \mathrm{D}(\mathcal{T}_{a,b}^n)$ and $B \in \mathcal{L}_2(\mathcal{H})$ be such that $U_J(B) = C_{a,b}^n \Psi$ with some $\Psi \in \mathrm{D}(\mathcal{T}_{a,b}^n)$. Then*

$$|W_{AB}(t)| \leq \frac{d_n^{\mathcal{T}_{a,b}}(U_J(A^*\rho), \Psi)}{|t|^n}, \quad t \in \mathbb{R} \setminus \{0\}. \tag{26}$$

Proof. Let $\Phi = U_J(A^*\rho)$. Then, by (22), we have $W_{AB}(t) = \left\langle \Phi, e^{it\mathcal{L}_H} C_{a,b}^n \Psi \right\rangle_{\mathcal{H}\otimes\mathcal{H}}$. Since Φ and Ψ are in $\mathrm{D}(\mathcal{T}_{a,b}^n)$, (22) follows from a simple application of Theorem 1.7-(iv). □

3 Strong Time Operators on Fock Spaces

A fundamental Hilbert space for the description of quantum fields is a *Fock space over a Hilbert space* $\mathcal{H}$:

$$\mathcal{F}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes^n \mathcal{H},$$

where $\otimes^0\mathcal{H} := \mathbb{C}$ and $\otimes^n\mathcal{H}$ $(n \geq 1)$ is the n-fold tensor product of $\mathcal{H}$. For a self-adjoint operator H on $\mathcal{H}$ and each $n \geq 1$, one can define a self-adjoint operator $H^{(n)}$ on $\otimes^n\mathcal{H}$ by

$$H^{(n)} := \overline{\sum_{j=1}^{n}(I \otimes \cdots I \otimes \overset{j\text{th}}{H} \otimes I \otimes \cdots \otimes I)|\hat{\otimes}^n \mathrm{D}(H)},$$

where $\hat{\otimes}^n$ denotes n-fold algebraic tensor product. Let $H^{(0)} := 0$ on $\mathbb{C}$. Then the direct sum of $\{H^{(n)}\}_{n=0}^{\infty}$

$$d\Gamma(H) := \oplus_{n=0}^{\infty} H^{(n)}$$

on $\mathcal{F}(\mathcal{H})$ is self-adjoint and called the *second quantization of* H. We show that, if H has a strong time operator, then so does $d\Gamma(H)$.

Before doing that, however, we make some preliminary remarkarks. Let T be a strong time operator of H with non-commutative factor C. For each $n \in \mathbb{N}$ and $j = 1, \cdots, n$, we define an operator $T_0^{(n,j)}$ on $\otimes^n\mathcal{H}$ by

$$T_0^{(n,j)} := I\hat{\otimes}I \cdots I\hat{\otimes}\overset{j\text{th}}{T}\hat{\otimes}I \cdots \hat{\otimes}I, \tag{27}$$

where $\hat{\otimes}$ means algebraic tensor product. It is easy to see that $T_0^{(n,j)}$ is symmetric.

We also introduce

$$C^{(n,j)} := I \otimes \cdots \otimes I \otimes \overset{j\text{th}}{C} \otimes I \cdots \otimes I, \tag{28}$$

which is a bounded self-adjoint operator on $\otimes^n\mathcal{H}$.

Lemma 3.1 *The operator $T_0^{(n,j)}$ is a strong time operator of $H^{(n)}$ with non-commutative factor $C^{(n,j)}$.*

Moreover, if (14) holds, then

$$C^{(n,j)}T_0^{(n,j)} \subset T_0^{(n,j)}C^{(n,j)}. \tag{29}$$

Proof. Note that $\mathrm{D}(T_0^{(n,j)}) = \mathcal{H}\hat{\otimes} \cdots \hat{\otimes}\, \overset{j\text{th}}{\mathrm{D}(T)}\, \hat{\otimes} \cdots \hat{\otimes}\mathcal{H}$. We have for all $t \in \mathbb{R}$

$$e^{-itH^{(n)}} = \otimes_{j=1}^{n} e^{-itH}.$$

Hence $e^{-itH^{(n)}}\mathrm{D}(T_0^{(n,j)}) \subset \mathrm{D}(T_0^{(n,j)})$ and, for all $\Psi \in \mathrm{D}(T_0^{(n,j)})$,

$$\begin{aligned} T_0^{(n,j)}e^{-itH^{(n)}}\psi &= \left(e^{-itH} \otimes \cdots e^{-itH} \otimes Te^{-itH} \otimes e^{-itH} \cdots \otimes e^{-itH}\right)\Psi \\ &= \left(e^{-itH} \otimes \cdots e^{-itH} \otimes e^{-itH}(T + tC) \otimes e^{-itH} \cdots \otimes e^{-itH}\right)\Psi \\ &= e^{-itH^{(n)}}(T_0^{(n,j)} + tC^{(n,j)})\Psi. \end{aligned}$$

Thus the desired result follows. □

Lemma 3.1 yields the following fact:

Lemma 3.2 *Let*

$$T_0^{(n)} := \sum_{j=1}^{n} T_0^{(n,j)}, \quad n \geq 1 \tag{30}$$

with $\mathrm{D}(T_0^{(n)}) := \hat{\otimes}^n \mathrm{D}(T)$. *Then* $T_0^{(n)}$ *is a strong time operator of* $H^{(n)}$ *with non-commutative factor*

$$C^{(n)} := \sum_{j=1}^{n} C^{(n,j)}. \tag{31}$$

Moreover, if (14) holds, then

$$C^{(n)} T_0^{(n)} \subset T_0^{(n)} C^{(n)}. \tag{32}$$

We are now ready to show that $d\Gamma(H)$ has strong time operators.

Let $j \geq 1$ and

$$T_j := a_0 \oplus a_1 I \oplus \cdots \oplus a_{j-1} I \oplus \left(\oplus_{n=j}^{\infty} T_0^{(n,j)} \right), \tag{33}$$

$$C_j := 0 \oplus 0 \oplus \cdots \oplus 0 \oplus \left(\oplus_{n=j}^{\infty} C^{(n,j)} \right), \tag{34}$$

where $a_1, \cdots, a_{j-1}$ are real constants.

Theorem 3.3 *For each* $j \geq 1$, T_j *is a strong time operator of* $d\Gamma(H)$ *with non-commutative factor* C_j.

Moreover, if (14) holds, then

$$|\left\langle \Phi, e^{-itd\Gamma(H)} C_j^n \Psi \right\rangle| \leq \frac{d_n^{T_j}(\Phi, \Psi)}{|t|^n}, \quad \Psi, \Phi \in \mathrm{D}(T_j^n), t \in \mathbb{R} \setminus \{0\}. \tag{35}$$

Proof. We have for all $t \in \mathbb{R}$

$$\begin{aligned} T_j e^{-itd\Gamma(H)} &= \left(\oplus_{n=0}^{j-1} a_j e^{-itH^{(n)}} \right) \oplus \left(\oplus_{n=j}^{\infty} T_0^{(n,j)} e^{-itH^{(n)}} \right) \\ &= \left(\oplus_{n=0}^{j-1} a_j e^{-itH^{(n)}} \right) \oplus \left(\oplus_{n=j}^{\infty} \left(e^{-itH^{(n)}} T_0^{(n,j)} + t e^{-itH^{(n)}} C^{(n,j)} \right) \right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 3.1)} \\ &= e^{-itd\Gamma(H)} T_j + t e^{-itd\Gamma(H)} C_j. \end{aligned}$$

Hence the first half of the theorem follows.

Assume (14). Then we have $C_j T_j \subset T_j C_j$. Hence we can apply Theorem 1.7-(iv) with (H, T, C) replaced by $(d\Gamma(H), T_j, C_j)$ to obtain (35). □

Let

$$N := d\Gamma(I), \tag{36}$$

the *number operator* on $\mathcal{F}(\mathcal{H})$. The vector $\Omega = \{1, 0, 0, \cdots\} \in \mathcal{F}(\mathcal{H})$ is called the *Fock vacuum* in $\mathcal{F}(\mathcal{H})$. Let P be the orthogonal projection onto the one dimensional subspace

$$\mathcal{F}_0 := \{\alpha\Omega | \alpha \in \mathbb{C}\} \tag{37}$$

Then the operator

$$Q := I - P \tag{38}$$

is the orthogonal projection onto the closed subspace

$$\mathcal{F}_0^\perp = \oplus_{n=1}^\infty \otimes^n \mathcal{H}. \tag{39}$$

It is easy to see that the number operator N is reduced by $\mathcal{F}_0^\perp$. We denote the reduced part of N to $\mathcal{F}_0^\perp$ by N_+. Explicitly we have

$$N_+ = \oplus_{n=1}^\infty nI. \tag{40}$$

Let

$$\Gamma_C := QN_+^{-1/2} d\Gamma(C) N_+^{-1/2} Q. \tag{41}$$

Lemma 3.4 *The operator Γ_C is a bounded self-adjoint operator with $\|\Gamma_C\| \leq \|C\|$.*

Proof. For all $\Psi \in \mathrm{D}(\Gamma_C)$, we have

$$(\Gamma_C \Psi)^{(n)} = \begin{cases} 0 & n = 0 \\ \frac{C^{(n)}}{n} \Psi^{(n)} & n \geq 1 \end{cases}$$

Hence a vector $\Psi \in \mathcal{F}(\mathcal{H})$ is in $\mathrm{D}(\Gamma_C)$ if and only if

$$\sum_{n=1}^\infty \frac{1}{n^2} \|C^{(n)} \Psi^{(n)}\|^2 < \infty.$$

It is easy to see that

$$\|C^{(n)}\| \leq n\|C\|.$$

Hence, for all $\Psi \in \mathcal{F}(\mathcal{H})$,

$$\sum_{n=1}^\infty \frac{1}{n^2} \|C^{(n)} \Psi^{(n)}\|^2 \leq \sum_{n=1}^\infty \|C\|^2 \|\Psi^{(n)}\|^2 < \infty.$$

Therefore it follows that $\mathrm{D}(\Gamma_C) = \mathcal{F}(\mathcal{H})$ and

$$\|\Gamma_C \Psi\|^2 \leq \|C\|^2 \|\Psi\|^2.$$

It is straightforward to see that Γ_C is symmetric. Thus the desired result follows. □

Remark 3.5 Note that, if $C = I$, then

$$\Gamma_I = Q. \tag{42}$$

Namely Γ_I is an orthogonal projection.

Let $a_0 \in \mathbb{R}$ and

$$\mathcal{T}_T := \{a_0\} \oplus \left(\oplus_{n=1}^{\infty} \frac{1}{n} T_0^{(n)} \right). \tag{43}$$

Theorem 3.6 *The operator $\mathcal{T}_T$ is a strong time operator of $d\Gamma(H)$ with non-commutative factor Γ_C.*

Moreover, if (14) holds, then

$$\Gamma_C \mathcal{T}_T \subset \mathcal{T}_T \Gamma_C. \tag{44}$$

and

$$|\left\langle \Phi, e^{-itd\Gamma(H)} \Gamma_C^n \Psi \right\rangle| \leq \frac{d_n^{\mathcal{T}_T}(\Phi, \Psi)}{|t|^n}, \quad \Psi, \Phi \in \mathrm{D}(\mathcal{T}_T^n), t \in \mathbb{R} \setminus \{0\}. \tag{45}$$

Proof. It is easy to see that $\mathcal{T}_T$ is symmetric. We have

$$\begin{aligned} \mathcal{T}_T e^{-itd\Gamma(H)} &= \{a_0\} \oplus \left(\oplus_{n=1}^{\infty} \frac{1}{n} T_0^{(n)} e^{-itH^{(n)}} \right) \\ &= \{a_0\} \oplus \left(\oplus_{n=1}^{\infty} (e^{-itH^{(n)}} \frac{1}{n} T_0^{(n)} + t e^{-itH^{(n)}} \frac{C^{(n)}}{n} \right) \quad \text{(by Lemma 3.2)} \\ &= e^{-itd\Gamma(H)} \mathcal{T}_T + t e^{-itd\Gamma(H)} \Gamma_C. \end{aligned}$$

Hence the first half of the theorem holds.

Assume (14). Then we have $C^{(n)} T_0^{(n)} \subset T_0^{(n)} C^{(n)}$ $(n \geq 0)$, which implies (44).

□

We have from Remark 3.5 and Theorem 1.7 the following fact:

Corollary 3.7 *Consider the case $C = I$. Then*

$$|\left\langle \Phi, e^{-itd\Gamma(H)} Q \Psi \right\rangle| \leq \frac{d_n^{\mathcal{T}_T}(\Phi, \Psi)}{|t|^n}, \quad \Psi, \Phi \in \mathrm{D}(\mathcal{T}_T^n), t \in \mathbb{R} \setminus \{0\}. \tag{46}$$

4 Strong Time Operators on Boson Fock Spaces and Fermion Fock Spaces

There are two important closed subspaces of $\mathcal{F}(\mathcal{H})$. The one is the *boson Fock space over* $\mathcal{H}$:

$$\mathcal{F}_{\mathrm{b}}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n} \mathcal{H},$$

where $\otimes_{\mathrm{s}}^{n} \mathcal{H}$ denotes the n-fold symmetric tensor product of $\mathcal{H}$ with $\otimes_{\mathrm{s}}^{0} \mathcal{H} := \mathbb{C}$, and the other is the *fermion Fock space over* $\mathcal{H}$:

$$\mathcal{F}_{\mathrm{f}}(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_{\mathrm{as}}^{n} \mathcal{H},$$

where $\otimes_{\mathrm{as}}^{n} \mathcal{H}$ denotes the n-fold anti-symmetric tensor product of $\mathcal{H}$ with $\otimes_{\mathrm{as}}^{0} \mathcal{H} := \mathbb{C}$.

The second quantization $d\Gamma(H)$ is reduced by $\mathcal{F}_{\rm b}(\mathcal{H})$ and $\mathcal{F}_{\rm f}(\mathcal{H})$. We denote the reduced part of $d\Gamma(H)$ to $\mathcal{F}_{\#}(\mathcal{H})$ by $d\Gamma_{\#}(H)$ ($\#$ =b, f).

It is easy to see that $\mathcal{T}_T$ and Γ_C are reduced by $\mathcal{F}_{\#}(\mathcal{H})$. We denote the reduced part of $\mathcal{T}_T$ (resp. Γ_C) to $\mathcal{F}_{\#}(\mathcal{H})$ by $\mathcal{T}_T^{(\#)}$ (resp. $\Gamma_C^{(\#)}$). Then Theorem 3.6 implies the following theorem:

Theorem 4.1 *The operator $\mathcal{T}_T^{(\#)}$ is a strong time operator of $d\Gamma_{\#}(H)$ with non-commutative factor $\Gamma_C^{(\#)}$. Moreover, if $C = I$, then (46) with $(d\Gamma(H), \mathcal{T}_T, Q)$ replaced by $(d\Gamma_{\#}(H), \mathcal{T}_T^{(\#)}, Q_{\#})$ holds, where $Q_{\rm b}$ (resp. $Q_{\rm f}$) is the orthogonal projection onto $\{0\} \oplus (\oplus_{n=1}^{\infty} \otimes_{\rm s}^n \mathcal{H})$ (resp. $\{0\} \oplus (\oplus_{n=1}^{\infty} \otimes_{\rm as}^n \mathcal{H})$) from $\mathcal{F}_{\rm b}(\mathcal{H})$ (resp. $\mathcal{F}_{\rm f}(\mathcal{H})$).*

5 Perturbations

In the context of quantum field theory based on the Fock spaces, the second quantization $d\Gamma_{\#}(H)$ of H represents the Hamiltonian of a quantum system of mutually independent infinitely many quantum particles each of which is described by the one particle Hamiltonian H. Hence $d\Gamma_{\#}(H)$ is not a Hamiltonian with interactions among the quantum particles. An interaction among the quantum particles can be introduced by perturbing $d\Gamma_{\#}(H)$:

$$\mathbb{H} := d\Gamma_{\#}(H) + \mathbb{H}_{\rm I}, \tag{47}$$

where $\mathbb{H}_{\rm I}$ is a symmetric operator on $\mathcal{F}_{\#}(\mathcal{H})$. Suppose that $\mathbb{H}$ is essentially self-adjoint and that there exist a closed subspace $\mathcal{K}$ of $\mathcal{F}_{\#}(\mathcal{H})$ which reduces $\overline{\mathbb{H}}$ and a unitary operator U from $\mathcal{K}$ to $\mathcal{F}_{\#}(\mathcal{H})$ satisfying

$$U\overline{\mathbb{H}}_{\mathcal{K}}U^{-1} = d\Gamma_{\#}(H),$$

where $\overline{\mathbb{H}}_{\mathcal{K}}$ denotes the reduced part of $\overline{\mathbb{H}}$ to $\mathcal{K}$.

Theorem 5.1 *Let*

$$\tau^{(\#)} := U^{-1}\mathcal{T}_T^{(\#)}U, \tag{48}$$

$$\gamma^{(\#)} := U^{-1}\Gamma_C^{(\#)}U. \tag{49}$$

Then $\tau^{(\#)}$ is a strong time operator of $\overline{\mathbb{H}}_{\mathcal{K}}$ with non-commutative factor $\gamma^{(\#)}$.

Moreover, if (14) holds, then

$$|\left\langle \Phi, e^{-it\overline{\mathbb{H}}_{\mathcal{K}}} \left(\gamma^{(\#)}\right)^n \Psi \right\rangle| \leq \frac{d_n^{\tau^{(\#)}}(\Phi, \Psi)}{|t|^n}, \quad \Psi, \Phi \in \mathrm{D}(\left(\tau^{(\#)}\right)^n), t \in \mathbb{R} \setminus \{0\}. \tag{50}$$

Remark 5.2 A unitary operator U satisfying the condition stated above may be constructed via the scattering theory for the pair $(\mathbb{H}, d\Gamma_{\#}(H))$ [10], depending on the form of $\mathbb{H}_{\rm I}$.

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